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A Mini-Course on Recent Progress in Algebraic K-Theory and its Relationship with Topology and Analysis

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Abstract. We attempt to survey some of the interrelationships between algebraic K-theory, topology, and analysis. The exposition is in two parts, the first dealing with K-theory of group rings and connections with topology, and the second dealing with flat bundles, secondary invariants, and connections with differential geometry.

Brief Outline

This mini-course is divided into two sections, each with its own bibliography. They can be read independently of one another, even though there are some connections between the two. The first describes a "modern" perspective on one of the most classical areas of algebraic K-theory, the study of Whitehead groups and K-theory of group rings. The second describes the connection between the theory of secondary classes in topology and differential geometry with algebraic K-theory, especially for fields, and provides an introduction to some recent work of Neumann and Yang on hyperbolic 3-manifolds.

Both parts of the mini-course are intended for beginners in these areas, with some previous exposure to higher K-theory, as might be found in [**R1**] or in [**LP**], for example. Therefore I hope the experts will forgive me for deliberately bypassing many interesting and intricate technical details, and for skipping over much of the history, in order to concentrate on what seem to me to be the key ideas of the subject. For this reason, some statements are not entirely accurate. The reader who wants to know more is urged to consult the papers in the bibliography.

I should also say that there is little if anything that is new in these notes, except for the way I have organized the material. My aim has been primarily to show how

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recent developments in algebraic K-theory are related to problems in other areas of mathematics.

Notation

k — a regular commutative ring. (The main case of interest, as we shall see, is $k = \mathbb{Z}$. We will also be interested in the case of k a field.)

kG — the group ring over k of a group G.

BG — the classifying space of G, a space with contractible universal cover and fundamental group G (and the homotopy type of a CW complex). Such a space is unique up to homotopy equivalence. It has the property that $H_*(BG)$ is the usual group homology of G.

$\mathbb{K}(R)$ — the (non-connective) K-theory spectrum of a ring R. By definition, its homotopy groups are the K-groups of R. The associated infinite loop space is $K_0(R) \times BGL(R)^+$, the result of applying the Quillen $+$-construction to the classifying space of the general linear group $GL(R)$, modified to have the correct π_0. The non-negative homotopy groups of $\mathbb{K}(R)$ are the same as those of $K_0(R) \times BGL(R)^+$. But in general, $\mathbb{K}(R)$ can have homotopy groups in negative degrees, the negative K-groups of R in the sense of Bass.

$K_i(R)$ — equal by definition to $\pi_i(\mathbb{K}(R))$.

I. "Assembly, Novikov Conjectures, and Control"

A. Applications of K-theory of group rings

Group rings arise in topology because if X is a space with universal cover $\widetilde{X}$, then the (singular, simplicial, or cellular) chain complex of $\widetilde{X}$, equipped with its action of $\pi_1(X)$, or equivalently, the chain complex of X with local coefficients, is

$$C_*(\widetilde{X};\, k) \cong C_*(X;\, k\pi_1(X)),$$

a chain complex of free $k\pi_1(X)$-modules.

I.A.1. Definition. Let

$$\begin{cases} \mathrm{Wh}_1(G) = K_1(\mathbb{Z}G)/\left(\{\pm 1\} \times G_{\mathrm{ab}}\right), \\ \mathrm{Wh}_0(G) = K_0(\mathbb{Z}G)/\mathbb{Z}. \end{cases}$$

Here $\{\pm 1\} \times G_{\mathrm{ab}}$ maps to $K_1(\mathbb{Z}G)$ via the obvious embedding of $\{\pm 1\} \times G$ into the units of the ring $\mathbb{Z}G$, and $\mathbb{Z}$ embeds in $K_0(\mathbb{Z}G)$ as $K_0(\mathbb{Z})$. The groups Wh_1 and

Wh_0 are related, since

$$\mathrm{Wh}_0(G) \hookrightarrow \mathrm{Wh}_1(G \times \mathbb{Z})$$

as a direct summand, by the Bass-Heller-Swan Theorem [**BHS**].

Classical applications of $K_1(\mathbb{Z}G)$.

Whitehead torsion. If $X \hookrightarrow Y$ is a (simplicial) homotopy equivalence of finite polyhedra, its Whitehead torsion in $\mathrm{Wh}(\pi_1(X))$ is the obstruction to "simplicity." Roughly speaking, this measures whether the homotopy equivalence is built out of elementary expansions and contractions or not. See [**Co**]. While, from its definition, Whitehead torsion appears to be a simplicial notion, Chapman [**Ch**] proved the remarkable fact that it is really a topological invariant, *i.e.*, it is preserved under homeomorphisms (even those that do not respect the simplicial structure).

The s-Cobordism Theorem. Suppose X^n, X'^n are closed connected manifolds, W^{n+1} is a compact manifold with boundary, and $\partial W = X \amalg X'$. Then W is called a *cobordism* between X and X'. W is called an *h-cobordism* if $X \hookrightarrow W$, $X' \hookrightarrow W$ are homotopy equivalences. The most obvious example of an h-cobordism is of course a cylinder, $W = X \times [0, 1]$. Observe that the Whitehead torsion of $X \hookrightarrow W$ in $\mathrm{Wh}(\pi_1(X))$ vanishes if $W \cong X \times [0, 1]$. The classification theory of manifolds is heavily based on:

I.A.2. The s-Cobordism Theorem (Barden-Mazur-Stallings; see [**Ke**]). *If $n \geq 5$, then an h-cobordism is diffeomorphic to a cylinder if and only if its Whitehead torsion vanishes. Furthermore, again assuming $n \geq 5$, every class in $Wh(\pi_1(X))$ can be realized by an h-cobordism.*

An immediate corollary is worth pointing out explicitly: if there is an h-cobordism W between X^n and X'^n, if $\mathrm{Wh}(\pi_1(X)) = 0$, and if $n \geq 5$, then W is diffeomorphic to $X \times [0, 1]$, and in particular, X' is diffeomorphic to X. However, the hypotheses here are stronger than they need to be to reach this conclusion. For example, it can happen that $X' \cong X$ even if $W \ncong X \times [0, 1]$. In this case, W is called an *inertial* h-cobordism based on X.

So Whitehead torsion plays a big role in manifold theory. The oldest example of interest is the classification of lens spaces X, quotients of S^{2k-1}, viewed as the unit sphere in $\mathbb{C}^k$, by a free (complex) linear action of $\mathbb{Z}/p$, where p is an odd prime. In this case, $\mathrm{Wh}(\pi_1(X))$ is known to be free abelian of rank $(p-3)/2$, and there can be many lens spaces with the same dimension and fundamental group that are h-cobordant and not homeomorphic. Theorem I.A.2 immediately implies that there must be plenty of "fake" lens spaces and/or non-trivial inertial h-cobordisms of lens spaces when n, $p \geq 5$, since the Whitehead group is infinite and there are only finitely many lens spaces for fixed n and p.

Of course, when the first work on lens spaces was done, the s-Cobordism Theorem did not exist. Instead, obstructions to diffeomorphisms (or PL homeomorphisms) between lens spaces were constructed using the classical theory of Reidemeister torsion, which is closely related to Whitehead torsion [**M**]. Thanks to the topological invariance of Whitehead torsion, we now know that it also obstructs homeomorphisms and not just PL homeomorphisms.

Classical applications of $K_0(\mathbb{Z}G)$. *The Wall obstruction.* If X is a connected CW complex and if X is "finitely dominated," that is, is homotopically a retract of a finite CW complex, then often one wants to know if X is homotopy equivalent to a *finite* complex. This is not always the case; a necessary and sufficient condition is the vanishing of its *Wall obstruction* in $\mathrm{Wh}_0(\pi_1(X))$. The idea is that if X were actually homotopically finite, its chain complex with local coefficients would be equivalent to a finite chain complex of finitely generated free $\mathbb{Z}G$-modules. Finite domination gives only that the chain complex is chain homotopy equivalent to a finite chain complex of finitely generated *projective* $\mathbb{Z}G$-modules. The alternating sum of the classes of these projective modules, computed in the stable class group $\widetilde{K}_0(\mathbb{Z}G) = \mathrm{Wh}_0(\pi_1(X))$, is the Wall obstruction. Since the class group $\widetilde{K}_0(\mathbb{Z}G) = \mathrm{Wh}_0(\pi_1(X))$ measures the (stable) difference between projective and free modules, if the obstruction is non-zero, then it's easy to see that the given chain complex cannot be chain homotopy equivalent to a finite chain complex of finitely generated free $\mathbb{Z}G$-modules, and thus X cannot be homotopy-equivalent to a finite complex. Wall [**Wa**] proved the much more difficult converse, that if the obstruction vanishes, then one can construct a finite CW complex homotopy-equivalent to X.

$K_0(\mathbb{Z}G)$ also arises in the *spherical space form problem* (Swan [**Sw**] *et al.*), the problem of determining what finite groups can act freely on spheres, and in the problem of determining if a non-compact manifold is homeomorphic to the interior of a compact manifold with boundary (Siebenmann). In both cases, a Wall obstruction is responsible.

B. Assembly

Out of the spectrum $\mathbb{K}(k)$ we can make a homology theory $H_*(\,\cdot\,;\mathbb{K}(k))$ such that

$$H_*(\text{point}; \mathbb{K}(k)) = K_*(k).$$

The definition is simply that

$$H_*(X; \mathbb{K}(k)) = \pi_*(X_+ \wedge \mathbb{K}(k)), \tag{I.B.1}$$

where X_+ denotes X with a disjoint basepoint adjoined,[1] and the basic properties of stable homotopy guarantee that this satisfies the Eilenberg-MacLane axioms for a homology theory. This somewhat intangible homology theory is close to ordinary homology with coefficients in $K_*(k)$. More precisely, there is a spectral sequence of Atiyah-Hirzebruch type

$$E^2_{p\,q} = H_p(X; K_q(k)) \Rightarrow H_{p+q}(X; \mathbb{K}(k)). \tag{I.B.2}$$

Since we are assuming k is regular, $K_q(k) = 0$ for $q < 0$, and so the spectral sequence is confined to the first quadrant.

Many results on K-theory of group rings make use of a natural map

$$H_*(BG; \mathbb{K}(k)) \xrightarrow{A} K_*(kG),$$

or to be fancier, a map of spectra

$$BG_+ \wedge \mathbb{K}(k) \to \mathbb{K}(kG),$$

called *assembly.* There are many possible definitions; here are a few:

[1] Those not familiar with spectra might be confused about what the definition means. Here π_* is computed in the category of spectra, and so really refers to stable homotopy, *i.e.*, to a *limit* of homotopy groups of spaces.

I.B.3. Definition (Loday [**L**]). Note that $G \hookrightarrow GL_1(kG) \hookrightarrow GL(kG)$, so one obtains an induced map

$$\alpha\colon BG \to BGL(kG) \to BGL(kG)^+ \to \mathbb{K}(kG).$$

Then if $\mu\colon \mathbb{K}(kG)\wedge\mathbb{K}(k) \to \mathbb{K}(kG)$ is the usual product coming from tensor products over k, A is the composite

$$BG_+ \wedge \mathbb{K}(k) \xrightarrow{\alpha\wedge\mathrm{id}} \mathbb{K}(kG) \wedge \mathbb{K}(k) \xrightarrow{\mu} \mathbb{K}(kG).$$

I.B.4. Definition (Weiss-Williams [**WW**]). Any homotopy functor (from spaces to spectra) can be shown to have a "best approximation" by a homology functor, and there is a natural transformation from this approximating functor to the original functor. Apply this to $X \mapsto \mathbb{K}(k\pi_1(X))$ and the result is assembly. (To be rigorous, one needs groupoids and ringoids because of basepoint problems.)

I.B.5. Examples.

(1) G arbitrary, $K_0(k) = \mathbb{Z}$ (which is the case if k is a PID). In degree 0, assembly is the map $\mathbb{Z} = K_0(k) \to K_0(kG)$. In degree 1, assembly is the map
$$K_1(k) \times G_{\mathrm{ab}} \to K_1(kG).$$
So $\mathrm{Wh}_0(G)$ and $\mathrm{Wh}_1(G)$ are the cokernels of assembly (with $k = \mathbb{Z}$) in degrees 0 and 1. To obtain a well behaved definition of higher and lower Whitehead groups, we could define $\mathbf{Wh}(G;\, k)$ to be the cofiber of $A : BG_+ \wedge \mathbb{K}(k) \to \mathbb{K}(kG)$ (in the stable homotopy category), and let $\mathrm{Wh}_*(G;\, k) = \pi_*(\mathbf{Wh}(G;\, k))$, $\mathrm{Wh}_*(G) = \mathrm{Wh}_*(G;\, \mathbb{Z})$. This would agree with our previous definitions of Wh_0 and Wh_1, and would also have some good functorial properties. In general we would obtain an exact sequence
$$\cdots \to \mathrm{Wh}_{*+1}(G;\, k) \xrightarrow{\partial} H_*(BG;\, \mathbb{K}(k)) \xrightarrow{A} K_*(kG) \to \mathrm{Wh}_*(G;\, k) \xrightarrow{\partial} H_{*-1}(BG;\, \mathbb{K}(k)) \xrightarrow{A} \cdots .$$

(2) $G = \mathbb{Z}$, $kG = k[t,\, t^{-1}]$. Assembly A is an isomorphism (recall we're assuming k is regular)
$$K_*(k) \oplus K_{*-1}(k) \xrightarrow{\cong} K_*(k[t,\, t^{-1}]),$$
by the "Fundamental Theorem" of K-theory (Quillen's generalization of the Bass-Heller-Swan Theorem [**Ql**, §6, Theorem 8]). If we were to drop the assumption that k is regular, assembly would still be a split injection, but would have as cokernel a direct sum of two copies of $NK_*(k)$.

(3) More generally, A an isomorphism for G free abelian or free (Quillen, Gersten [**Ge**]).

(4) G finite. A can be interesting, but is usually very far from being an isomorphism. For example, if $k = \mathbb{Z}$, $H_p(G;\, K_q(\mathbb{Z})) \otimes \mathbb{Q} = 0$ for $p > 0$, so
$$\operatorname{rank} H_j(BG;\, \mathbb{K}(\mathbb{Z})) = \operatorname{rank} K_j(\mathbb{Z}) = \begin{cases} 1, & j = 0 \\ 1, & j \equiv 1 \pmod 4,\ j \ge 5, \\ 0, & \text{otherwise,} \end{cases}$$

while rank $K_j(\mathbb{Z}G)$ depends on the representation theory of G, and can be rather large. (To see this, relate $K_j(\mathbb{Z}G)$ to $K_j(\mathbb{Q}G)$ using the localization sequence. Since $\mathbb{Q}G$ is a semi-simple algebra over a field, its K-theory splits as a direct sum of K-groups of finite-dimensional division algebras.)

For an interesting subcase, suppose $k = \mathbb{R}$. Then by a theorem of Suslin [**Su1**], $\mathbb{K}(\mathbb{R})$ is equivalent to *bo*, the connective real *topological* K-theory spectrum, provided one takes finite coefficients. Since $\mathbb{R}G$ is a direct sum of matrix algebras over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ when G is finite, $\mathbb{K}(\mathbb{R}G)$ is similarly equivalent (with finite coefficients) to a direct sum of copies of *bo*, *bu*, and *bsp* (a shifted version of *bo*). One gets the cleanest results when $|G|$ is a power of 2. Then assembly with $\mathbb{Z}/2^\infty = \varinjlim \mathbb{Z}/2^m$ coefficients in periodic topological K-theory is an isomorphism [**R2**], by a "dual" variant of the Atiyah-Segal Theorem in topological K-theory. Putting this together with Suslin's theorem implies that $A\colon H_*(BG;\, \mathbb{K}(\mathbb{R})(\mathbb{Z}/2^\infty)) \to K_*(\mathbb{R}G)(\mathbb{Z}/2^\infty)$ becomes an isomorphism after inverting the Bott element. Untangling this statement for the integral assembly map shows that $A\colon H_*(BG;\, \mathbb{K}(\mathbb{R})) \to K_*(\mathbb{R}G)$ has many $\mathbb{Z}/2$ summands in its image not coming from $K_*(\mathbb{R})$, and thus is definitely a non-trivial map.

C. Novikov Conjectures

In the section above, we defined the assembly map

$$H_*(BG;\, \mathbb{K}(k)) \xrightarrow{A} K_*(kG).$$

All the known results on assembly for the K-theory of group rings are consistent with the following conjectures:

I.C.1. Conjecture. *For G torsion-free, A is an isomorphism.* (*This includes the conjecture that $Wh_*(G) = 0$.*)

I.C.2. K-Theory Novikov Conjecture. *For general G, A is rationally injective.*

Connections with Topology. Conjectures I.C.1–2 are analogues of conjectures of Borel and Novikov (respectively) about "rigidity" of manifolds with large fundamental group. For simplicity, we state these only for closed aspherical manifolds. When applied to a manifold, *closed* means "compact without boundary," and *aspherical* means "having contractible universal cover." The most familiar examples of closed aspherical manifolds are double coset spaces $\Gamma\backslash G/K$, where G is a connected unimodular Lie group, K is a maximal compact subgroup, and Γ is a discrete, torsion-free, cocompact subgroup of G. Compact Riemann surfaces are of this form with $G = PSL(2, \mathbb{R})$, and more generally, compact hyperbolic n-manifolds are of this form with $G = SO_0(n, 1)$.

Borel's conjecture in its simplest form, which was motivated by the Mostow Rigidity Theorem, states that any homotopy equivalence between closed aspherical manifolds should be homotopic to a homeomorphism. Novikov's conjecture, when specialized to closed aspherical manifolds, says that any homotopy equivalence between closed aspherical manifolds should preserve rational Pontrjagin classes. This is weaker than, but consistent with, Borel's conjecture, since by a famous theorem of Novikov, homeomorphisms preserve rational Pontrjagin classes. (In general, homotopy equivalences between non-aspherical manifolds do *not* necessarily preserve rational Pontrjagin classes.)

The Borel and Novikov conjectures are equivalent to statements similar to Conjectures I.C.1–2 for the analogous assembly map in "L-theory" of group rings:

$$H_*(BG; \mathbb{L}(\mathbb{Z})) \xrightarrow{A} L_*(\mathbb{Z}G).$$

L-theory is a sort of K-theory for rings with involution that arises in surgery theory. For more details, one can consult the papers in [**NC**].

Note an analogy: $\mathrm{Wh}_1(G)$ classifies h-cobordisms modulo trivial ones. Similarly, surgery theory says the homotopy groups of the cofiber of the L-theory assembly map classify homotopy-equivalent manifolds modulo homeomorphism. So we have the dictionary:

K**-theory**	$\Longleftrightarrow$	L**-theory**
Wh_*		"Structure sets"
$\mathrm{Wh}_* = 0$		Borel Conjecture
$A \otimes \mathbb{Q}$ injective		Novikov Conjecture.

NOTE. For groups with torsion, as we saw already in the last section, the usual assembly map cannot possibly be an isomorphism, because of what happens for finite groups. Instead, Farrell and Jones [**FJ**] conjecture (and prove in a few cases) that a more complicated assembly map is an isomorphism. Earlier results in the same direction were also given by Quinn [**Qn**]. The Farrell-Jones Conjecture also has a counterpart in analysis, called the Baum-Connes Conjecture [**BCH**], with $\mathbb{K}(kG)$ replaced by the topological K-theory of the reduced group C^*-algebra of G. Very roughly speaking, the idea of all of these modifications of the assembly map is to accept the K-theory of group rings of finite groups (or of virtually cyclic groups, extensions of a finite group by $\mathbb{Z}$) as a "black box," and then to reduce everything to this case.

D. Some Recent Results

I.D.1. THEOREM (Bökstedt-Hsiang-Madsen [**BHM**]). *The K-theory Novikov Conjecture (I.C.2) is true for $k = \mathbb{Z}$ if $H_i(G)$ is finitely generated for all i.*

The method of proof of this theorem uses Waldhausen's A-theory and the cyclotomic trace. It is still unknown if one could prove it using only methods "internal to K-theory"; most experts seems to doubt this. It is also unknown if the condition on G is necessary.

I.D.2. THEOREM (Farrell-Jones [**FJ**]). *The K-theory isomorphism conjecture is rationally true for $k = \mathbb{Z}$, G a cocompact discrete subgroup in a connected Lie group.*

I.D.3. THEOREM (Carlsson-Pedersen [**CP1**], [**CP2**]). *The K-theory assembly map is split injective if one can find a compact model for BG whose universal cover, EG, has a G-equivariant contractible compactification X which is "small at ∞."*

The "small at ∞" condition means that if K is a compact set in EG, then its translates under the G-action become arbitrarily small near $Y = X \setminus EG$. More precisely, if $y \in Y$ and if U is a neighborhood of y in X, then there exists a smaller neighborhood V of y in X such that $gK \cap V \neq \emptyset \Rightarrow gK \subseteq U$. The prototype for this situation arises when BG is a closed manifold of negative curvature. Then EG has a natural disk compactification, obtained by adjoining the "sphere at infinity," and the condition is satisfied.

E. K-Theory with Control

Many of the proofs of the results above are based on the idea of *K-theory with control*. This is motivated by *controlled topology*, of which a prototype result is the Chapman-Ferry Theorem we now state. We will need the following concept: suppose one has maps $f, g\colon M \to N$ and a reference map $N \to X$, where X is a metric space. Then a homotopy H from f to g is called *δ-controlled* if for each $m \in M$, the image in X of the "track" $H_t(m)$, $0 \le t \le 1$, has diameter at most δ. In this case, we also say that f and g are δ-homotopic.

I.E.1. Theorem (Chapman-Ferry [**CF**]). *Given a closed topological manifold X^n with $n \ge 5$, which we equip with a metric d (in the sense of metric spaces, not necessarily in the sense of Riemannian geometry), and given $\varepsilon > 0$, there exists a $\delta > 0$ such that any δ-controlled homotopy equivalence $f : {X'}^n \to X^n$ is ε-homotopic to a homeomorphism. (A homotopy equivalence $f\colon X' \to X$ is called δ-controlled if there is a homotopy inverse $g\colon X \to X'$ to f such that $f \circ g$ is δ-homotopic to 1_X and $g \circ f$ is δ-homotopic to $1_{X'}$. We use X as the reference metric space in both cases.)*

The Idea of Controlled K-Theory. Let X be a space with some structure that enables us to measure distances, for example a metric space. We consider *locally finite* free or projective k-modules $A = \bigoplus_{x\in X} A_x$, where each A_x is a free or projective k-module, and locally finite means that each $x \in X$ has a neighborhood U such that $\bigoplus_{x\in U} A_x$ is finitely generated. The morphisms

$$\phi : A \to B, \quad \phi = \bigoplus_{x,y} \phi^x_y : A_x \to B_y$$

are required to satisfy $\phi^x_y = 0$ if x and y are "not close." Out of this category, we manufacture K-theory $\mathbb{K}(k; X)$ the usual way. There are two main examples.

I.E.2. Example 1 (The Bounded Case). We take for X a metric space, and we require that given ϕ, $\exists d_\phi > 0$ such that $\phi^x_y = 0$ if $d(x, y) > d_\phi$.

I.E.3. Example 2 (The Continuously Controlled Case [**ACFP**]). Suppose X is a topological space which is open and dense in a larger space $\overline{X}$. (Often this will be some sort of compactification of X.) We suppose that for all $z \in \overline{X} \smallsetminus X$ and for any neighborhood V of z in $\overline{X}$, there exists a neighborhood U of z in $\overline{X}$ with $\phi^x_y = 0$, $\phi^y_x = 0$ for $x \in V$, $y \notin U$.

Connection with Assembly. The idea of how to apply controlled K-theory to the study of assembly comes from the way K-theory of group rings arises in topology in the first place. In other words, we equate a configuration of kG-modules over a space having G as fundamental group to a configuration of k-modules over the universal cover. Then we try to apply control on the universal cover (for example, using some good compactification). When we can choose BG to be a finite CW-complex, the usual assembly map for G is the map on G-fixed points of a G-equivariant bounded assembly map for $X = EG$:

$$\mathbb{H}^{\mathrm{lf}}(X; \mathbb{K}(k)) \to \mathbb{K}(k; X).$$

Here, on the right, $\mathbb{K}(k; X)$ is the spectrum for K-theory with bounded control, as in I.E.2. On the left, $\mathbb{H}^{\mathrm{lf}}$ is the spectrum corresponding to locally finite homology. If X has a compactification $\overline{X}$, this is the relative homology spectrum $\mathbb{H}^{\mathrm{St}}(\overline{X}, X; \mathbb{K}(k))$ for "Steenrod K-homology." Now the idea is to compare actual G-fixed points with "homotopy fixed points." If Z is a G-space, G-fixed points in Z are in one-to-one correspondence with $\mathrm{Maps}^G(\mathrm{pt}, Z)$, the G-maps from a point to Z. The *homotopy G-fixed points* $Z^{hG} = \mathrm{Maps}^G(EG, Z)$ are obtained by "thickening" a point to the contractible G-space EG. Via the obvious map $EG \to \mathrm{pt}$ we obtain a canonical map from G-fixed points to homotopy G-fixed points. Doing this on the spectrum level, we obtain a commutative diagram

$$\begin{array}{ccc}
\mathbb{H}(BG; \mathbb{K}(k)) & \xrightarrow{A} & \mathbb{K}(kG) \\
\| & & \| \\
\mathbb{H}^{\mathrm{lf}}(X; \mathbb{K}(k))^G & \xrightarrow{A} & \mathbb{K}(k; X)^G \\
\downarrow & & \downarrow \\
\mathbb{H}^{\mathrm{lf}}(X; \mathbb{K}(k))^{hG} & \xrightarrow{A} & \mathbb{K}(k; X)^{hG}.
\end{array}$$

The vertical arrow on the left is shown fairly easily in [**CP1**] to be an equivalence, so to prove injectivity of the assembly map, it is enough to obtain a (homotopy) splitting of the horizontal arrow at the bottom. This can be done by studying the "forget control" map from bounded control on X to continuous control at ∞.

An Additional Application. The idea of controlled K-theory has also been used to give new proofs of Chapman's theorem [**Ch**] on the topological invariance of Whitehead torsion (Ranicki-Yamasaki [**RY**], Ferry-Pedersen [**FP**]). The way this works is roughly as follows. Suppose X and Y are finite polyhedra and $f\colon X \to Y$ is a homeomorphism which is not necessarily a simplicial map. Then f is in particular a homotopy equivalence. By simplicial approximation, f can be approximated as closely as desired by a simplicial map g in the same homotopy class. One needs to show that the Whitehead torsion of g is trivial. The idea is to make use of the fact that the lift of g to the universal covers, $\widetilde{g}\colon \widetilde{X} \to \widetilde{Y}$, is a boundedly controlled homotopy equivalence, since f^{-1} can also be approximated closely by a simplicial map that will be a homotopy inverse to g.

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II. "K-Theory, Secondary Invariants, and Differential Geometry"

A. Flat Vector Bundles

II.A.1. Definition. If X is a manifold, a vector bundle E on X is called *flat* if it has a connection with zero curvature. This implies $E = \widetilde{X} \times_\sigma V$, where $\widetilde{X}$ is the universal cover of X and σ is a representation of $G = \pi_1(X)$ on V.

Thus flat bundles correspond to representations σ of G. Once this has been noticed, we can also define a flat vector bundle over an arbitrary space X (not necessarily a manifold) to be a bundle of the form $E = \widetilde{X} \times_\sigma V$, again with $\widetilde{X}$ the universal cover of X. The stable class of such an E gives a map

$$X \to BG \xrightarrow{B\sigma} BGL(k) \hookrightarrow BGL(k)^+,$$

where $k = \mathbb{R}$ or $\mathbb{C}$ (with the *discrete*) topology. This can be viewed as a cohomology class for the generalized cohomology theory defined by the algebraic K-theory spectrum $\mathbb{K}(k)^{\mathrm{alg}}$. Ignoring the flat structure amounts to looking at the map $X \to BGL(k)^{\mathrm{top}}$, which defines the associated class in topological K-theory.

By Chern-Weil theory (see, for example, [**D1**] or [**MT**]), the rational Chern (resp., Pontrjagin) classes of a complex (resp., real) flat bundle over a manifold must vanish. It is then easy to extend this result to the case where the base space X is any space with the homotopy type of a finite-dimensional CW complex. The bundle can still be non-trivial topologically, but we have:

II.A.2. Proposition. *If E is a flat vector bundle over a finite CW complex X, then $E \oplus E \oplus \cdots \oplus E$ (m summands) is trivial for some m, and $E \otimes E \otimes \cdots \otimes E$ (n factors) is trivial for some n.*

Proof. We may assume X connected. (Otherwise work one component at a time.) Say E has rank r. Then $[E] - r$ lies in the augmentation ideal of $K(X)$ and is a torsion element since its rational characteristic classes vanish. The first statement follows from this since $E \oplus E \oplus \cdots \oplus E$ is eventually in the "stable range." As for the second statement, if $r = 1$, then since $c_1(E)$ (or $w_1(E)$ in the real case) is torsion, some tensor power of E is trivial. If $r > 1$, then in K-theory we have

$$[E]^n = (r + ([E] - r))^n = \sum_{j=0}^{n} \binom{n}{j} r^{n-j} ([E] - r)^j,$$

and we can make all the terms with $j > 0$ vanish for suitably high n since $[E] - r$ is both torsion and nilpotent. Again, $E \otimes E \otimes \cdots \otimes E$ is eventually in the stable range, so we can go back from K-theory to bundles. □

All this suggests studying the relationship between $\mathbb{K}(k)^{\mathrm{alg}}$ and $\mathbb{K}(k)^{\mathrm{top}}$.

II.A.3. Theorem (Suslin [**Su1**]). *If $k = \mathbb{R}$ or $\mathbb{C}$, the natural map $\mathbb{K}(k)^{\mathrm{alg}} \to \mathbb{K}(k)^{\mathrm{top}}$ induces an isomorphism on K-groups with finite coefficients* (*in positive degrees*).

However, for $i > 0$, $K_i^{\mathrm{alg}}(k) \to K_i^{\mathrm{top}}(k)$ is very far from an isomorphism rationally. In fact, every class in $K_2(k)$ (which is a huge group) arises from some flat vector bundle over T^2, so the map $K_2^{\mathrm{alg}}(k) \to K_2^{\mathrm{top}}(k)$ has a huge kernel. This is a consequence of the fact that $K_2(k)$ is generated by Steinberg symbols, for if $\{a,b\} \in K_2(k)$, a, $b \in k^\times$, then the class $\{a,b\}$ arises from the rank-3 flat bundle corresponding to the representation of $\pi_1(T^2) = \mathbb{Z}^2$ sending the two generators to

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b^{-1} \end{pmatrix}.$$

When $k = \mathbb{C}$, this bundle is topologically trivial. When $k = \mathbb{R}$, the only obstruction to its topological triviality is the second Stiefel-Whitney class $w_2 \in H^2(T^2;\, \mathbb{Z}/2) = \mathbb{Z}/2$. So in either case, the kernel of the map

$$H^0(T^2;\, \mathbb{K}(k)^{\mathrm{alg}}) \to H^0(T^2;\, \mathbb{K}(k)^{\mathrm{top}}) = KU^0(T^2) \quad \text{or} \quad KO^0(T^2)$$

is huge.

Chern-Simons Invariants. If E is a vector bundle over X (say over $k = \mathbb{C}$) with a flat connection θ, then $d\theta + \frac{1}{2}[\theta,\, \theta] = 0$ (since the left-hand side of this equation is the formula for the curvature Ω). Out of $[\theta,\, \theta]$, Chern-Simons [**ChS**] (and later, Cheeger-Simons [**CgS**] and Cheeger [**Cg**]) manufactured *secondary invariants* $\widehat{C}_m(E,\theta) \in H^{2m-1}(X;\mathbb{C}/\mathbb{Z})$ with $\beta\widehat{C}_m(E,\theta) = -c_m(E) \in H^{2k}(X;\mathbb{Z})$. Of course, this "transgression relation" does not specify $\widehat{C}_m(E,\theta)$ uniquely, so the point of the construction is to have a specific choice, depending on the flat connection, of classes with this property. When $m = 1$, it is obvious what to do. The flat connection corresponds to a map $\pi_1(X) \to GL(n,\, \mathbb{C})$ (where n is the rank of the bundle), and taking the determinant gives a map $\pi_1(X) \to \mathbb{C}^\times \cong \mathbb{C}/\mathbb{Z}$, which amounts to a class in $H^1(X;\mathbb{C}/\mathbb{Z})$. With the correct choice of the isomorphism $\mathbb{C}^\times \cong \mathbb{C}/\mathbb{Z}$, this is $\widehat{C}_1$.

In higher dimensions, the definition is more complicated, but is based on the fact that if P_m is the invariant polynomial defining the m-the Chern class, the real part of the coefficient of λ^{n-m} in $\det(\lambda I + \frac{i}{2\pi}A)$, and if

$$Tc_m(E,\theta) = m\int_0^1 P(\theta \wedge \phi_t^{m-1})\, dt,$$

where

$$\phi_t^m = t\Omega + \frac{1}{2}(t^2-2)[\theta,\theta],$$

then $dTc_m(E,\theta)$ is the form defining the mth Chern class on the principal bundle associated to E.[2] This "transgressed Chern class" lives on the principal bundle associated to E, but it descends to give the class $\widehat{C}_m(E,\theta)$ on M. A slick but somewhat non-constructive proof of the existence of the characteristic classes $\widehat{C}_m(E,\theta)$ may be found in [**CgS**, Theorem 2.2].

It is perhaps worth mentioning that sometimes one can do something similar with $\mathbb{C}$ replaced by an arbitrary k-algebra R (with unit; $k = \mathbb{R}$ or $\mathbb{C}$). One can define an R-bundle over X to be a locally trivial fiber bundle over X with fibers that are finitely generated projective R-modules, where the transition functions are R-linear. A *flat* R-bundle (E,θ) over X then corresponds to a homomorphism $\pi_1(X) \to GL(n,\, R) \to K_1(R)$. Applying the Chern character $K_1(R) \to HC_1^-(R)$, which is really just the logarithmic derivative (see [**R**, §6.2]), one obtains a homomorphism

[2]Caution: There is a rather bewildering assortment of sign and normalization conventions in use in the literature, so the formulas here may differ by some constants from those you will find in some other references.

$\pi_1(X) \to HC_1^-(R)$ and thus a characteristic class $\widehat{C}_1(E,\theta) \in H^1(X; HC_1^-(R))$. However, one has to be careful; the analogue of Proposition II.A.2 is false if one replaces k by R.

The Chern-Simons invariants can be related to invariants coming from $K_*(\mathbb{C})$. For example:

II.A.4. THEOREM (Dupont [**D2**]). *$2\widehat{C}_2$ coincides with a homomorphism*

$$H_3(SL(2,\mathbb{C});\,\mathbb{Z}) \to \mathbb{C}/\mathbb{Q}$$

given by the dilogarithm (*cf. Bloch-Wigner and* [**G**]).

B. Secondary Invariants of Elliptic Operators

Roughly speaking, elliptic operators are to K-homology what vector bundles are to K-cohomology, and there are similar secondary invariants. The most famous are the η and ρ invariants for odd-dimensional manifolds. The η-invariant of a self-adjoint elliptic operator D is a measure of the asymmetry of the spectrum, and is defined by

$$\eta(D) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2}\,\mathrm{Tr}\,(De^{-tD^2})\,dt. \tag{II.B.1}$$

Since the spectrum of an elliptic operator over a closed manifold consists entirely of a discrete set of eigenvalues, each with finite multiplicity, this is formally

$$\begin{aligned} &\frac{1}{\sqrt{\pi}} \sum_{\lambda\in\mathrm{Spec}\,D} \int_0^\infty t^{-1/2}(\lambda e^{-t\lambda^2})\,dt \\ &\quad = \sum_{\lambda\in\mathrm{Spec}\,D} \mathrm{sign}(\lambda), \end{aligned}$$

but the sum is not convergent. The main applications of the η-invariant stem from:

II.B.2. THEOREM (Atiyah-Patodi-Singer [**APS**]). *If M^{2n} is compact Riemannian with boundary N^{2n-1}* (*and with metric that is a Riemannian product near the boundary*), *and D_M is the Dirac operator on M, then for suitable boundary conditions on D_M, the operator is Fredholm with*

$$\mathrm{Ind}\,D_M = -\tfrac{1}{2}\eta(D_N) + \int_M \widehat{\mathcal{A}}(M).$$

Thus $\eta(D_N)$ measures the deviation between the index of D_M and the formula $\int_M \widehat{\mathcal{A}}(M)$ that would give this index if M had no boundary.

The η-invariant is closely related to Chern-Simons classes. Suppose we twist D both by a flat bundle (E,θ) and by a trivial flat bundle of the same dimension. If one subtracts the index formulas for the two cases, the integral cancels, whereas the difference of the indices on the left is certainly an integer. So one gets a well defined invariant $\eta(E,\theta)$ in $\mathbb{C}/\mathbb{Z}$ ($\mathbb{R}/\mathbb{Z}$ if the associated representation is unitary). This invariant is related to Chern-Simons theory as follows:

II.B.3. THEOREM (Atiyah-Patodi-Singer [**APS**]). *A flat bundle (E,θ) on N^{2n-1} defines a class in*

$$K^{-1}(N;\mathbb{C}/\mathbb{Z})$$

(basically the same as Chern-Simons). The invariant $\eta(E, \theta)$ is this class paired with the class in $K_1(N)$ of the relevant elliptic operator.

Other Secondary Invariants Related to K-Theory. There are other secondary invariants related to K-theory, which can be regarded as more complicated variations on the themes we have discussed. Without going into details, let us just mention the Helton-Howe invariants of almost-commuting operators [**B**], the Connes-Karoubi multiplicative character of a p-summable Fredholm operator [**CK**], and recent work of Kaminker [**Km1**, **Km2**].

C. Invariants of Hyperbolic 3-Manifolds

Our story here begins with:

II.C.1. Hilbert's Third Problem. *If P_1 and P_2 are polyhedra of the same volume, can P_1 be cut into finitely many pieces congruent to pieces that can be reassembled to give P_2?*

More precisely, what Hilbert seems to have had in mind can be expressed in modern language as follows. Form the abelian group $\mathcal{SC}(\mathbb{R}^3)$ with one generator $[P]$ for each finite polyhedron P in $\mathbb{R}^3$, subject to the following:

II.C.2. Relations.
1. $[P] = [P_1] + [P_2]$ if $P = P_1 \cup P_2$ and if P_1 and P_2 intersect only along lower-dimensional subpolyhedra. We write this situation as $P = P_1 \cup_f P_2$, where $\cup_f$ stands for "union along faces." Of course, disjoint union is a special case of union along faces, so $[P \amalg Q] = [P] + [Q]$.
2. $[P] = [Q]$ if $P \cong Q$ in the sense of Euclidean geometry, i.e., if there is a rigid motion of Euclidean space (possibly orientation-reversing) taking P onto Q.
3. If $P = P_1 \cup_f P_2$ and if $Q = Q_1 \cup_f Q_2$, and if $[P_1] = [Q_1]$, then $[P] = [Q]$ if and only if $[P_2] = [Q_2]$.

This is really the Grothendieck group of an abelian monoid (with cancellation) defined by the same relations, where $[P] = [Q]$ is designed to systematize some obvious conditions for P and Q to "have the same volume." Axiom (1) says volume is additive for unions along faces, axiom (2) says it's a congruence invariant, and axiom (3) says one can "subtract." When $[P] = [Q]$, we say (for obvious reasons) that P and Q are *scissors congruent*, and we call $\mathcal{SC}(\mathbb{R}^3)$ the *scissors congruence group* (of Euclidean 3-space). Thus there is an obvious map $\mathcal{SC} \to \mathbb{R}$ obtained by mapping $[P] \mapsto \mathrm{vol}(P)$. Hilbert's question was then: is this map an isomorphism? Lurking in the background was Hilbert's knowledge of the fact, proved by Bolyai (though it could have been done by Euclid), that the answer to the corresponding question is "yes" in $\mathbb{R}^2$ (if we replace volume by area). Here's a sketch of the proof.

II.C.3. Theorem (Bolyai). *If $\mathcal{SC}(\mathbb{R}^2)$ is the group defined generated by polygons in $\mathbb{R}^2$, subject to Relations II.C.2, then the map $\mathcal{SC}(\mathbb{R}^2) \to \mathbb{R}$, defined by $[P] \mapsto \mathrm{area}(P)$, is an isomorphism.*

Proof (Sketch). First of all, $\mathcal{SC}(\mathbb{R}^2)$ is generated by those $[P]$ with P a triangle, since any polygon can be decomposed (along faces) into triangles, as illustrated in Figure II.C.4(a) on the next page.

So it's basically enough to show that the class of a triangle in the Euclidean plane only depends on its area. Also note that we can easily go back and forth between triangles and parallelograms, since given any triangle $\triangle ABC$, we can find $\triangle ACD$ congruent to $\triangle ABC$, and then $[\varnothing ABCD] = 2[\triangle ABC]$, as illustrated in Figure II.C.4(b).

Next, we show that two triangles $\triangle ABC$ and $\triangle ABD$ (with the same base $\overline{AB}$) define the same class in $\mathcal{SC}(\mathbb{R}^2)$ if they have the same altitude, *i.e.*, if the perpendicular distances from C and D to the line $\overline{AB}$ are the same. By what we just proved, it's enough to show that two parallelograms with a side in common and the same height define the same class in $\mathcal{SC}(\mathbb{R}^2)$. This can be seen from the construction in Figure II.C.5(a), together with Relation II.C.2(3), since $[\triangle ADF] = [\triangle BCE]$ (by Relation II.C.2(2)), and

$$[\bigtriangledown ABCF] = [\varnothing ABCD] + [\triangle ADF] = [\varnothing ABEF] + [\triangle BCE].$$

Since we have shown that the class of any triangle in $\mathcal{SC}(\mathbb{R}^2)$ only depends on its base and height, we will be done if we can show two right triangles with the same area define the same class in $\mathcal{SC}(\mathbb{R}^2)$. The construction for doing this is illustrated in Figure II.C.5(b). Here $\triangle ABD$ and $\triangle ACE$ are right triangles with the same area. Thus $|\overline{AC}| \cdot |\overline{AE}| = |\overline{AB}| \cdot |\overline{AD}|$, which means

$$\frac{|\overline{AC}|}{|\overline{AB}|} = \frac{|\overline{AD}|}{|\overline{AE}|},$$

i.e., $\triangle ACD$ is similar to $\triangle ABE$. That means $\overline{EB}$ and $\overline{DC}$ are parallel line segements, and so $\triangle BDE$ and $\triangle BCE$ have the same base and height. By what we just proved,

$$[\triangle BDE] = [\triangle BCE].$$

But now

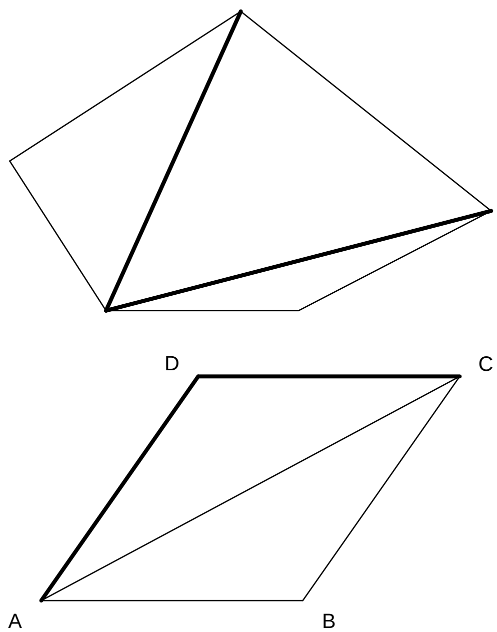
$$[\triangle ABD] = [\triangle BDE] + [\triangle ABE] = [\triangle BCE] + [\triangle ABE] = [\triangle ACE]. \quad \square$$

In dimension 3, Hilbert conjectured that the answer to his question was "no," and this was proven by Dehn (1900) even before the published version of Hilbert's question appeared. But the negative answer itself raises a question, to describe the equivalence classes for *scissors congruence*. This can be done in any dimension, not only in Euclidean space, but also in hyperbolic space. The question is especially interesting for polyhedra in hyperbolic 3-space.

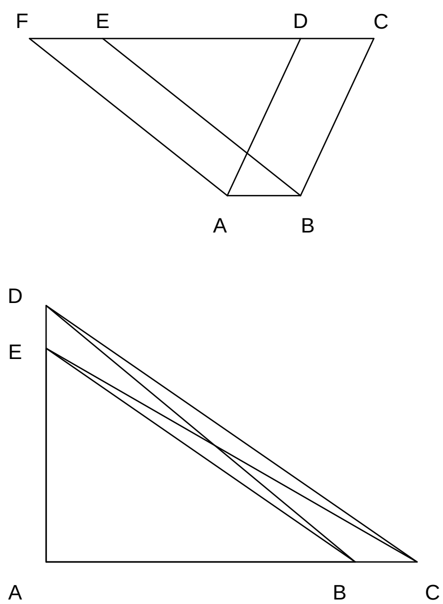
II.C.6. Theorem (Dupont-Sah [**DS**]). *The scissors congruence group of polyhedra in hyperbolic 3-space is divisible without 2-torsion. It is the* (-1)*-eigenspace for complex conjugation acting on* $\mathcal{P}(\mathbb{C})$*, where* $\mathcal{P}(\mathbb{C})$ *fits into the exact sequence*

$$0 \to \mathbb{Q}/\mathbb{Z} \to H_3(SL(2,\mathbb{C})^d;\mathbb{Z}) \to \mathcal{P}(\mathbb{C}) \to \mathbb{C}^\times \wedge \mathbb{C}^\times \xrightarrow{\text{Steinberg symbol}} K_2(\mathbb{C}) \to 0.$$

Aspherical Manifolds and K-Theory. Now suppose G is a group for which BG can be taken to be a compact oriented manifold M^n, and one has a representation $G \xrightarrow{\sigma} GL(k)$. This induces a map $BG \xrightarrow{B\sigma} BGL(k)$ and thus a class $B\sigma_*([M])$ in $H_n(BGL(k)) = H_n(\mathbb{K}(k))$. We can detect it via any invariant of $K_n(k)$ that factors through the Hurewicz homomorphism.



II.C.4. Figure. (a) Dividing a polygon into triangles.
(b) Triangles making up a parallelogram.



II.C.5. Figure. (a) Parallelograms with the same base and height.
(b) Right triangles with the same area.

Application to Hyperbolic 3-Manifolds. We apply this idea to the situation of a compact oriented hyperbolic 3-manifold M^3. The fundamental group G comes with map to $SO_0(3, 1) \cong PSL(2, \mathbb{C})$, the connected component of the identity in $SO(3, 1)$. Modulo the [small] issue of lifting from $PSL(2, \mathbb{C})$ to $SL(2, \mathbb{C})$, one basically gets a class in $H_3(GL(\mathbb{C}))$.

Now K-theory of fields, roughly speaking, splits into two parts: *Milnor K-theory* $K_*^M(k)$, defined by higher-order analogues of Steinberg symbols, and the more mysterious *indecomposable K-theory*. Recall:

THEOREM (Suslin [**Su2**]). *For a field k,*

$$K_n^M(k) \cong H_n(GL(n, k))/\mathrm{im}\,\big(H_n(GL(n-1, k)) \to H_n(GL(n, k))\big).$$

While indecomposable K-theory is not very well understood in high degrees, we have a good approximation to it in degree 3, the *Bloch group* $\mathcal{B}(k)$, a group with explicit generators and relations. Modulo torsion, $K_3(k)$ is built out of $K_3^M(k)$ and the Bloch group $\mathcal{B}(k)$ [**Su3**].

Since it comes from $H_3(GL(2))$, the homology invariant of a hyperbolic 3-manifold maps to 0 in Milnor K-theory and should basically live in the Bloch group.

This is confirmed by:

THEOREM (Neumann-Yang [**NY1**], [**NY2**], [**NY3**]). *An oriented finite-volume hyperbolic 3-manifold M^3 has an invariant $\beta(M) \in \mathcal{B}(\mathbb{C})$ (roughly speaking, its scissors congruence class). In fact, $\beta(M)$ can be defined to live in $\mathcal{B}(k)$ for a number field $k(M)$ associated to M. (Roughly speaking again, this is the smallest field k for which the defining representation of $G = \pi_1(M)$ can be defined over k.) Under the (normalized) Bloch regulator $\mathcal{B}(\mathbb{C}) \to \mathbb{C}/\mathbb{Q}$, $\beta(M)$ goes to* $\mathrm{vol}\,(M) + i\,\mathrm{CS}\,(M)$*, where* $\mathrm{vol}\,(M)$ *is the volume of M for the hyperbolic metric of constant curvature -1, and where* $\mathrm{CS}\,(M)$ *is the Chern-Simons invariant (gotten by integrating the Chern-Simons class $\widehat{C}_2$ over M).*

If $k(M)$ is embedded in $\mathbb{C}$ as an imaginary quadratic extension of a totally real number field, then $CS(M)$ is rational.

(Conjecturally, $CS(M)$ is irrational if $k(M) \cap \overline{k(M)} \subset \mathbb{R}$. This seems to be backed by numerical evidence.)

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