Characters of Connected Lie Groups

(Mathematical Surveys and Monographs 71)

By Lajos Pukánszky: xvii+128 pp., ISBN 0 8218 1088 X

List: \$59, Institutional Member: \$47, Individual Member: \$35

(American Mathematical Society, 1999)

Lajos Pukánszky [3] devoted most of his life to the study of a single subject: the unitary representation theory of solvable, or even completely arbitrary, connected Lie groups. (Any connected Lie group is (at least locally) a semidirect product of a semisimple group and a solvable group.)

A locally compact group G is said to be 'type I' if each of its unitary representations generates a von Neumann algebra of type I. This condition is implied by the stronger condition of being 'CCR' ('liminaire' in French), or having that property that for each irreducible representation π, $\pi(C_c(G))$ is contained in the compact operators. Back in the 1950's, Harish-Chandra proved that semisimple Lie groups are CCR, and Dixmier proved that nilpotent groups are CCR. But even the simplest solvable Lie group, the '$ax+b$ group', or the two-dimensional affine group of the line, is not CCR, and starting in dimension 5, there are solvable Lie groups that are not type I. Pukánszky set himself the task of understanding why and how this is the case, and of trying to make order out of the seemingly 'wild' aspects of Lie group representation theory. This (posthumous) book is a summary of his main accomplishments, originally published in a long series of papers, of which the most notable are [8], [9], and [10].

Chapter I of this book, based largely on the technical results in [8], is devoted first to proving that *locally algebraic* connected Lie groups are type I. This class includes the nilpotent Lie groups, the semisimple Lie groups, and a few of the most familiar solvable groups, such as the '$ax+b$ group' and the 'diamond group'. Then Pukánszky goes on to prove a theorem of Dixmier, that the regular representation of a connected Lie group always generates a semifinite von Neumann algebra. (In other words, it has a central direct integral decomposition into irreducible representations and type

II factor representations. Pukánszky eventually showed that the representations occurring in the central decomposition of the regular representation are among the *normal representations* studied in Chapter III.)

In contrast to semisimple Lie groups, which have a rigid structure theory, solvable Lie groups are quite 'flexible,' and there is no good classification of them. Thus it is not possible to study their representation theory 'case by case,' as is sometimes done with semisimple Lie groups. Nevertheless, Pukánszky discovered (though he never stated things in these terms) that the phenomenon of 'non-type I-ness" in solvable Lie groups can arise for exactly two different reasons, which are typified by two basic examples:

1. the *Mautner group* of dimension 5, the semidirect product $\mathbb{R} \ltimes \mathbb{C}^2$, where $\mathbb{R}$ acts on $\mathbb{C}^2$ by $t \cdot (z, w) = (e^{it}z, e^{i\lambda t}w)$, where λ is irrational;

2. the *Dixmier group* of dimension 7, the simply connected Lie group with Lie algebra spanned by $e_1, \dots, e_7$, satisfying the bracket relations

$$\begin{aligned} [e_1, e_2] &= e_7, \quad [e_1, e_3] = e_4, \quad [e_1, e_4] = -e_3, \\ [e_2, e_5] &= e_6, \quad [e_2, e_6] = -e_5, \quad [e_i, e_j] = 0 \text{ for } i,\, j \geq 3. \end{aligned}$$

In both of these cases, it is natural to try to analyze the representation theory of the solvable Lie group G by applying the 'Mackey method' to the action of G on the Pontryagin dual $\widehat{N}$ of the (abelian) commutator subgroup N. (Since N acts trivially on $\widehat{N}$, the action factors through G/N.) In the case of the Mautner group, $N = \mathbb{C}^2$, and the action of $G/N \cong \mathbb{R}$ on $\widehat{N} \cong \mathbb{C}^2$ is given by $t \cdot (z, w) = (e^{-it}z, e^{-i\lambda t}w)$. The key feature here is that each torus $|z| = c_1$, $|w| = c_2$ (c_1, $c_2 > 0$) is invariant, and on it $\mathbb{R}$ acts ergodically. Or in Mackey's terminology, there are 'non-transitive quasi-orbits' giving rise to non-type I behavior. In Dixmier's example, something rather different happens. The action of $G/N \cong \mathbb{R}^2$ on $\widehat{N} \cong \mathbb{R}^5$ has nice orbits, but for generic points in $\widehat{N}$, the stabilizer is disconnected, and the character χ of N does not extend to a character of its stabilizer G_χ in G. The non-type I-ness in this case arises from the fact that G_χ is a non-type I central extension of a *discrete* abelian group.

Pukánszky showed that the Dixmier example typifies a feature of the general case: that the failure of Lie groups to be type I can always be traced to the representation theory of central extensions of abelian groups, the subject of Chapter II. From this analysis, Pukánszky arrives at the main results of the book, which are in Chapter III. The basic result can be summarized by saying that connected Lie groups have a good representation theory, provided one is willing to view the basic objects of this theory as being quasi-equivalence classes of *normal* representations rather than unitary equivalence classes of irreducible representations. A normal representation is a factor representation of type I or II, for which the image of the group C^*-algebra has non-trivial intersection with the trace-class operators in the sense of von Neumann algebras. The trace on such a representation makes possible a character formula of Kirillov type, which Pukánszky proved but does not discuss in detail in this book. However, he does show in Chapter IV how, in the solvable case, to parameterize the normal representations via generalized coadjoint orbits.

Unfortunately, Pukánszky did not live to finish and fully polish this book, so certain important parts of his theory are missing. And the book does everything from Pukánszky's own, and rather idiosyncratic, point of view. Thus the author does not mention the general theory of multiplier representations of abelian groups, due to Baggett and Kleppner [1], from which the results of Chapter II easily follow, nor does he mention the alternative proof by Green [4] of the main results of Chapter III. The important results of Charbonnel [2] and Poguntke [6], which amplify many of Pukánszky's results, are only mentioned in passing.

Nevertheless, the book is a useful reference for Pukánszky's theory, and is somewhat more convenient than reading [8], [9], and [10], since some of the duplication between papers has been eliminated. The reader should be warned of two things, however. First, the style of this book is totally different from that of Pukánszky's earlier book [7] on nilpotent groups. Whereas that book was intended for beginners, this book is intended only for those who already know quite a bit about Lie group representations. Unlike Kirillov's text [5], which is rather informal and tries to avoid technicalities, this book almost relishes in them. Secondly, Pukánszky's notation takes some getting used to. The notation $\overset{\wedge}{\mathcal{G}}_g$ of [9] has here been typeset as $\overset{A}{\mathcal{G}}_g$. And the notation $G = (I)$ is supposed to mean not that G is equal to anything, but that it is a type I group.

References

[1] L. Baggett and A. Kleppner, 'Multiplier representations of abelian groups', *J. Funct. Anal.* 14 (1973) 299–324.

[2] J.-Y. Charbonnel, 'Sur les caractères des groupes de Lie', *J. Funct. Anal.* 72 (1987) 94–150.

[3] J. Dixmier, M. Duflo, A. Hajnal, R. Kadison, A. Korányi, J. Rosenberg, and M. Vergne, 'Lajos Pukánszky (1928–1996)', *Notices*

Amer. Math. Soc. 45 (1998) 492–499.

[4] P. Green, 'The local structure of twisted covariance algebras', *Acta Math.* 140 (1978) 191–250.

[5] A. A. Kirillov, 'Elements of the theory of representations' transl. by Edwin Hewitt, Grundlehren der Math. Wissenschaften 220 (Springer-Verlag, Berlin-New York, 1976).

[6] D. Poguntke, 'Simple quotients of group C^*-algebras for two-step nilpotent groups and connected Lie groups', *Ann. Sci. École Norm. Sup.* (4) 16 (1983) 151–172.

[7] L. Pukánszky, *Leçons sur les représentations des groupes*, Monographies de la Soc. Math. de France, No. 2 (Dunod, Paris 1967).

[8] L. Pukánszky, 'Action of algebraic groups of automorphisms on the dual of a class of type I groups', *Ann. Sci. École Norm. Sup.* (4) 5 (1972), 379–395.

[9] L. Pukánszky, 'Unitary representations of solvable Lie groups', *Ann. Sci. École Norm. Sup.* (4) 4 (1971), 457–608.

[10] L. Pukánszky, ‘Characters of connected Lie groups’, *Acta Math.* 133 (1974) 81–137.

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