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Higher G-Signatures for Lipschitz Manifolds*

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Abstract. Using the Teleman signature operator and Kasparov's KK-theory, we prove a strong De Rham theorem and a higher G-signature theorem for Lipschitz manifolds. These give in particular a substitute for the usual G-signature theorem that applies to certain nonsmooth actions on topological manifolds. Then we present a number of applications. Among the most striking are a proof that 'nonlinear similarities' preserve 'renormalized Atiyah–Bott numbers', and a proof that under suitable gap, local flatness, and simple connectivity hypotheses, a compact (topological) G-manifold M is determined up to finite ambiguity by its isovariant homotopy type and by the classes of the equivariant signature operators on all the fixed sets $M^H, H \subseteq G$. These could also be proved using joint work of Cappell, Shaneson, and the second author on topological characteristic classes.

Key words. Lipschitz manifold, Teleman signature operator, G-signature theorem, Novikov Conjecture, equivariant K-theory, KK-theory, nonlinear similarity, Atiyah–Bott number, surgery.

0. Introduction

As its title suggests, this paper is a continuation of our article [35]. In fact, it originally constituted the final section of the first two drafts of [35] (written in 1986 and 1987), until the length and complexity of the manuscript led us to divide it into two largely independent papers. Our goal here is to show that some of the theory of higher G-indices (as developed in [35]) goes over to topological manifolds and nonsmooth actions, using the Teleman signature operator of [45, 46, 14]. In particular, we discuss how to exploit Teleman's construction in the presence of a group action, a nontrivial fundamental group, or both.

The first section of this paper, which in many ways is a 'warm-up' for the rest of the paper, gives a proof of the Novikov Conjecture for topological manifolds with suitable fundamental groups that is independent of surgery theory, via the method of Kasparov and Kaminker–Miller. The treatment here includes a strong form of a

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Lipschitz De Rham Theorem for cohomology with local coefficients, and also generalizes to the equivariant case so as to apply to the Equivariant Novikov Conjecture in the sense of [35, 36].

Section 2 focuses on actions of compact, and especially finite, groups on Lipschitz manifolds. We begin by showing that when a finite group G acts on a topological manifold M, M very often admits a G-invariant Lipschitz structure, even if the action is rather pathological. Thus a Lipschitz G-Signature Theorem can be quite useful in studying nonsmooth transformation groups. We prove such a theorem and its 'higher G-signature' analogue. Then in Section 3 we give a number of applications. The main thrust of this part of the paper is to apply the techniques of Lipschitz analysis of signature operators to construct a useful topologically invariant characteristic class for certain G-manifolds. (The topological invariance stems from the (weak) existence and uniqueness of invariant Lipschitz structures for group actions [38].) This class is a generalization of the L-class of smooth manifolds whose topological invariance is a famous theorem of Novikov. The topological invariance of this class has a number of applications to distinguishing *smooth* G-manifolds topologically; this is of interest even for representations on vector spaces. For instance, we obtain a fairly direct proof of some of the known results on the 'nonlinear similarity' problem (see especially [16] and [6]), such as the following from [6]:

3.3. THEOREM. *The renormalized Atiyah–Bott numbers $AB(\gamma, \rho)$ (see 3.2 for the definition), $\gamma \in G$, are (up to a sign, depending on whether orientation is preserved or not) topological conjugacy invariants of a linear representation ρ of a finite group G.*

That our characteristic class comes from a signature operator enables one to bring in the fundamental group as we did in the previous paper in this series. In fact, the deduction of the above theorem uses a torus trick (as in [6]) to implicate the fundamental group and our higher G-signature theorem (which [6] also invoked).

The class studied here has an alternate definition [8] using purely topological methods. This approach has many of the same consequences, and the two approaches interact significantly. In [8] topologically invariant classes are defined to live in a certain cosheaf homology group, *a priori*, and then the group is computed to be KO-homology (by topological methods). Using [52] a manifold-theoretic interpretation of that cosheaf homology group can be given for which the signature operator gives the isomorphism.

This is significant because in the theory of [52] the cosheaf homology group represents the normal invariants in surgery, and thus connects the signature operator with positive classification results. A consequence of this sort of reasoning is the following:

3.12. THEOREM. *A topological orientation-preserving action of a finite group G on a compact simply connected topological manifold M, such that for all subgroups $H \subseteq K \subseteq G$, the fixed sets M^K and M^H are simply connected submanifolds of* *dimension $\neq 3$ and the inclusion of M^K in M^H is locally flat and of codimension $\geqslant 3$, is determined up to finite indeterminacy by its isovariant homotopy type and the classes of the equivariant signature operators on all the fixed sets M^H.*

Here, as usual in the subject, 'isovariant' means both 'equivariant' and 'preserving isotropy groups'.

This is essentially due to Sullivan for G trivial (via Browder–Novikov surgery theory) and to Madsen and Rothenberg for G of odd order and the actions satisfying the 'gap hypothesis'. In 1987 Cappell and the second author had conjectured this theorem on the basis of the announcement of the main results of this paper (and [38]). [8] was written in an attempt to approach this conjecture. (It too leads to a proof of this theorem in conjunction with [52]. [52] itself does not lead to a proof of this without the input of either this paper or [8]. Interestingly enough, now [38] can be avoided in the proofs of the topological invariance results and in the classification results. This is sketched in chapter 13 of [52], but key steps in the proof of [52] rely on constructions developed first for use in [38].) We should remark that unlike the $G = \{e\}$ case (for dimension $\neq 4$), this theorem is false in the smooth category (in part because of the difference between $\mathrm{Wh}^{\mathrm{Diff}}$ and $\mathrm{Wh}^{\mathrm{Top}}$)!

It might be possible to combine the results of the approach developed here and that in [8], which deals with nicer actions, but on spaces more general than manifolds, by extending Cheeger's work on Witt spaces [10] to obtain results on Lipschitz actions on Witt spaces. We will not pursue this here.

Some of the results of this article were announced in [33] and [38], and we apologize to our readers for preoccupation with other projects that caused us to delay its publication until now. Many of the results that we were originally going to include were proved simultaneously (though more elegantly) by Michel Hilsum, and the original draft of his article [15] was circulated at about the same time as the first draft of this article. We have therefore reorganized our exposition to remove the duplications with his work and to take advantage of several very useful technical results in his paper [15]. We are very grateful to Michel Hilsum, Jerry Kaminker, Jim McClure, John Miller, Mel Rothenberg, and Georges Skandalis for helpful suggestions with this work, and especially thankful to John Miller, Georges Skandalis, and Michel Hilsum for detecting errors in the preliminary drafts and for suggesting ways of overcoming them.

1. Higher Signatures for Lipschitz Manifolds

We begin by stating and proving a version of the Novikov Conjecture on higher signatures for topological manifolds. This result by itself may not at first seem particularly exciting, since the (rational) Novikov Conjecture in all three of the categories Diff, PL, and Top can be reduced down to the same algebraic statement in L-theory (in fact, one can get results in still more general categories: rational homology manifolds (see [19]) or certain categories of stratified spaces (see [52])).

But there is still some interest in having a direct, non L-theoretic proof, and as we shall see, our method can be extended to study some problems on group actions in the PL and Top categories. Furthermore, our method gives not just the rational but also (for suitable torsion-free fundamental groups) an *integral* version of the topological Novikov Conjecture. We might point out, therefore, that to obtain *rational* results about the equivariant Novikov Conjecture by our method of [36] for actions of a compact (continuous) group G, it is necessary first to prove integral results for actions of the finite subgroups H of G. Many of the methods and results of this section will be needed again in Section 2. We are grateful to Michel Hilsum and to Georges Skandalis for some suggestions on how to deal with the technical aspects of unbounded Kasparov bimodules (see Theorem 1.8 below).

It is necessary first to recall some fundamental results on Lipschitz manifolds. For further details, the reader should consult [44, 45, 46, 15, 33, 48].

1.1. THEOREM (Sullivan [44]). *Any topological manifold (without boundary) of dimension* $\neq 4$ *has a Lipschitz structure, which is unique up to a locally bi-Lipschitz homeomorphism isotopic to the identity.*

Remarks. By work of Donaldson and Sullivan [11], it is now known that there are four-dimensional Top manifolds without Lipschitz structures. The most remarkable part of Theorem 1.1 is the uniqueness, since for a PL manifold, existence but not uniqueness of a Lipschitz structure is obvious. In the compact case, the word 'locally' can be removed.

1.2. THEOREM (Teleman [45] and [46], Hilsum [14]). *Let* M^{2n} *be a closed oriented Lipschitz manifold of even dimension. Then from the 'de Rham' complex of* L^2-*differential forms on* M *(with respect to some choice of a Lipschitz 'Riemannian metric'* g*) one obtains a signature operator* D_g *which is closed and self-adjoint, and* D_g *(together with the grading on forms introduced by Atiyah–Singer* [2], [1, *formula* (6.9)]*) determines a class* $[D]$ *in* $K_0(M)$ *which is independent of the choice of the metric* g*. The image of* $[D]$ *in* $K_0(pt)$ *(i.e., the index of* D_g*) is the usual signature of the manifold.*

From 1.1 and 1.2 together, one sees that any closed oriented topological manifold M^{2n} has a signature class $[D]$ in $K_0(M)$, which depends only on the homeomorphism class of M (at least if $n \neq 2$). We may regard this as a characteristic class for topological manifolds, and as such, it can be identified in terms of more traditional invariants. We shall do this in [37]. For odd-dimensional manifolds, D_g is still defined the same way but D_g preserves the grading and its class lives in $K_1(M)$. Thus, if M has dimension four, we may still unambiguously define its signature class $[D_M] \in K_0(M)$ by using the unique Lipschitz structure on $M \times S^1$ and taking the projection of $[D_{M\times S^1}] \in K_1(M \times S^1) \cong K_1(M) \oplus K_0(M)$ into $K_0(M)$.

Theorem 1.2 has been generalized by Hilsum in [15, §2] to the case of noncompact manifolds, provided that one considers only metrics g for which M is metrically complete in the induced metric space structure.

For π a countable group, we let $\beta: K_*(B\pi) \to K_*(C_r^*(\pi))$ be the assembly map defined by Kasparov in [24, §6.2]. The following theorem proves the 'Strong Novikov Conjecture' (cf. [32, 23, 24]) in the topological category for many fundamental groups.

1.3. THEOREM. *Let M^{2n} be a closed, connected, oriented Lipschitz manifold, and let $[D] \in K_0(M)$ be the class of the Teleman signature operator. Assume that for some group π (often, but not always, this will be $\pi_1(M)$) one is given a continuous map $f: M \to B\pi$. Then $\beta(f_*([D])) \in K_0(C_r^*(\pi))$ is an oriented homotopy invariant of M. If β is injective (for instance, if π is solvable with a composition series with torsion-free abelian composition factors* [32], *or if π is the fundamental group of a complete Riemannian manifold of non-positive curvature* [24], *or if π is a torsion-free subgroup of $GL(N, \mathbb{Q})$ for some N* [25]), *then the 'higher signature' $f_*([D]) \in K_0(B\pi)$ is an oriented homotopy invariant.*

1.4. COROLLARY (Novikov Conjecture). *If $\beta: K_*(B\pi) \to K_*(C_r^*(\pi))$ is injective after tensoring with $\mathbb{Q}$ (for instance, if π is a discrete subgroup of a Lie group with finitely many connected components* [24], *or if π has an embedding into $GL(N, \mathbb{Q})$ for some N* [25]), *then the higher signatures $\langle \mathbb{L}(M) \cup f^*(a), [M] \rangle$,* where $\mathbb{L}$ *is the Teleman L-class* ([46]) *and $a \in H^*(\pi, \mathbb{Q})$, are oriented homotopy invariants for connected, closed, oriented topological manifolds M equipped with maps $f: M \to B\pi$ (and in particular for manifolds with π as fundamental group).*

Proof of Theorem 1.3. The key step will be to identify $\beta(f_*([D]))$ with the symmetric signature of M, by duplicating the argument in [20]. There is a genuine technical difficulty in doing this, which comes from the fact that Hilsum in [14] defines $[D]$ as coming from the unbounded (but self-adjoint) signature operator, whereas the theory of [20] requires the use of chain complexes of bounded operators. The simplest way to get around this seems to be to work with L^2-differential forms, which are canonically defined on a Lipschitz manifold, but to modify the exterior derivative to obtain a bounded differential. As we shall see, this amounts to developing a theory of 'Sobolev spaces' on M.

Fix a Riemannian metric g on M and let $A = C_r^*(\pi)$. Let $\tilde{M}$ be the (possibly disconnected) covering of M, with covering group π, defined by f and let $\mathscr{V} = \tilde{M} \times_\pi A$ be the associated flat A-vector bundle over M. We give $\mathscr{V}$ a flat connection and define the signature operator $D_{\mathscr{V}}$ (based on the metric g) and develop its properties as in [14]. First, let $K^j_{\mathscr{V}}$ be the Hilbert A-module of L^2-differential forms of degree $j + n$ on M with values in $\mathscr{V}$. (Here j will run from $-n$ to n; recall M has dimension $2n$.) Locally, this will look like $K_g \otimes_{\mathbb{C}} A$, where K_g is the Hilbert space $L^2(M, \wedge^\cdot)$ constructed as in [14, §2.6], with A-valued inner product

$$\langle \varphi, \psi \rangle_A = \int \varphi^* \wedge *\psi; \quad \varphi, \psi \in L^2(M, \wedge^{j+n} \otimes \mathscr{V}).$$

As in [14, §2.7 and §4], exterior differentiation defines a closed adjointable A-linear operator $d_{\mathscr{V}}: K^j_{\mathscr{V}} \to K^{j+1}_{\mathscr{V}}$, which satisfies $d^2_{\mathscr{V}} = 0$ (since the connection on $\mathscr{V}$ is flat) but is unbounded.

We need to establish for $D_{\mathscr{V}} = d_{\mathscr{V}} + d^*_{\mathscr{V}}$ some analogues of the properties of $d + d^*$ established in [14]. Since it already takes a fair amount of work to do this on an *ad hoc* basis, and since we will eventually need to be able to compute indices of signature operators with coefficients on Lipschitz manifolds, we make a digression at this point and do things in somewhat greater generality. The machinery that follows for dealing with unbounded Kasparov bimodules was shown to us by Georges Skandalis. Presumably one could handle the case where A is graded or nonunital, but we stick to the simplest case, as this will suffice for our applications.

1.5. LEMMA (Skandalis). *Let A and B be two C^*-algebras, with A unital. (B can be graded but we grade A trivially.) Let $(\mathscr{E}, D)$ be an unbounded Kasparov (A, B)-bimodule in the sense of* [3, Definition 2.1]. *Then*

$$\mathscr{A} =_{\text{def}} \{a \in A: a(\operatorname{dom} D) \subseteq \operatorname{dom} D \text{ and } [a, D] \text{ extends to an element of } \mathscr{L}(\mathscr{E})\}$$

is a dense $*$-subalgebra of A, stable under holomorphic functional calculus.

Proof. It's clear $\mathscr{A}$ is an algebra. It is a $*$-subalgebra because of self-adjointness of D, for if $a \in \mathscr{A}$ and $\xi \in \operatorname{dom} D$, then $a^*\xi \in \operatorname{dom} D$ with $Da^*\xi = a^*D\xi + [a, D]^*\xi$, as can be seen by showing $a^*\xi \in \operatorname{dom} D^*$. We norm it by defining

$$\|a\|_{\mathscr{A}} = \|a\| + \|[a, D]\|.$$

Then for $a, b \in \mathscr{A}$,

$$\begin{aligned} \|ab\|_{\mathscr{A}} &= \|ab\| + \|[ab, D]\| \\ &\leqslant \|a\|\,\|b\| + \|a\|\,\|[b, D]\| + \|b\|\,\|[a, D]\| \\ &\leqslant \|a\|_{\mathscr{A}}\|b\|_{\mathscr{A}}. \end{aligned}$$

To show $\mathscr{A}$ becomes a Banach $*$-algebra in this norm, we need to check completeness. Suppose (a_n) is a sequence in $\mathscr{A}$ which is Cauchy for $\|\ \|_{\mathscr{A}}$. Then, in particular, (a_n) is Cauchy in A and converges to some $a \in A$. If $\xi \in \operatorname{dom} D$, then

$$\begin{aligned} &\|Da_n\xi - Da_m\xi\| \\ &\quad = \|a_nD\xi - a_mD\xi + [D, a_n]\xi - [D, a_m]\xi\| \\ &\quad \leqslant \|a_n - a_m\|\,\|D\xi\| + \|[D, a_n] - [D, a_m]\|\,\|\xi\|, \end{aligned}$$

so $(Da_n\xi)$ is Cauchy in $\mathscr{E}$, hence $a\xi \in \operatorname{dom} D$ since D is closed. This shows $a(\operatorname{dom} D) \subseteq \operatorname{dom} D$, and the boundedness of $[a, D]$ and the fact that $a_n \to a$ in $\|\ \|_{\mathscr{A}}$ follow easily.

For the last statement, it's enough to show that if $a \in \mathscr{A}$ and $\|a\|_{\mathscr{A}} < 1$, then $1 - a$ is invertible not just in A but in $\mathscr{A}$. If $\xi \in \operatorname{dom} D$, then

$$\|[D, a^n]\xi\| \leqslant n\|[D, a]\|\,\|\xi\|\,\|a\|^{n-1},$$

and, hence, $\Sigma[D, a^n]\xi$ is absolutely convergent, say to η. Then since $\Sigma a^n D\xi$ also converges absolutely (to $(1-a)^{-1}D\xi$), $\Sigma D a^n \xi$ converges absolutely to $(1-a)^{-1}D\xi - \eta$, and since D is a closed operator, $(1-a)^{-1}\xi \in \operatorname{dom} D$ and

$$\|(1-a)^{-1}D\xi - D(1-a)^{-1}\xi\| = \|\eta\| \leqslant \text{const}\|\xi\|,$$

so that $(1-a)^{-1} \in \mathscr{A}$, as required. □

1.6. COROLLARY. *Any finitely generated projective right A-module P is isomorphic to one of the form $\mathscr{P} \otimes_{\mathscr{A}} A$, where $\mathscr{P}$ is a finitely generated projective right $\mathscr{A}$-module.*

Proof. This is the Karoubi density theorem ([22, Exercise II.6.5]). □

Now let P and $\mathscr{P}$ be as in Corollary 1.6. We want to 'extend' D so as to get an unbounded Kasparov $(\mathbb{C}, B)$-bimodule based on $\mathscr{E}_1 = P \hat{\otimes}_A \mathscr{E}$. If $P = A^r$, then $\mathscr{E}_1 = \mathscr{E}^r$ and we can just take a direct sum of r copies of D. But in general, there is no such canonical way to do this and we are led to the following definition.

1.7. DEFINITION (Skandalis). Let $A, B, \mathscr{E}$, and D be as in Lemma 1.5. Define $\mathscr{A}$ as in that lemma and let $\mathscr{P}$ be a finitely generated projective right $\mathscr{A}$-module, $P = \mathscr{P} \otimes_{\mathscr{A}} A$. Let $\mathscr{E}_1 = P \hat{\otimes}_A \mathscr{E}$. For $\xi \in P$, let $T_\xi(\eta) = \xi \otimes \eta \in \mathscr{E}_1$ for $\eta \in \mathscr{E}$, so $T_\xi \in \mathscr{L}(\mathscr{E}, \mathscr{E}_1)$. A *$D$-connection* $\tilde{D}$ is a symmetric (i.e., $\langle \tilde{D}x, y\rangle_B = \langle x, \tilde{D}y\rangle_B$) B-linear operator $\mathscr{P} \otimes_{\mathscr{A}} \operatorname{dom} D \to \mathscr{E}_1$, with the property that for $\xi \in \mathscr{P}$, the commutator

$$\left[\begin{pmatrix} D & 0 \\ 0 & \tilde{D} \end{pmatrix}, \begin{pmatrix} 0 & T_\xi^* \\ T_\xi & 0 \end{pmatrix}\right],$$

which is defined on $(\operatorname{dom} D) \oplus (\mathscr{P} \otimes_{\mathscr{A}} \operatorname{dom} D)$, extends to an element of $\mathscr{L}(\mathscr{E} \oplus \mathscr{E}_1)$. These are useful because of the following observation of Skandalis:

1.8. THEOREM (Skandalis). *In the situation of Definition 1.7, D-connections exist, and any two differ by an element of $\mathscr{L}(\mathscr{E}_1)$. Any D-connection $\tilde{D}$ is closed, self-adjoint (i.e., $\tilde{D}^* = \tilde{D}$ as closed operators), and regular (i.e., $1 + \tilde{D}^2$ has dense range and an inverse in $\mathscr{L}(\mathscr{E}_1)$). Furthermore, $(\mathscr{E}_1, \tilde{D})$ is an unbounded Kasparov bimodule representing the Kasparov product of $[P] \in K_0(A) \cong KK(\mathbb{C}, A)$ and $(\mathscr{E}, D) \in KK(A, B)$.*

First we need a lemma.

1.9. LEMMA. *Let $\mathscr{E}$ be a Hilbert B-module, where B is a (possibly graded) C^*-algebra, and let D be a self-adjoint regular operator on $\mathscr{E}$. Then for any $a = a^* \in \mathscr{L}(\mathscr{E})$, $\tilde{D} = D + a$ (with domain* $\operatorname{dom} D$*) is self-adjoint and regular. Furthermore, if $(1 + D^2)^{-1}$ extends to an element of $\mathscr{K}(\mathscr{E})$, then the same holds for $(1 + \tilde{D}^2)^{-1}$.*

Proof. First we check self-adjointness. Obviously $\tilde{D}$ is symmetric (i.e., $\tilde{D} \subseteq \tilde{D}^*$). So assume $\xi \in \operatorname{dom} \tilde{D}^*$; this means that there exists an element $\tilde{D}^*\xi \in \mathscr{E}$ such that for all $\eta \in \operatorname{dom} \tilde{D} = \operatorname{dom} D$,

$$\langle \tilde{D}\eta, \xi\rangle_B = \langle \eta, \tilde{D}^*\xi\rangle_B.$$

But then

$$\langle (D+a)\eta, \xi\rangle_B = \langle D\eta, \xi\rangle_B + \langle \eta, a\xi\rangle_B = \langle \eta, \tilde{D}^*\xi\rangle_B,$$

or

$$\langle D\eta, \xi\rangle_B = \langle \eta, \tilde{D}^*\xi - a\xi\rangle_B.$$

Hence, $\xi \in \operatorname{dom} D^* = \operatorname{dom} D$ and $D^*\xi = D\xi = \tilde{D}^*\xi - a\xi$, so $\tilde{D}^*\xi = (D+a)\xi$ and $\tilde{D}^* \subseteq \tilde{D}$.

Next we check regularity. We will use the fact that the closure of $1 + D^2$ has a bounded inverse in $\mathscr{L}(\mathscr{E})$. First observe that it is possible to choose λ with $0 < \lambda < 1$ and with $1 - \lambda < \lambda^2/\|a\|^2$, by taking λ close enough to 1. We use the operator inequalities

$$(D+a)^2 \leqslant 2(D^2 + a^2) \leqslant 2(D^2 + \|a\|^2) \quad \text{and} \quad (\lambda^{1/2}D + \lambda^{-1/2}a)^2 \geqslant 0$$

or

$$\begin{aligned}(D+a)^2 &\geqslant (1-\lambda)D^2 + (1-\lambda^{-1})a^2 \\ &\geqslant (1-\lambda)D^2 - (\lambda^{-1} - 1)\|a\|^2\end{aligned}$$

to obtain the estimates

$$\begin{aligned}0 &< 1 - \lambda \\ &= (1+D^2)^{-1/2}(1-\lambda)(1+D^2)(1+D^2)^{-1/2} \\ &\leqslant (1+D^2)^{-1/2}[(1-\lambda)(1+D^2) + \lambda - (\lambda^{-1}-1)\|a\|^2](1+D^2)^{-1/2} \\ &\leqslant (1+D^2)^{-1/2}(1+\tilde{D}^2)(1+D^2)^{-1/2} \\ &\leqslant 2(1+D^2)^{-1/2}(1+D^2+\|a\|^2)(1+D^2)^{-1/2} \\ &\leqslant 2(1+\|a\|^2).\end{aligned}$$

Thus, there exists a well-defined invertible element $b = b^* \in \mathscr{L}(\mathscr{E})$, in fact with $1 - \lambda \leqslant b^{-1} \leqslant 2(1 + \|a\|^2)$, which is an inverse for $(1+D^2)^{-1/2}(1+\tilde{D}^2)(1+D^2)^{-1/2}$. Since the closure of $(1+D^2)^{1/2}$ is surjective, this proves $1 + \tilde{D}^2$ has dense range. The statement about compactness follows easily since we have

$$(1+\tilde{D}^2)^{-1} = (1+D^2)^{-1/2}b(1+D^2)^{-1/2}. \qquad \square$$

Now we can proceed to the proof of Theorem 1.8.

Proof of Theorem 1.8. First we check existence. If $\mathscr{P} = \mathscr{A}^r$, we may take

$$\tilde{D} = \left.\begin{pmatrix} D & & & 0 \\ & D & & \\ & & \ddots & \\ 0 & & & D \end{pmatrix}\right\} r \text{ rows.}$$

This satisfies the condition since if

$$\xi_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix},$$

with the 1 in the ith slot, $T^*_{\xi_i}$ just selects out the ith coordinate and thus

$$\left[\begin{pmatrix} D & 0 \\ 0 & \tilde{D} \end{pmatrix}, \begin{pmatrix} 0 & T^*_{\xi_i} \\ T_{\xi_i} & 0 \end{pmatrix}\right]$$

$$= \left[\begin{pmatrix} D & & & 0 \\ & D & & \\ & & \ddots & \\ 0 & & & D \end{pmatrix}, \begin{pmatrix} 0 & \dots & 1 & \dots & 0 \\ \vdots & & & & \\ 1 & & 0 & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}\right] = 0.$$

For the general case, we can first replace $\mathscr{E}$ by $\mathscr{E}^r$ and $\mathscr{A}$ by $M_n(\mathscr{A})$; then since $\mathscr{A}$ is a symmetric Banach $*$-algebra, we may assume $\mathscr{P} = e\mathscr{A}$ with e a self-adjoint projection [21, pp. 35–36]. Then we take $\tilde{D}(e\xi) = eDe\xi$ for $\xi \in \operatorname{dom} D$. (This makes sense since $e(\operatorname{dom} D) \subseteq \operatorname{dom} D$ by definition of $\mathscr{A}$.) Note that if $a \in \mathscr{A}$, $T_{ea}(\xi) = ea\xi$ and thus

$$\left[\begin{pmatrix} D & 0 \\ 0 & \tilde{D} \end{pmatrix}, \begin{pmatrix} 0 & T^*_{ea} \\ T_{ea} & 0 \end{pmatrix}\right] = \left[\begin{pmatrix} D & 0 \\ 0 & eDe \end{pmatrix}, \begin{pmatrix} 0 & a^*e \\ ea & 0 \end{pmatrix}\right]$$

$$= \begin{pmatrix} 0 & Da^*e - a^*eDe \\ eDea - eaD & 0 \end{pmatrix}.$$

This is bounded since $a^*e, ea \in \mathscr{A}$ and thus have bounded commutator with D. So this proves existence.

Next, if $\tilde{D}_1$ and $\tilde{D}_2$ are any two D-connections and if $v_1, \dots, v_r$ generate $\mathscr{P}$ as an $\mathscr{A}$-module, we see from boundedness of

$$\left[\begin{pmatrix} D & 0 \\ 0 & \tilde{D}_1 \end{pmatrix}, \begin{pmatrix} 0 & T^*_{v_i} \\ T_{v_i} & 0 \end{pmatrix}\right] - \left[\begin{pmatrix} D & 0 \\ 0 & \tilde{D}_2 \end{pmatrix}, \begin{pmatrix} 0 & T^*_{v_i} \\ T_{v_i} & 0 \end{pmatrix}\right]$$

that $\tilde{D}_1 - \tilde{D}_2$ is bounded and has an adjoint on $v_i\mathscr{A} \otimes_{\mathscr{A}} \operatorname{dom} D$ for each i, hence on all of $\mathscr{P} \otimes_{\mathscr{A}} \operatorname{dom} D$.

Now let $\tilde{D}$ be a D-connection. Since $\tilde{D}$ is densely defined and symmetric (as an unbounded operator on $\mathscr{E}_1$), it is closable. To prove self-adjointness and regularity, we check these first in the case of the specific ('Grassmann') connection constructed above. These properties are trivial in the case where

$$\mathscr{P} = \mathscr{A}^r \quad \text{and} \quad \tilde{D} = \begin{pmatrix} D & & 0 \\ & \ddots & \\ 0 & & D \end{pmatrix},$$

since they just come down to properties of D, so we need only consider the case $\mathscr{P} = e\mathscr{A}$, $\tilde{D} = eDe$. Of course, we could just as well consider $(1-e)\mathscr{A}$ and $(1-e)D(1-e)$ simultaneously, which amounts to checking the case where we've kept $\mathscr{E}$ the same but replaced D by

$$\tilde{D} = eDe + (1-e)D(1-e), \quad e \in \mathscr{A}.$$

However,

$$\begin{aligned} D - \tilde{D} &= eD(1-e) + (1-e)De \\ &= e^2 D - eDe + De^2 - eDe \\ &= e[e, D] + [D, e]e, \end{aligned}$$

which is self-adjoint and lies in $\mathscr{L}(\mathscr{E})$ since $e \in \mathscr{A}$. Hence, self-adjointness and regularity of $\tilde{D}$ follow from Lemma 1.9.

For self-adjointness and regularity in the general case, we may use Lemma 1.9 again, for by what we have already proved, any D-connection $\tilde{D}$ differs by a self-adjoint element of $\mathscr{L}(\mathscr{E}_1)$ from a Grassmann connection. To conclude, we need to show that $(\mathscr{E}_1, \tilde{D})$ is an unbounded Kasparov bimodule, which follows instantly from the last part of Lemma 1.9, and that it represents the Kasparov product $[P] \otimes_{\mathscr{A}} [(\mathscr{E}, D)]$. For this last assertion, it's once again enough to show that the class of $(\mathscr{E}_1, \tilde{D})$ in $KK(\mathbb{C}, B)$ is independent of the choice of connection, and then to compute for a Grassmann connection. The independence of connection will follow again from Lemma 1.9, for if $\tilde{D}_1$ and $\tilde{D}_2$ are any two connections, we have $\tilde{D}_2 = \tilde{D}_1 + a, a = a^* \in \mathscr{L}(\mathscr{E}_1)$, and then Lemma 1.9 shows that $\tilde{D}_1 + ta, 0 \leqslant t \leqslant 1$, gives an operatorial homotopy between the two Kasparov elements. Finally, to check that we get the correct product in the case of a Grassmann connection, observe that this is obvious in the case $\mathscr{P} = \mathscr{A}^r$, so we only have to deal with the case $\mathscr{P} = e\mathscr{A}$. Since we saw above that D and $eDe + (1-e)D(1-e)$ differ by a bounded self-adjoint operator, the same homotopy trick shows that we may replace D by $eDe + (1-e)D(1-e)$, i.e., assume D commutes with e. Then e commutes with $D(1+D^2)^{-1/2}$ and $eD(1+D^2)^{-1/2} = eDe(1+(eDe)^2)^{-1/2}$, so converting our unbounded Kasparov bimodules to bounded ones (by the Baaj–Julg procedure in [3]) reduces us to a fairly trivial case of the usual product, where everything is obvious. So this completes the proof. □

Proof of Theorem 1.3 (*continued*). We will apply Theorem 1.8 to the situation where P is the space of sections of $\mathscr{V}$, which is a finitely generated projective module over the algebra $C(M) \otimes A$. The Teleman–Hilsum signature operator D defines an unbounded $(C(M), \mathbb{C})$-bimodule, but we may 'dilate' it by tensoring with A, i.e., by considering differential forms with values in A, to get an unbounded Kasparov $(C(M) \otimes A, A)$-bimodule. We may give $\mathscr{V}$ a 'smooth' structure, either directly, using locally constant transition functions for the bundle, or else using Corollary 1.6. Then it is clear that $D_{\mathscr{V}} = d_{\mathscr{V}} + d_{\mathscr{V}}^*$ (defined on a suitable domain) gives a D-connection in the sense of Definition 1.7. Hence we may apply Theorem 1.8 and conclude that $D_{\mathscr{V}}$ is self-adjoint and regular. Define $\Delta_{\mathscr{V}} = D_{\mathscr{V}}^2$ and also define

$$d^j : K_{\mathscr{V}}^j \to K_{\mathscr{V}}^{j+1} \quad \text{by} \quad d^j = d_{\mathscr{V}}(1 + \Delta_{\mathscr{V}})^{-1/2}.$$

Note that d^j is a *bounded* A-linear operator which agrees on the domain of $d_{\mathscr{V}}$ with $(1 + \Delta_{\mathscr{V}})^{-1/2} d_{\mathscr{V}}$ (since the fact that $d_{\mathscr{V}}^2 = 0$ implies that $d_{\mathscr{V}}$ commutes with bounded functions of $\Delta_{\mathscr{V}}$).

1.10. LEMMA. $\{K_{\mathscr{V}}^j, d^j\}_{|j| \leqslant n}$, *together with the hermitian pairing*

$$K_{\mathscr{V}}^j \times K_{\mathscr{V}}^{-j} \to A : \langle \varphi, \psi \rangle = \int \varphi^* \wedge \psi,$$

is a regular hermitian Fredholm complex in the sense of [20]. *The Hodge* $*$*-operator defines a grading* τ *on* $K_{\mathscr{V}}^{\bullet}$, *and the associated signature operator in the sense of* [20] *is* $D_{\mathscr{V}}(1 + \Delta_{\mathscr{V}})^{-1/2}$.

Proof. This is almost immediate, once one notices that $d^{j+1} d^j = 0$, since on a dense domain,

$$d^{j+1} d^j = d_{\mathscr{V}}(1 + \Delta_{\mathscr{V}})^{-1/2} d_{\mathscr{V}}(1 + \Delta_{\mathscr{V}})^{-1/2} = d_{\mathscr{V}}^2(1 + \Delta_{\mathscr{V}})^{-1} = 0,$$

and $d^{j+1} d^j$ is bounded. The regularity of the complex follows from the first sentence of [20, §4]. The Fredholm property is checked just as in [20, §5]. The signature operator is obtained from $d + d^*$, which is the bounded operator coinciding on a suitable dense domain with

$$\begin{aligned} &d_{\mathscr{V}}(1 + \Delta_{\mathscr{V}})^{-1/2} + (1 + \Delta_{\mathscr{V}})^{-1/2} d_{\mathscr{V}}^* \\ &\quad = d_{\mathscr{V}}(1 + \Delta_{\mathscr{V}})^{-1/2} + d_{\mathscr{V}}^*(1 + \Delta_{\mathscr{V}})^{-1/2} = D_{\mathscr{V}}(1 + \Delta_{\mathscr{V}})^{-1/2}, \end{aligned}$$

as required. □

Proof of Theorem (continued). As in [14], we can use $D_{\mathscr{V}}$ and τ to define an unbounded Kasparov module in the sense of [3]. By Theorem 1.8, it is clear that the A-index of $[D_{\mathscr{V}}]$, the element of $KK(C(M), A)$ so obtained, is just the Kasparov product $[\mathscr{V}] \otimes_{C(M)} [D]$, where

$$[\mathscr{V}] \in K^0(M; A) = KK(\mathbb{C}, C(M) \otimes A)$$

is the K-theoretic class of the bundle $\mathscr{V}$. On the other hand, the Baaj–Julg recipe for converting an unbounded Kasparov module to an ordinary bounded one involves taking $D_{\mathscr{V}}(1 + \Delta_{\mathscr{V}})^{-1/2}$, which by Lemma 1.10 is just the signature operator of our Fredholm complex. In other words, the A-signature of the complex in (1.10) is just

$$[\mathscr{V}] \otimes_{C(M)} [D] = \beta(f_*([D]))$$

(cf. [23]).

To show this is an oriented homotopy invariant, we identify it with the symmetric signature of the 'simplicial cochain complex' of M. However, since it is not obvious that M is triangulable, this requires a word of explanation. Since M is a compact ANR, it is dominated by a finite polyhedron, so there is an algebraic Poincaré complex $C^{\bullet}$ of finitely generated projective A-modules homotopy-equivalent to its singular cochain complex. (In fact, M has the same homotopy type as a compact polyhedron, say L (either using [26, Essay III, §2] or else by [53, Corollary 5.3]), and one can use the simplicial cochain complex of L with local coefficients in A.)

Our theorem will follow from [20, Proposition 3.8 and Theorem 4.1] if we can show that that $K^{\bullet}_{\mathscr{V}}$ and $C^{\bullet}$ are homotopy-equivalent, and this in turn will follow if we can prove a 'Lipschitz de Rham Theorem' for $K^{\bullet}_{\mathscr{V}}$. The method for doing this is based on ideas of Kaminker and Miller, some of which are unpublished.

First we replace $K^{\bullet}_{\mathscr{V}}$ with a chain-equivalent complex of 'Sobolev spaces'. Namely, let

$$H^i_{n-i} = K^i_{\mathscr{V}} \cap \bigcap_{j=0}^{\infty} \operatorname{dom}(D^j_{\mathscr{V}}),$$

completed with respect to the A-valued inner product defined by

$$\langle \eta, \omega \rangle_{n-i} = \langle \eta, (1 + \Delta_{\mathscr{V}})^{n-i} \omega \rangle.$$

This is roughly speaking a Sobolev space of $(n+i)$-forms with $n-i$ derivatives in L^2. Then $d_{\mathscr{V}}: H^i_{n-i} \to H^{i+1}_{n-i-1}$ is a bounded operator since

$$\begin{aligned} \langle d_{\mathscr{V}}\omega, d_{\mathscr{V}}\omega \rangle_{n-i-1} &= \langle d_{\mathscr{V}}\omega, (1 + \Delta_{\mathscr{V}})^{n-i-1} d_{\mathscr{V}}\omega \rangle \\ &= \langle \omega, (1 + \Delta_{\mathscr{V}})^{n-i-1} d^*_{\mathscr{V}} d_{\mathscr{V}}\omega \rangle \end{aligned}$$

is dominated by

$$\langle \omega, (1 + \Delta_{\mathscr{V}})^{n-i} \omega \rangle = \langle \omega, \omega \rangle_{n-i-1}$$

and $V^i: H^i_{n-i} \to K^i_{\mathscr{V}}$ defined by $\omega \mapsto (1 + \Delta_{\mathscr{V}})^{(n-i)/2}\omega$ is an isometry for each i. It is immediate that $V^{\bullet}: (H^{\bullet}, d_{\mathscr{V}}) \to (K^{\bullet}_{\mathscr{V}}, d^{\bullet})$ is a chain equivalence.

Now observe that the condition for a form to lie in H^i_{n-i}, once we have fixed the Riemannian metric g, is local, since the Hodge $*$-operator and $d_{\mathscr{V}}$ are local and M is compact. Therefore, H^i_{n-i} may be identified with the global sections of a sheaf $H^i_{n-i,\mathrm{loc}}$, where for any open set U,

$$\Gamma(U, H^i_{n-i,\mathrm{loc}}) = \left\{ \begin{array}{l} \text{measurable } (n+i) \text{ forms } \omega \text{ on } U \text{ such that for each open set} \\ W \text{ with compact closure in } U,\ \omega|_W = \eta|W \text{ for some } \eta \in H^i_{n-i} \end{array} \right\}.$$

1.11. THEOREM (sheaf-theoretic Lipschitz de Rham). *For each i, the sheaf $H^i_{n-i,\mathrm{loc}}$ is fine, $(H^\bullet_{n-\bullet,\mathrm{loc}}, d_{\mathscr{V}})$ is a fine resolution of the locally constant sheaf of germs of sections of $\mathscr{V}$, and the complex $(H^\bullet_{n-\bullet}, d_{\mathscr{V}})$ is homotopy-equivalent to the singular cochain complex of M with twisted coefficients and to the algebraic Poincaré complex $C^\bullet$.*

Proof. By [15, Corollaire 1.11 and Remarque 1.12], there are 'blip' functions inside $\bigcap_{j=0}^{\infty} \mathrm{dom}(D^j)$ of arbitrarily small compact support, hence one can construct partitions of unity using such functions just as with C^∞ functions on a smooth manifold. This shows that the sheaves are fine. To show that we have a resolution of the locally constant sheaf of germs of sections of $\mathscr{V}$, it is necessary to check the 'Poincaré Lemma', that $(H^\bullet_{n-\bullet,\mathrm{loc}}, d_{\mathscr{V}})$ is an acyclic complex of sheaves. (The idea of this was alluded to in the somewhat terse comments in [20, proof of Theorem 4.1].) Since this is a local problem, it's enough to consider the case of the open unit cube $U = (-1, 1)^{2n}$ in $\mathbb{R}^{2n}$ with a Lipschitz Riemannian metric g. Then the bundle $\mathscr{V}$ is trivial and $\Delta_{\mathscr{V}}$ is the usual Laplacian (for the metric g) on A-valued forms. We must show that the complex $(H^\bullet_{n-\bullet,\mathrm{loc}}, d_{\mathscr{V}})$ is chain homotopy equivalent to the constant sheaf. The usual proof (cf. [5, pp. 33–35]) works: define a chain homotopy by

$$\omega \mapsto \eta, \quad \eta(x) = \begin{cases} 0, & \omega = \sum\limits_{1 \leqslant i_1 < \cdots < i_q \leqslant 2n-1} f_{(i_1,\ldots,i_q)}\, dx_{i_1} \wedge \cdots \wedge dx_{i_q}, \\ \displaystyle\int_0^{x_{2n}} v(x,t)\, dt, & \omega = v \wedge dx_{2n}, \\ & v = \sum\limits_{1 \leqslant i_1 < \cdots < i_{q-1} \leqslant 2n-1} f_{(i_1,\ldots,i_{q-1})}\, dx_{i_1} \wedge \cdots \wedge dx_{i_{q-1}}. \end{cases}$$

The proof that this works formally is as in [5]; one only needs to check that this is well-defined as a map of local Sobolev spaces $H^q_{n-q,\mathrm{loc}} \to H^{q-1}_{n-q+1,\mathrm{loc}}$. This is immediate since, as usual, one gains one distributional derivative by integrating.

The rest then follows from the uniqueness theorem for sheaf cohomology and comparison of resolutions. □

Proof of Theorem 1.3 (concluded). It is evident that the homotopy equivalences of Theorem 1.11 are compatible with Poincaré duality, so that Definition 2.2 of [20] is satisfied. The theorem now follows just as in [20, Theorem 4.1]. □

2. The Lipschitz G-Signature Theorem

The next step is to generalize all of the above in the context of a G-action on M, where G is a compact Lie group. (Noncompact group actions are of course also interesting but are much more awkward to handle analytically. On the other hand, there is no point in considering actions of non-Lie groups because of [15, Remarque 1.6(2)], which shows that the isometry group of a Lipschitz Riemannian manifold is a Lie group.) To obtain an equivariant Kasparov element from the Teleman signature operator, it is obviously necessary to assume that G acts by Lipschitz homeomorphisms.

Before proceeding, it is worth addressing the question of how the category of Lipschitz G-actions is related to other more familiar categories of G-actions on manifolds. Theorem 1.1 shows that any free properly discontinuous action of a discrete group on a topological manifold of dimension $\neq 4$ can be made Lipschitz (by pulling back a Lipschitz structure on the quotient). Also, any PL G-action on a PL manifold is obviously locally Lipschitz (or globally Lipschitz if the manifold is compact) since it is Lipschitz on each simplex (when the manifold is compact, there are only finitely many simplices so there is a uniform upper bound on the Lipschitz constants). On the other hand, the following example shows that Lipschitz actions, even of finite groups on smooth manifolds, can be rather nasty. In particular, if M is a Lipschitz G-manifold, it need not be a G-ANR in the sense of [30].

2.1. EXAMPLE. For each prime p, there is an orientation-preserving action of $\mathbb{Z}/p$ on a (standard) sphere S^n by Lipschitz homeomorphisms, for which the fixed set F does not have finitely generated homology (and thus cannot be an ANR).

Construction. We begin by choosing a diffeomorphism α of the annulus $D^n(2)\backslash\mathring{D}^n(1)$ (here $D^n(r)$ denotes the closed n-disk of radius r) so that

(a) α has period p, sends the boundary components to themselves, and is orientation-preserving,
(b) the restrictions of α to the boundary spheres $S^{n-1}(2)$ and $S^{n-1}(1)$ are conjugate under the homothety $x \mapsto \frac{1}{2}x$,
(c) the fixed set of $\alpha|_{S^{n-1}(1)}$ is an integral homology sphere, and
(d) the fixed set of α has torsion in its integral homology (of order prime to p, of course).

There are many possibilities for such an α, at least if n is large enough – see [17, Corollary 3.1]. Now extend α to all of $D^n(2)$ by 'rescaling'; more precisely, define

$$\alpha(x) = \begin{cases} 0, & \text{if } x = 0, \\ 2^{-r}\alpha(2^r x), & \text{if } 2^{-r} \leqslant \|x\| \leqslant 2^{-(r-1)}, \quad r \geqslant 0. \end{cases}$$

Note that α is a well-defined homeomorphism of $D^n(2)$ since $\alpha(x) = \frac{1}{2}\alpha(2x)$ when $\|x\| = 1$. Furthermore, α restricted to the annulus $D^n(2^{-(r-1)})\backslash\mathring{D}^n(2^{-r})$ is Lipschitz *with the same Lipschitz constant* as α on $D^n(2)\backslash\mathring{D}^n(1)$. We claim α is Lipschitz on $D^n(2)$. Indeed, $\alpha|_{D^n(2)\backslash\mathring{D}^n(1/2)}$ has some Lipschitz constant K, and by the remark we just made,

$$d(\alpha(x), \alpha(y)) \leqslant Kd(x, y)$$

if x and y lie in the same or adjacent annuli $D^n(2^{-(r-1)})\backslash\mathring{D}^n(2^{-r})$. On the other hand,

$$d(0, \alpha(y)) \leqslant 2d(0, y)$$

(since α preserves the annuli $D^n(2^{-(r-1)})\backslash \mathring{D}^n(2^{-r})$), and similarly

$$\frac{d(\alpha(x), \alpha(y))}{d(x, y)} \leqslant \frac{2+\frac{1}{2}}{1-\frac{1}{2}} = 5$$

if x and y are non-zero and not in the same or adjacent annuli. (The worst case is illustrated in Figure 1.) Thus α is Lipschitz, with Lipschitz constant $\leqslant \max(K, 5)$, and periodic of period p. By construction, the integral homology of the fixed set is not finitely generated, since torsion is contributed from each annulus. Doubling α gives an action on a sphere instead of on a disk. □

This example shows that existence of a G-invariant Lipschitz structure is quite a weak condition. In fact, one can even prove:

2.2. THEOREM [38]. *Suppose M^n is a (closed, say) topological manifold and G is a finite group acting on M by homeomorphisms.*

(a) *If the action is locally linear and no component of any fixed set M^H is of dimension 4 (for any subgroup H of G—note that $H = \{1\}$ is included, which means $n \neq 4$), then M has a Lipschitz structure for which the action is by Lipschitz locally linear homeomorphisms. The Lipschitz action so obtained is unique up to Lipschitz conjugacy.*

(b) *If for each subgroup H of G, all components of the fixed set M^H are locally flat submanifolds of M, then for some k, the action of G on $M^n \times T^k$ (where G acts trivially on the second factor) is topologically conjugate to a Lipschitz action on*

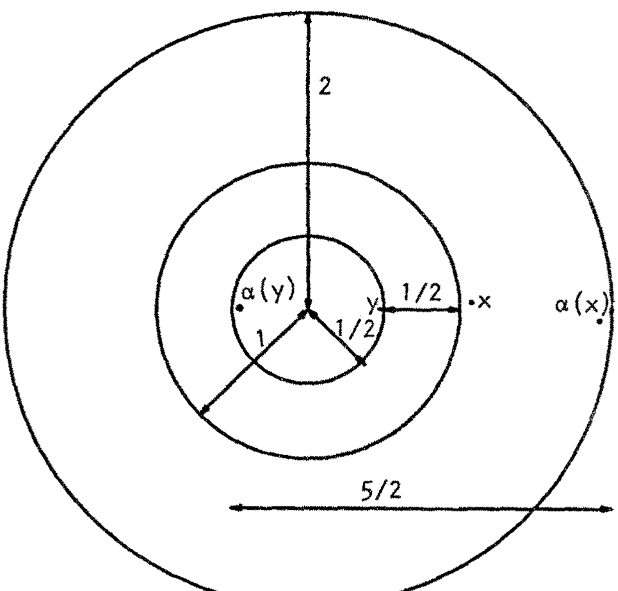

Fig. 1. Estimating the Lipschitz constant for α when x and y are not in the same or adjacent annuli.

a Lipschitz manifold. Any two such Lipschitz actions on $M^n \times T^k$ can be taken Lipschitz conjugate after crossing with another torus.

We are now ready for versions of the G-signature theorem and higher G-signature theorem for Lipschitz manifolds. Note in particular that parts (iii) and (v) of the following theorem establish the Equivariant Novikov Conjecture ([35, Theorem 3.8] and [36]) in the Lipschitz category.

2.3. THEOREM. *Let M^{2n} be a closed, oriented, connected Lipschitz manifold of even dimension, and suppose a compact Lie group G acts on M by Lipschitz homeomorphisms preserving the orientation.*

(i) *If g is a G-invariant Riemannian metric, then the Teleman-Hilsum signature operator D_g defines a class $[D]$ in $K_0^G(M)$ which is independent of the choice of Riemannian metric g.*

(ii) *The index of D (i.e., the image of $[D]$ under the map $f_*: K_0^G(M) \to K_0^G(pt) \cong R(G)$ induced by the map $f: M \to pt$) is just the G-signature of M defined the usual way from the action of G on $H^n(M; \mathbb{C})$ and from the cup-product pairing.*

(iii) *If π is a countable group (for instance the fundamental group of M), if $f: M \to B\pi$ is a map (e.g., the classifying map for the universal cover $\tilde{M}$ of M) which factors through M/G as in* [35, *(1.1)*], *and if $\mathscr{V}$ is the flat $C_r^*(\pi)$-bundle defined by f, then one can do the same with $D_{\mathscr{V}}$, and*

$$[D_{\mathscr{V}}] = [\mathscr{V}] \otimes_{C(M)} [D] \in KK^G(C(M), C_r^*(\pi))$$

(here $C_r^*(\pi)$ has the trivial G-action). The index of $[D_{\mathscr{V}}]$ in

$$K_0^G(C_r^*(\pi)) \cong K_0(C_r^*(\pi)) \otimes_{\mathbb{Z}} R(G)$$

may be obtained from the action of G on the cochain complex of M with local coefficients in $\mathscr{V}$.

(iv) *If G is finite, π is the fundamental group***oid** *of M, and $\mathscr{Y}$ is the flat $C_r^*(\pi)$-bundle on M defined in* [36, Proposition 3.4], *then one can do the same with $D_{\mathscr{Y}}$, and*

$$[D_{\mathscr{Y}}] = [\mathscr{Y}] \otimes_{C(M)} [D] \in KK^G(C(M), C_r^*(\pi)).$$

The index of $[D_{\mathscr{Y}}]$ in $K_0^G(C_r^*(\pi))$ may be obtained from the action of G on the cochain complex of M with local coefficients in $\mathscr{Y}$. Alternatively, one can use modules for $C_r^*(\Gamma)$, where Γ is the discrete group of transformations of the universal cover $\tilde{M}$ of M generated by covering transformations and lifts of elements of G.

(v) *If M is a G-ANR, in particular if the G-action is locally linear, then in the context of* (iii) *or* (iv), *the index of $[D_{\mathscr{V}}]$ or $[D_{\mathscr{Y}}]$ may also be obtained as the symmetric signature of the complex of finitely generated projective $C_r^*(\pi)$-modules (resp. $C_r^*(\Gamma)$-modules) obtained from a G-domination of M by a finite G-CW complex. It is therefore invariant under orientation-preserving pseudo-equivalences.*

Proof. (i) Assuming the metric g is G-invariant, G acts isometrically on the L^2-forms on M and commutes with the Hodge $*$-operator. Hence, it preserves the Hilbert space and the grading used to define D_g. If the action is Lipschitz, then G preserves the space of Lipschitz functions on M, which is a dense $*$-subalgebra $\mathscr{M}$ of $C(M)$ such that $[D_g, a]$ is bounded for all $a \in \mathscr{M}$. This is the $*$-subalgebra used by Hilsum [14] to check the Baaj–Julg conditions for a Kasparov element. So it is evident that copying the Baaj–Julg procedure gives an equivariant Kasparov element. (See also [15, Remarque 4.4(2)].) The argument of [14] for producing an operatorial homotopy between D_{g_0} and D_{g_1} goes over to the equivariant case if we stay in the class of G-invariant metrics. And of course, such metrics exist by averaging if G is compact.

(ii) This follows from the fact that if g is chosen G-invariant, then exactly the same Fredholm operator is obtained from the ordinary and from the equivariant signature operators. Its Fredholm index is the usual signature; its Fredholm G-index (i.e., the formal difference of the G-actions on its kernel and cokernel) is the index of the equivariant $[D]$.

(iii)–(v) The calculations of the Kasparov products work in the same way as in the nonequivariant case which we did above in Theorem 1.8 and the proof of Theorem 1.3. (See [47] for the necessary technicalities.) For (v), note that when M is G-dominated by a finite G-CW complex L, then in the context of (iii), the singular cochain complex of M with local coefficients in $\mathscr{V}$, together with the obvious G-action on this complex, is equivalent to a finite algebraic Poincaré G-complex of projective $C_r^*(\pi)$-modules with G-action, sitting as a summand in the cellular complex of L with local coefficients. (In general, the equivariant finiteness obstruction for M need not vanish, so that M itself doesn't necessarily have the G-homotopy type of a finite G-complex ([30], [51], [12, §12]). Fortunately this doesn't matter for our purposes.) In the situation of (iv), we argue the same way with complexes of $C_r^*(\Gamma)$-modules; compare Lemma 3.2 and Proposition 3.3 in [36]. □

Erratum. Since we have mentioned here the results on the Equivariant Novikov Conjecture in [36], we take this opportunity to correct a small error in that paper. Corollary 4.2 of that paper is simply wrong, since an affine action of a finite group on a flat cubical torus need not be isometric. The same error might seem at first to cause problems when it comes time to apply [36, Theorem 4.1], since that theorem gives a G-equivariant map to a flat cubical torus with an affine G-action, while [36, Theorem 5.1] requires a manifold of non-positive curvature with an isometric G-action. However, it is easy to see that given a flat cubical torus with an affine G-action, one can change the flat metric on the torus so as to make the action isometric, simply by averaging the metric under the finite 'holonomy group' (the image of G in $\mathrm{GL}(n, \mathbb{Z})$). The resulting flat torus is no longer cubical.

For purposes of applications, it is highly desirable to replace the index theorem (2.3) by a local formula involving fixed sets of subgroups of G. First, it is necessary to prove the analogue in equivariant *K-homology* of Segal's Localization Theorem [41, Proposition 4.5]. Though the following theorem probably applies if G is an arbitrary compact Lie group, the necessary universal coefficient theorem isn't worked out yet, so the proof would have to be more complicated. Besides, for our applications, we will only need the case where G is topologically cyclic, so what we do here is sufficient.

2.4. THEOREM (*Localization in K-homology*). *Let X be a locally compact, second-countable G-space, where G is either a finite group or else a connected compact Lie group with $\pi_1(G)$ torsion-free, for instance, a torus. Let $\mathfrak{p}$ be a prime ideal of $R(G)$ supported on a conjugacy class S of subgroups of G (in the sense of* [40, §3]). *Then if $K_G^*(X)$ is finitely generated as an $R(G)$-module, or in particular, if the one-point compactification of X is G-dominated by a finite G-CW-complex, the inclusion $X^{(S)} \hookrightarrow X$ induces an isomorphism*

$$K_*^G(X^{(S)})_{\mathfrak{p}} \xrightarrow{\cong} K_*^G(X)_{\mathfrak{p}}.$$

Proof. Because of the exact sequence of the pair $(X, X^{(S)})$, it is enough to show that if $X^{(S)} = \emptyset$, then $K_*^G(X)_{\mathfrak{p}} = 0$. Now by [4] or [28], if G is finite, there is a universal coefficient short exact sequence of $R(G)$-modules

$$0 \to \operatorname{Ext}^1_{R(G)}(K_G^{*-1}(X), R(G)) \to K_*^G(X) \to \operatorname{Hom}_{R(G)}(K_G^*(X), R(G)) \to 0,$$

and similarly, by [34] or unpublished work of Bökstedt briefly discussed in [28], if G is a connected compact Lie group with $\pi_1(G)$ torsion-free, there is a strongly convergent spectral sequence

$$\operatorname{Ext}^*_{R(G)}(K_G^*(X), R(G)) \Rightarrow K_*^G(X).$$

(**Caution:** Bökstedt appears to have K_G^* and K_*^G reversed, but that is because he restricts attention to finite G-CW-complexes. For present purposes, as in [34] or [28], we want to use the 'Steenrod K-homology theory' (cf. [18]) coming from equivariant Brown-Douglas-Fillmore or equivariant Kasparov theory. This behaves differently under limits than does representable homology for infinite complexes.)

By [41, Proposition 4.1], $K_G^*(X)_{\mathfrak{p}} = 0$ if $X^{(S)} = \emptyset$. But since $R(G)$ is a Noetherian ring [40, Corollary 3.3], localization commutes with Ext [9, Ch. VI, Exercise 11 and Ch. VII, Exercise 10] provided we restrict attention to finitely generated modules. Hence

$$\operatorname{Ext}^*_{R(G)_{\mathfrak{p}}}(K_G^*(X)_{\mathfrak{p}}, R(G)_{\mathfrak{p}}) \Rightarrow K_*^G(X)_{\mathfrak{p}}$$

and the result follows. □

Remark. In fact, Theorem 2.4 is *false* without the finite generation hypothesis. Here is a simple counterexample. Let $G = S^1$, so $R(G) = \mathbb{Z}[t, t^{-1}]$, and give S^{2n-1} the free G-action coming from viewing S^{2n-1} as the unit sphere in $\mathbb{C}^n$, with $G = S^1$ acting by multiplication by scalars of modulus one. Then

$$K_G^i(S^{2n-1}) = \begin{cases} R(G)/(t-1)^n, & \text{if } i = 0, \\ 0, & \text{if } i = 1. \end{cases}$$

Let $X = \amalg_{n=1}^{\infty} S^{2n-1}$ (infinite disjoint union!) and let $\mathfrak{p} = (t+1)$, so $\mathfrak{p}$ has support $H = \{+1, -1\}$. Clearly $X^H = \emptyset$; on the other hand, by the universal coefficient theorem,

$$K_i^G(S^{2n-1}) = \begin{cases} 0, & \text{if } i = 0, \\ R(G)/(t-1)^n, & \text{if } i = 1, \end{cases}$$

so

$$K_i^G(X) = \begin{cases} 0, & \text{if } i = 0, \\ \prod_{n=1}^{\infty} R(G)/(t-1)^n, & \text{if } i = 1. \end{cases}$$

Since the infinite *product* is not a torsion module, it's easy to see that $K_*^G(X)_{\mathfrak{p}} \neq 0$. □

Now we may apply Theorem 2.4 to the situation of Theorem 2.3. The following result, in view of Theorem 2.2, improves [49, Theorem 14.B.2] and in several respects is more satisfying than [16, Theorem 6.8]. It also applies slightly more generally than [15, Proposition 4.6] and [29, Theorem 7.2].

2.6. THEOREM. *Let M^{2n} be a closed, connected oriented Lipschitz manifold on which a compact Lie group G acts by Lipschitz homeomorphisms preserving the orientation. Assume further that $K_G^*(M)$ is finitely generated as an $R(G)$-module, which is the case if M is a G-ANR or if the action is locally linear.*

(i) *Then*

$$\operatorname{Sign}(M, \gamma) = \sum_{i=1}^{r} \sigma(F_i^\gamma), \quad \gamma \in G,$$

where the left-hand side denotes the G-signature of M, viewed as a (virtual) character on G and evaluated at γ, where F_i^γ runs over the components of the fixed set M^γ, and where $\sigma(F_i^\gamma)$ is a term depending only on the germ of M (with its action of γ) around F_i^γ. In particular, if any F_i^γ is a smooth manifold with a smooth γ-normal bundle in M, then $\sigma(F_i^\gamma)$ is given by the formula of Atiyah and Singer [2, Theorem 6.12].

(ii) *In the non-simply connected case, if there is a G-map f from M to an equivariantly aspherical G-space X in the sense of* [36] *or* [27] *(this is a G-space such that each connected component of X^H is aspherical, for each subgroup H of G) such that $K_G^*(X)$ is finitely generated as an $R(G)$-module, a similar formula holds for the higher G-signature in $K_*^G(X)$ computed at γ, viewed (via the Chern character) as an element of $H_*(X^\gamma; \mathbb{C})$.*

In the special case where $X = B\pi$ with trivial G-action, for some group π, the higher G-signature may be viewed as living in

$$H_*(B\pi; \mathbb{C}) \otimes_{\mathbb{Z}} R(G).$$

Proof. Apply the localization theorem to the image of $[D] \in K_0^G$ (as constructed in (2.3)) in $K_0^G(M)_{\mathfrak{p}}$, where $\mathfrak{p}$ is the prime ideal consisting of characters that vanish on the closure H of the cyclic subgroup generated by γ. (The group H must be of the form $(\mathbb{Z}/m) \times \mathbb{T}^k$ for some m and k, and it is easy to see that the statement of (2.4) holds for such groups if it holds for both tori and finite abelian groups.) We deduce that this is a sum of local contributions $\sigma(F_i^\gamma)$ from the F_i^γ. If we now map into $K_0^G(pt)_{\mathfrak{p}}$ or into $K_0^G(X)_{\mathfrak{p}}$ to compute the G-signature or higher G-signature, we get a local formula for the latter. (In the 'higher' case, we need to use the localization theorem for X as well.)

The crucial thing is to show that the contribution $\sigma(F_i^\gamma)$ depends only on the germ of M (as an H-space) around F_i^γ. This may be regarded as the 'infinitesimal H-equivariant normal bundle' of F_i^γ, but of course this need not be a bundle in the usual sense; furthermore, F_i^γ need not be a manifold. However, since $K_G^*(M)$ is finitely generated as an $R(G)$-module, there can be only finitely many components F_i^γ. Choose disjoint H-invariant open sets $V_i \supseteq F_i^\gamma$ and smaller H-invariant open sets U_i with $F_i^\gamma \subset U_i \subset\subset V_i$. (It's obviously no loss of generality to assume the inclusions are strict, since the only case when we can't arrange this is the trivial case $M^\gamma = M$.) We assume we've already fixed a G-invariant Lipschitz Riemannian metric g and thus the operator D. Then choose 'cut-off functions' $f_i \geqslant 0$ which are Lipschitz (hence, in the domain of D) and H-invariant, with $f_i \equiv 1$ on U_i, $f_i \equiv 0$ on the complement of V_i. The localization theorem says that the restriction map

$$KK^H(C(M), \mathbb{C}) \to KK^H\left(\bigoplus_{i=1}^r C_0(U_i), \mathbb{C} \right) \cong \bigoplus_{i=1}^r KK^H(C_0(U_i), \mathbb{C})$$

becomes an isomorphism after localization. The idea is therefore to show that the unbounded Kasparov $(C_0(U_i), \mathbb{C})$-bimodule $(L^2(M, \wedge^\bullet), D)$ can be replaced by $(L^2(M, \wedge^\bullet), f_iDf_i)$ without changing its KK^H-class. Then it's evident that, since $C_0(U_i)$ and f_iDf_i act trivially on forms supported off V_i, we can replace the Hilbert space $L^2(M, \wedge^\bullet)$ by $L^2(V_i, \wedge^\bullet)$. Since V_i was an arbitrarily small H-invariant neighbourhood of F_i^γ, this proves the desired localization statement.

To make the argument work, we need the following technical lemma involving standard techniques from the theory of elliptic partial differential equations.

2.7. LEMMA. *With notation as above, the operator f_iDf_i on its natural domain*

$$\mathrm{dom}(f_iDf_i) = \{\omega : f_i^2\omega \in \mathrm{dom}\, D\}$$

is self-adjoint, hence regular in the sence of Baaj and Julg. If φ is a Lipschitz function, φ maps the domain of f_iDf_i into itself, and $[\varphi, f_iDf_i]$ is bounded. Finally, for $\varphi \in C_0(U_i)$, $\varphi(1+(f_iDf_i)^2)^{-1}$ is a compact operator.

Proof. Observe that by the Leibnitz rule, Lipschitz functions on M map $\mathrm{dom}\, D$ to itself and have compact commutator with D. It follows that

$$\mathrm{dom}(f_iDf_i) = \{\omega: f_i^2\omega \in \mathrm{dom}\, D\}$$

and that

$$f_i D f_i = D f_i^2 + [f_i, D] f_i$$

as closed operators, where $[f_i, D]f_i$ is bounded. It is also clear that $f_i D f_i$ is symmetric. For self-adjointness, let $\omega \in \operatorname{dom}(f_i D f_i)^*$ and $v \in \operatorname{dom} D$. Then if $\eta = (f_i D f_i)^* \omega$, we have

$$\langle \eta, v \rangle = \langle \omega, f_i D f_i v \rangle = \langle \omega, (f_i^2 D + f_i [f_i, D]) v \rangle,$$

so

$$\langle f_i^2 \omega, D v \rangle = \langle \eta, v \rangle + \langle [D, f_i] f_i \omega, v \rangle = \langle \eta + [D, f_i] f_i \omega, v \rangle,$$

which shows

$$f_i^2 \omega \in \operatorname{dom} D^* = \operatorname{dom} D$$

so that

$$\omega \in \operatorname{dom}(f_i D f_i) \quad \text{and} \quad D f_i^2 \omega = \eta + [D, f_i] f_i \omega,$$

and thus

$$(f_i D f_i)\omega = (D f_i^2 - [D, f_i] f_i)\omega.$$

The fact that $[\varphi, f_i D f_i]$ is bounded when φ is Lipschitz follows from the same property for D.

The compactness has to do with 'pseudolocality' of $(1 + (f_i D f_i)^2)^{-1}$. We may suppose φ has compact support in some set $W \subset\subset U_i$. Choose another Lipschitz cut-off function ζ with $\zeta \equiv 1$ on W, $\zeta \equiv 0$ on the component of U_i. Thus $\zeta \equiv 0$ where $f_i \neq 1$. Let $v = (1 + (f_i D f_i)^2)\omega$. Then

$$\| \omega \|_{L^2(M, \Lambda)} \leqslant \| v \|_{L^2(M, \Lambda)},$$

so if we can prove an estimate of the form

$$\| D\omega \|_{L^2(W, \Lambda)} \leqslant C \| v \|_{L^2(M, \Lambda)},$$

it will follow from the Sobolev embedding theorem (which still applies, in effect, in the Lipschitz case, as proved in [14] – this is really the same as the compactness of the resolvent of D) that $v \mapsto \omega|_{L^2(W,\Lambda)} = (1 + (f_i D f_i)^2)^{-1} v|_{L^2(W,\Lambda)}$ is compact, and thus that $\varphi(1 + (f_i D f_i)^2)^{-1}$ is compact.

Now, using the fact that $\zeta = 0$ where $f_i \neq 1$, we get

$$\begin{aligned}
&\langle \zeta^2 v, \omega \rangle|_{L^2(U_i, \Lambda)} \\
&\quad = \langle \zeta^2 (1 + (f_i D f_i)^2)\omega, \omega \rangle \\
&\quad = \langle \zeta^2 (1 + D^2)\omega, \omega \rangle \\
&\quad = \langle \zeta^2 \omega, \omega \rangle + \langle \zeta^2 D\omega, D\omega \rangle - \langle 2\zeta L_{d\zeta}(D\omega), \omega \rangle \\
&\quad = \| \zeta\omega \|^2 + \| \zeta(D\omega) \|^2 + 2\left\langle \frac{1}{\sqrt{2}} \zeta(D\omega), \sqrt{2} L_{d\zeta}\omega \right\rangle,
\end{aligned}$$

where $L_{d\zeta} = e(d\zeta) - e(d\zeta)^*$, e denoting exterior multiplication. Using the fact that

$$|2\langle A, B\rangle| \leqslant \|A\|^2 + \|B\|^2,$$

we get

$$\begin{aligned}
&|\langle \zeta^2 v, \omega\rangle_{L^2(U_i,\Lambda)}| \\
&\quad \geqslant \|\zeta\omega\|^2 + \|\zeta(D\omega)\|^2 - \tfrac{1}{2}\|\zeta(D\omega)\|^2 - 2\|L_{d\zeta}\omega\|^2 \\
&\quad = \|\zeta\omega\|^2 + \tfrac{1}{2}\|\zeta(D\omega)\|^2 - 2\|L_{d\zeta}\omega\|^2,
\end{aligned}$$

which on rearrangement gives

$$\langle D\omega, D\omega\rangle_{L^2(W,\Lambda)} \leqslant C_1\|\omega\|^2 + C_2\|v\|\,\|\omega\| \leqslant C^2\|v\|^2$$

for a suitable constant C, as required. The completes the proof of the lemma. □

Proof of Theorem 2.6 (*continued*). The lemma shows that $(L^2(M, \Lambda^\cdot), f_i D f_i)$, or if one prefers, $(L^2(V_i, \Lambda^\cdot), f_i D f_i)$, gives a well-defined class in $KK^H(C_0(U_i), \mathbb{C})$. We have only to show that this class is the same as that defined by $(L^2(M, \Lambda^\cdot), D)$. But this follows from the fact that the argument in the lemma shows that

$$(t + (1-t)f_i)D(t + (1-t)f_i), \quad 0 \leqslant t \leqslant 1,$$

provides a homotopy between them. □

3. Applications of the Lipschitz G-Signature Theorem

Using Theorem 2.6, we can now obtain a number of geometric consequences similar to those obtainable from the usual G-signature theorem. Of course, our information is less precise, since (2.6) does not give an explicit formula for the local contributions to the G-signature except when the fixed sets are smooth manifolds with smooth G-normal bundles. However, in many cases of interest, these could be obtained by computing directly for standard examples and applying cobordism arguments (as in [49, §14.B] or [16, §6]). In addition, the fact that our theorem applies specifically to non-smooth situations makes it useful for analyzing such problems as topological conjugacy of linear representations or defining and computing characteristic classes in non-smooth categories.

The following theorem generalizes an observation of Schafer [39], which is (i) in the smooth case. However, we believe the PL case of (ii) to be new.

3.1. THEOREM. (i) *Suppose M^{4n} is a closed, connected, oriented topological manifold admitting a free orientation-preserving action of a non-trivial finite group G. Then the signature of M is divisible by $|G|$, and the signature vanishes if G acts trivially on $H^{2n}(M; \mathbb{Q})$.*

(ii) *Suppose M^{4n} is a closed, connected, oriented topological manifold admitting an orientation-preserving action of $G = \mathbb{Z}/p^r$ (p any prime, $r \geqslant 1$), such that no point of M is fixed by all of G. Assume further that all fixed sets of subgroups are locally flatly* *embedded topological manifolds, or such that M and the action are PL. Then the signature of M is divisible by p, and the signature vanishes if G acts trivially on $H^{2n}(M; \mathbb{Q})$.*

(iii) *If, in the context of* (i) *or* (ii), $\pi_1(M) \to \pi_1(M/G)$ *splits, the Strong Novikov Conjecture holds for* $\pi_1(M)$ (β: $K_*(B\pi_1(M)) \to K_*(C_r^*(\pi_1(M)))$ *is rationally injective), and G acts trivially on* $H^*(M; \mathbb{R}[\pi_1(M)])$, *then the higher signature of M vanishes in* $H_*(\pi_1(M); \mathbb{Q})$.

Proof. In case (i), M/G is a topological manifold, hence it admits a Lipschitz structure by Theorem 1.1 (unless $n = 1$, in which case we can replace M by $M \times \mathbb{CP}^2$ without changing the signature). Pulling this back to M, we get a G-invariant Lipschitz structure. Also since M/G has the homotopy type of a finite polyhedron, $K_G^*(M) \cong K^*(M/G)$ is finitely generated. Thus we can apply Theorem 2.6, and the G-signature of M (viewed as a character on G) vanishes at all $\gamma \in G \setminus \{e\}$. This shows the G-signature representation is a multiple of the regular representation of G, so the ordinary signature of M is divisible by $|G|$. On the other hand, if G acts trivially on middle cohomology, then the G-signature representation must also be a multiple of the trivial representation. This can happen only if the signature vanishes.

Next we do the case of (ii) when M admits a G-invariant Lipschitz structure and $K_G^*(M)$ is finitely generated. (These conditions are automatic in the PL case.) By Theorem 2.6 (applied in turn to each cyclic generator of G), the G-signature character must vanish on all generators of $\mathbb{Z}/p^r$. However, the nongenerators are exactly the elements of the subgroup isomorphic to $\mathbb{Z}/p^{r-1}$, and so the G-signature representation is induced from this subgroup. This shows that the signature is divisible by p, and vanishes if the G-action on middle cohomology is trivial.

The argument for (iii) is exactly the same as for (i) or (ii), using part (ii) of Theorem 2.6, together with (iii) of Theorem 2.3 and the Novikov Conjecture to translate a statement about $K_*(C_r^*(\pi_1(M)))$ to one about $H_*(\pi_1(M); \mathbb{C})$.

We must still deal with (ii) and (iii) when the fixed sets are locally flatly embedded topological submanifolds. Then by [30], M is a G-ANR, and so $K_G^*(M)$ is finitely generated. However, we don't know that M has a G-invariant Lipschitz structure, so we replace M by $M \times T^k$, which has such a structure by Theorem 2.2 if k is sufficiently large. So apply the case of (iii) which we already know to $M \times T^k$, and observe that the higher G-signature of M can be read off from that of $M \times T^k$ in

$$K_*^G(B\pi_1(M) \times T^k) \cong K_*^G(B\pi_1(M)) \otimes_{\mathbb{Z}} K_*(T^k). \qquad \square$$

Nonlinear Similarities. Next we discuss the application of our higher G-signature theorem to the famous *nonlinear similarity* problem. We recall the basic question: suppose G is a finite group and ρ_1 and ρ_2 are linear representations of G on finite-dimensional real vector spaces V_1 and V_2. One wants to know when there is a *homeo*morphism h: $V_1 \to V_2$ interwining ρ_1 and ρ_2. Without loss of generality one may assume $h(0) = 0$. Then if h is a diffeomorphism, the differential of h at 0 gives a linear conjugacy from ρ_1 to ρ_2. However, there are cases (with dim $V_i \geqslant 6$) where ρ_1

and ρ_2 are topologically, but not linearly, conjugate. The following constraint on this phenomenon was announed in [6] and in [52, Ch. 14, §4].

3.2. DEFINITION. Let ρ be a finite-dimensional linear representation of a group G, and let γ be an element of G of order k. Let $\zeta = e^{2\pi i/k}$, a primitive kth root of unity; then the restriction of ρ to the cyclic subgroup generated by γ is equivalent to $\oplus_{j=0}^{k-1} n_j t^j$, where $t^j(\gamma)$ is multiplication by ζ^j on $\mathbb{C}$ if $j \neq 0$ or $k/2$, multiplication by ζ^j on $\mathbb{R}$ if $j = 0$ or $k/2$. The *renormalized Atiyah–Bott number* of γ and ρ is defined by the formula

$$AB(\gamma, \rho) = \prod_{j \neq 0} \left(\frac{\zeta^j + 1}{\zeta^j - 1}\right)^{n_j}.$$

Remark. Note that the decomposition we have given of $\rho(\gamma)$ is not exactly unique, since for $0 < j < k/2$, t^j and t^{k-j} are equivalent as real-linear representations, though only via an orientation-reversing intertwining operator. However, one can check that $AB(\gamma, \rho)$ is an invariant of orientation-preserving real-linear conjugacies, and changes sign under orientation-reversing real-linear conjugacies.

3.3. THEOREM. *The renormalized Atiyah–Bott numbers $AB(\gamma, \rho)$, $\gamma \in G$, are (up to a sign, depending on whether orientation is preserved or not) topological conjugacy invariants of a linear representation ρ of a finite group G.*

Proof. Without loss of generality we may fix a group element $\gamma \in G$ of order k and assume it generates G. Let ρ_1 and ρ_2 be linear representations of G on finite-dimensional real vector spaces V_1 and V_2 and suppose $h: V_1 \to V_2$ is a homeomorphism intertwining ρ_1 and ρ_2 and satisfying $h(0) = 0$. We may assume dim V_j is even; otherwise, add on a trivial one-dimensional representation and take the product of the original h with the identity map on $\mathbb{R}$. If γ reverses orientation on V_1, then it must do so on V_2 as well, so -1 occurs as an eigenvalue of the action in both cases and both renormalized Atiyah–Bott numbers vanish. Therefore we may as well assume that G is orientation-preserving. Note that h must send the fixed set $W_1 = V_1^\gamma$ homeomorphically onto $W_2 = V_2^\gamma$, hence these fixed sets have the same dimension, n_0 in the notation of (3.2). We may assume ρ_1 and ρ_2 act orthogonally, in which case we have orthogonal direct sum decompositions $V_j = V'_j \oplus W_j$ $(j = 1, 2)$, with G acting trivially on the second factor. Let ρ'_j be the restriction of ρ_j to V'_j. Now we need the following lemma, which of course is trivial if $n_0 = 0$ (the case where we do not need to 'renormalize' the Atiyah–Bott number):

3.4. LEMMA. *After perhaps increasing n_0, there is a G-equivariant homeomorphism h' from* $(V'_1, \rho'_1) \times (T^{n_0}$, trivial action) *to* $(V'_2, \rho'_2) \times (T^{n_0}$, trivial action).

We defer the proof for the moment and use this and Theorem 2.6(ii) to complete the proof of the Theorem. We use h' to identify $(V'_1 \backslash \{0\}) \times T^{n_0}$ with $(V'_2 \backslash \{0\}) \times T^{n_0}$, and form the identification space

$$X = (\{0\} \times T^{n_0}) \cup \left((V'_1 \backslash \{0\}) \times T^{n_0} \underset{h'}{\cong} (V'_2 \backslash \{0\}) \times T^{n_0}\right) \cup T^{n_0},$$

where the last T^{n_0} should be viewed as

$$\text{(point at } \infty \text{ in } V_2') \times T^{n_0}.$$

Note that X is a topological manifold homeomorphic to $S^{\dim V_1'} \times T^{n_0}$. It carries an action of G by homeomorphisms, with two fixed-set components F_1 and F_2, each homeomorphic to T^{n_0}. The germ of F_1 in X (with its G-action) is clearly the same as that of $\{0\} \times T^{n_0}$ in $V_1' \times T^{n_0}$ under ρ_1, while the germ of F_2 in X is by construction the same as that of $\{0\} \times T^{n_0}$ in $V_2' \times T^{n_0}$ under the *conjugate action* to ρ_2. Note also that there is an equivariant G-map from X to T^{n_0}, the latter having trivial G-action. Theorem 2.2 now says that after increasing n_0 if necessary (and we can do this keeping the dimension of X even), we may assume that X has a G-invariant Lipschitz structure.

We may now apply Theorem 2.6(ii) to compute the higher G-signature of X in

$$H_*(T^{n_0}; \mathbb{C}) \otimes_{\mathbb{Z}} R(G),$$

since the action of G on X is locally linear. But this higher G-signature must vanish since a sphere has no middle cohomology. If n_0 were zero, X would be a sphere, γ would have two isolated fixed points, and the calculation of [1, formula (7.6)] would give

$$\begin{aligned} 0 &= G\text{-sign}(X)(\gamma) \\ &= AB(\gamma, \rho_1) + AB(\gamma, \overline{\rho_2}) \\ &= AB(\gamma, \rho_1) - AB(\gamma, \rho_2), \end{aligned}$$

proving that $AB(\gamma, \rho_1) = AB(\gamma, \rho_2)$. In general, we need to use the higher G-signature; but for a G-manifold Y, the higher G-signature of $Y \times T^{n_0}$ in $H_*(T^{n_0}; \mathbb{C}) \otimes_{\mathbb{Z}} R(G)$ is just a shifted version of the ordinary G-signature of Y. This tells us that the local contributions of F_1 and F_2 to the higher G-signature of X are just shifted versions of the renormalized Atiyah–Bott numbers $AB(\gamma, \rho_1)$ and $AB(\gamma, \overline{\rho_2}) = -AB(\gamma, \rho_2)$, respectively. Thus, as before, the renormalized Atiyah–Bott numbers are equal. □

3.5. COROLLARY (Hsiang-Pardon [16, Theorem A], Madsen-Rothenberg [29, Theorem 6.6]). *If ρ_1 and ρ_2 are topologically conjugate repesentations for which no element γ has eigenvalue -1, then ρ_1 and ρ_2 are linearly conjugate. In particular, if G is of odd order, then topologically conjugate representations of G are linearly conjugate.*

Proof. To show ρ_1 and ρ_2 are equivalent as linear representations, it suffices to show that their characters agree on any element γ. Without loss of generality, assume there is an orientation-preserving topological conjugacy between them. Then by Theorem 3.3, $AB(\gamma, \rho_1)$ and $AB(\gamma, \rho_2)$ coincide. The rest of the argument, which depends on the Franz Lemma, is exactly the same as the Atiyah–Bott proof that if G acts smoothly on a sphere with exactly two fixed points, then the representations at the fixed points are conjugate to one another [1, Theorems 7.15 and 7.27]. □

Remark. The problem with the case where -1 appears as an eigenvalue is obviously that it kills off the renormalized Atiyah–Bott number, regardless of what the other eigenvalues are. Thus in this case the equality of renormalized Atiyah–Bott numbers carries no information (except for the fact that if -1 appears as an eigenvalue for one representation, then it does for the other). This is the ultimate source of all examples of nonlinear similarity.

Now we have to go back and fill in the proof of Lemma 3.4. If we let Y_j be the unit sphere in V_j', then V_j' is (equivariantly) the (infinite) cone cY_j, and $X_j = Y_j/G$ is a (compact) stratified space. So in fact Lemma 3.4 is a special case of something much more general:

3.6. THEOREM ('Belt Buckle Trick'). *If X_j, $j = 1, 2$ are stratified spaces (in the category of (topological) manifold-stratified spaces discussed in* [52, Ch. 5] *or* [31]) *and there is a stratified homeomorphism between $cX_1 \times \mathbb{R}^i$ and $cX_1 \times \mathbb{R}^i$* that is the identity on $\{0\} \times \mathbb{R}^i$, *then there is a stratified homeomorphism between $cX_1 \times T^n$ and $cX_1 \times T^n$ for some n. (Actually, one can take $n = i$ except in a few cases where low-dimensional difficulties come in.)*

Proof. A complete geometric proof will appear in [7]. The result is essentially due to Morton Brown (unpublished), at least when X_j are manifolds. For the related fact that if A and B are compacta, then $A \times \mathbb{R} \cong B \times \mathbb{R}$ implies $A \times S^1 \cong B \times S^1$, see [42, Corollary 5.4] and [13, p. 85]. Here we will explain the case needed for Lemma 3.4 using the equivariant h-cobordism theorem and then use the same idea together with the main result of [52] to get the general case.

First consider the situation of Lemma 3.4, so that $i = \dim W_j$ is what we called n_0 there. In the situation of that Lemma, h restricts to a homeomorphism $h_1: \{0\} \times W_1 \to \{0\} \times W_2$. Replacing h by $h \circ (\mathrm{id}_{V_1'} \times h_1^{-1})$, we may assume $h: V_1' \times \mathbb{R}^i \to V_2' \times \mathbb{R}^i$ is an equivariant homeomorphism restricting to the identity on $\{0\} \times \mathbb{R}^i$, as in the statement of 3.6. We may also assume $h(Y_1 \times \mathbb{R}^i)$ is disjoint from $Y_1 \times \mathbb{R}^i$, and that the region between them in $V_2' \times \mathbb{R}^i$ defines a G-equivariant h-cobordism 'bounded over $\mathbb{R}^i$' in the sense of controlled topology (see [52, Ch. 9]), meaning that the inclusions of the boundary components are proper bounded homotopy equivalences. (Otherwise, compose with the homeomorphism $(v, t) \mapsto (f(t) \cdot v, t)$, $v \in V_2'$, $t \in \mathbb{R}^i$, defined by a suitable map $f: \mathbb{R}^i \to \mathbb{R}_+^\times$.) If necessary, we may increase i to be $\geqslant 5$. Then the h-cobordism theorem in the appropriate category (the statement and proof are almost the same as for [43, Corollary 4], except that one keeps track of boundedness over $\mathbb{R}^i$, and the obstruction group is slightly different; Wh groups are replaced by $\tilde{K}_{1-i}$) says this h-cobordism is boundedly classified by its torsion. Whatever this is, it will be killed by taking a product with S^1, so we obtain a G-homeomorphism

$$Y_1 \times S^1 \times \mathbb{R}^i \to Y_2 \times S^1 \times \mathbb{R}^i$$

bounded over $\mathbb{R}^i$. Choose any hyperplane $\mathbb{R}^{i-1} \hookrightarrow \mathbb{R}^i$. Then one obtains in a similar way a G-equivariant h-cobordism from $Y_1 \times S^1 \times \mathbb{R}^{i-1}$ to $Y_2 \times S^1 \times \mathbb{R}^{i-1}$ bounded over $\mathbb{R}^{i-1}$. As before this gives a G-homeomorphism

$$Y_1 \times T^2 \times \mathbb{R}^{i-1} \to Y_2 \times T^2 \times \mathbb{R}^{i-1}$$

bounded over $\mathbb{R}^{i-1}$. Continuing by induction, one gets the result by trading one copy of $\mathbb{R}$ at a time for a copy of S^1.

The general case of stratified sets and stratified maps works in the same way, except that we need the machinery of [31] or of [52] ('Stable Classification Theorem', Ch. 6, §2 and 'Main Theorem of Controlled Topology', Ch. 9, §2) to handle the h-cobordisms. □

Computation of Topological Normal Invariants. Now we will apply the topological G-signature operator to compute the normal invariant term in the stratified surgery sequence of [52], in the case of orientation-preserving actions of finite groups. This will enable us to prove Theorem 3.12 as stated in the Introduction. It would also be interesting to work out the cases of non-oriented manifolds or of actions that do not preserve orientation. At least in some cases, it should be possible to proceed in a similar fashion, but using the topological G-signature operator with coefficients in the flat real line bundle defined by w_1. We begin by recalling the following surgery sequences from [52]:

3.8. 'STABLE CLASSIFICATION THEOREM' [52, Ch. 6, §2]. *Let X be a manifold-stratified space with no four-dimensional strata. Then there is a fibration*

$$\mathscr{S}^{\mathrm{Top},-\infty}(X) \to \mathbf{H}_0(X, \mathscr{L}^{\mathrm{BQ}}) \to \mathbf{L}^{\mathrm{BQ}}(X) \times \oplus\, \mathbb{Z}$$

for computing the 'spacified' stable structure set of stratified spaces (simple-)homotopy-equivalent to X modulo homeomorphism.

Here the middle term is the spectral cosheaf homology of a cosheaf $\mathscr{L}^{\mathrm{BQ}}$, and the map on the right should be viewed as a suitable surgery assembly map for 'Browder-Quinn' surgery theory. If X is a manifold, the space on the right is the usual Wall surgery spectrum (depending only on the fundamental group of X). The $\mathbb{Z}$'s are a reflection of a nontrivial restriction on the local algebraic topology of manifolds that does not follow merely from local Poincaré duality. The actual structure set of stratified spaces (simple-) homotopy-equivalent to X modulo homeomorphism is π_0 of a related spectrum obtained by destabilizing:

3.9. 'DESTABILIZATION THEOREM' [52, Ch. 10, §3]. *With X as in Theorem 3.8, there is a fibration*

$$\mathscr{S}^{\mathrm{Top},s}(X) \to \mathscr{S}^{\mathrm{Top},-\infty}(X) \to \mathbf{H}^*(\mathbb{Z}/2; \mathbf{Wh}^{\mathrm{Top}}_{\leqslant 0}(X)).$$

We will refer to the middle term in (3.8) as the *stratified normal invariant set.* In fact, when X is a manifold,

$$\pi_0(\mathbf{H}_0(X, \mathscr{L}^{\mathrm{BQ}})) = H_0(X, \mathbf{L})$$

is the group of possible normal invariants, which away from the prime 2 is the KO-homology of X, the receiver group for the class of the signature operator. In the more general present context, certain homotopy-transverse maps with bundle data give rise to elements of the stratified normal invariant set. In particular, for any stratified space X, the degree-2 map

$$X \coprod X \xrightarrow{(\mathrm{id}_X, \mathrm{id}_X)} X$$

gives rise to a characteristic class of X away from 2, which should be viewed as the difference of the signature classes for $X \amalg X$ and X, in other words, as the signature class of X. When X is a manifold, this can be checked to be exactly the class $[D]$ of Theorem 1.2 in $KO_*(X) \otimes \mathbb{Z}[\frac{1}{2}] \hookrightarrow K_*(X) \otimes \mathbb{Z}[\frac{1}{2}]$. We shall prove the analogue of this for certain topological orientation-preserving actions of a finite group on a manifold.

First we need a few additional features of the construction in [52] that goes into Theorems 3.8 and 3.9.

(1) The spectra involved are functorial, with respect to open inclusions and restrictions to lower strata, in the local geometry.
(2) The stratification gives rise to a sequence of fibrations for computing the L-spectra, where the fibers are the L-spectra of the pure strata (which are manifolds), which only depend on the fundamental group.

These facts imply that $\mathbf{L}^{\mathrm{BQ}}(X) = 0$ when X is a stratified space with boundary such that each 'stratum with boundary' is relatively connected and simply connected (the 'π-π vanishing condition'). Now we are ready to state the main result. A purely topological calculation of the L-cosheaf homology groups appears in [8], where the applications to defining characteristic classes are also discussed. The present method also defines these classes for actions of compact Lie groups which aren't finite, although in this case the characteristic classes do not capture all the homology.

3.10. THEOREM. *If M is an oriented closed topological manifold with an orientation-preserving action of a finite group G and $X = M/G$, such that for all subgroups $H \subseteq K \subseteq G$, the fixed sets M^K and M^H are simply connected submanifolds of dimension $\neq 3$ and the inclusion of M^K in M^H is locally flat and of codimension $\geqslant 3$, then there are natural equivalences away from 2:*

$$\mathbf{H}_0(X, \mathscr{L}^{\mathrm{BQ}}) \to \oplus_H \mathbf{KO}^{N_G(H)/H}_{\dim M^H}(M^H), \qquad \mathbf{L}^{\mathrm{BQ}}(X) \to \oplus_H \mathbf{KO}^{N_G(H)/H}_{\dim M^H},$$

where H runs over a set of representatives up to conjugacy for the isotropy groups for the action of G on M. After substituting these into the long exact homotopy sequence associated with Theorem 3.8, the map

$$\theta\colon S^{\mathrm{Top}}(X) \otimes \mathbb{Z}[\tfrac{1}{2}] \to \oplus\, KO^{N_G(H)/H}_{\dim M^H}(M^H) \otimes \mathbb{Z}[\tfrac{1}{2}]$$

becomes simply the map sending

$$[N \xrightarrow{f} M] \in S^{\mathrm{Top}}(X),$$

where N is a G-manifold and f is an isovariant homotopy equivalence, to

$$\oplus_H([D_{M^H}] - f_*([D_{N^H}])),$$

where the signature classes in equivariant K-homology are defined using Theorem 2.3.

(There is no problem with strata of dimension 4 since by the work of Freedman, topological surgery works well in dimension 4 if the fundamental group is not too large. Here all strata have finite π_1.)

Proof. We need the facts (see [50]) that there is no difference between L^s and L^h after inverting 2, and that if π is a finite group, the odd L-groups of π vanish away from 2, and the canonical maps

$$L_{2k}(\mathbb{Z}\pi) \otimes \mathbb{Z}[\tfrac{1}{2}] \to L_{2k}(\mathbb{R}\pi) \otimes \mathbb{Z}[\tfrac{1}{2}] \to KO_{2k}(\mathbb{R}\pi) \otimes \mathbb{Z}[\tfrac{1}{2}] \to KO^{\pi}_{2k}(pt) \otimes \mathbb{Z}[\tfrac{1}{2}]$$

are isomorphisms. The first of these is induced by the inclusion $\mathbb{Z}\pi \hookrightarrow \mathbb{R}\pi$. The isomorphism from L-theory to KO-theory of $\mathbb{R}\pi$ is obtained by diagonalizing a hermitian or skew-hermitian form and taking the formal difference of the positive-definite and negative-definite parts. The isomorphism from $KO_*(\mathbb{R}\pi)$ to $KO^{\pi}_*(pt) \cong KO_{\pi}^{-*}(pt)$ comes from identifying a finite-dimensional representation of π with a π-equivariant vector bundle over a point. Note of course that $\mathbb{R}\pi$ is a direct sum of matrix algebras over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, corresponding to the various irreducible real representation of π, and that the composition of the above maps is what Wall calls the multi-signature, decomposing a hermitian or skew-hermitian form over the group ring into pieces corresponding to the various simple summands, and taking the corresponding signature of each piece.

Now the strata of X correspond to the various orbit types for the action of G on M, and since we are assuming all fixed sets are simply connected, these have finite fundamental groups $N_G(H)/H$, H running over a set of representatives up to conjugacy for the isotropy groups for the action of G on M. The L-spectrum of X is therefore an iterated fibration of Wall L-spectra for these finite groups. After tensoring with $\mathbb{Z}[\frac{1}{2}]$, we can replace these by equivariant KO-spectra, and all the odd homotopy groups vanish, so the even homotopy groups simply add together and the spectrum splits. (One can also write down an explicit map to the sum that realizes the splitting, by using the natural maps from the L-spectrum of X to the L-spectra of closed unions of strata, together with the maps corresponding to the multi-signatures of the associated fixed sets M^H.) Similar comments apply to the L-cosheaf $\mathscr{L}^{\mathrm{BQ}}(X)$, so the cosheaf homology of X is just the direct sum of the indicated equivariant KO-homology groups. This proves the first part of the theorem.

Next we can define a functorial map θ sending

$$[N \xrightarrow{f} M] \in S^{\mathrm{Top}}(X),$$

where N is a G-manifold and f is an isovariant homotopy equivalence, to

$$\oplus_H([D_{M^H}] - f_*([D_{N^H}])),$$

where the signature classes in equivariant K-homology are defined using Theorem 2.3. (Note that the stratified structure set of X is the same as the *isovariant* G-structure set of M! Furthermore, by the Destabilization Theorem (3.9), the stable and unstable structure sets agree away from 2.)

We already know that θ is the normal invariant map in the surgery sequence for manifolds, so by functoriality and induction on the number of strata, it has to be the correct map in the general case. □

3.11. COROLLARY. *If M is an oriented closed topological manifold with an orientation-preserving action of a finite group G, such that for all subgroups $H \subseteq K \subseteq G$, the fixed sets M^K and M^H are simply connected submanifolds of dimension $\neq 3$ and the inclusion of M^K in M^H is locally flat and of codimension $\geqslant 3$, then the isovariant G-structure set of M is given by*

$$S^{G\text{-isovar}}(M) \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \bigoplus \widetilde{KO}^{N_G(H)/H}_{\dim M^H}(M^H) \otimes \mathbb{Z}[\tfrac{1}{2}],$$

the isomorphism being defined by the classes of the appropriate signature operators on fixed sets.

Proof. By the Destabilization Theorem (3.9), the stable and unstable structure sets agree after tensoring with $\mathbb{Z}[\frac{1}{2}]$. Now substitute the calculation of (3.10) into the long exact sequence in homotopy groups coming from the fibration in (3.8). □

Remark. Since M^H need not have a $N_G(H)/H$-fixed point, the reader is cautioned that in odd dimensions, the 'reduced' equivariant KO-group (the relative group for a map to a point) might actually be bigger than the unreduced group. The additional piece can be identified with a sort of eta-invariant or Atiyah–Bott number.

This immediately implies the main theorem stated in the Introduction:

3.12. THEOREM. *A topological orientation-preserving action of a finite group G on a compact simply connected topological manifold M, such that for all subgroups $H \subseteq K \subseteq G$, the fixed sets M^K and M^H are simply connected submanifolds of dimension $\neq 3$ and the inclusion of M^K in M^H is locally flat and of codimension $\geqslant 3$, is determined up to finite indeterminacy by its isovariant homotopy type and the classes of the equivariant signature operators on all the fixed sets M^H.*

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