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J ROSENBERG

C*-algebras, positive scalar curvature and the Novikov conjecture, II

Contents

0. Introduction and acknowledgements
1. Positive scalar curvature and the index theory of the Dirac operator
2. Towards a conjecture of Gromov and Lawson
3. Behavior under finite coverings and the transfer map in spin bordism

0. Introduction and acknowledgements

In this lecture I hope to turn the title of this Seminar around and discuss "operator-algebraic methods in geometry" rather than "geometric methods in operator algebras." The intention is to provide an introduction to some of the literature on topological obstructions to positive scalar curvature (including [25], [32], [12], [13], and [14]), with emphasis on the index-theoretic method of [30]. The first section of this paper will thus be expository, and biased toward topics likely to be of interest to those interested in applications of C*-algebras in differential geometry. While I was preparing this survey, I decided to attempt a deeper analysis of a conjecture of Gromov and Lawson that if true would provide a nice framework for the whole subject; this accounts for Sections 2 and 3 of this article. I suspect that much of the content of Section 2 may be known to the experts, but I have not seen any of this material written down, and the treatment here is my own. In any event, the results of this part are needed for Section 3, which I believe to be new. In fact, by way of advertisement for "non-commutative differential geometry," I might add that I can think of no technique that would yield the example of Theorem 3.1 without methods similar to those of [30].

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I would like to thank Professors Edward Effros and Huzihiro Araki for the opportunity to speak at this Seminar, and for their excellent organizational work. In addition, I would like to thank Professors Blaine Lawson and Mikhaël Gromov for teaching me about this subject, and the members of the University of Pennsylvania geometry seminar for suggesting the problem treated in Section 3. (In a discussion at Penn in November, 1982, I naively suggested that if a manifold were to have a metric of positive scalar curvature, one ought to be able to "average" this metric under a freely acting finite group of diffeomorphisms, and so get a similar metric on any manifold (regularly) finitely covered by this original one. When Chris Croke, Herman Gluck, Jerry Kazdan and Wolfgang Ziller countered in unison that this certainly wasn't obvious, I slowly began to suspect that some topological obstruction might be involved.) I also thank G. G. Kasparov for sending me preprints of his recent work, which plays a vital role in the methods of [30] and thus in some of the results discussed here. Finally, I would like to thank Drs. Iain Raeburn and Colin Sutherland for their hospitality at the University of New South Wales in July-August, 1983, during which time some of this work was completed.

Added in proof: After this paper was completed, it was pointed out to me that T. Miyazaki [On the existence of positive scalar curvature metrics on non-simply-connected manifolds, J. Fac. Sci. Univ. Tokyo, Sect. IA 30 (1984), 549-561], had done some work along the lines of Section 2 below (duplicating, for instance, our Theorems 2.2, 2.14, and 2.15), and that L. Bérard Bergery had found an example similar to that of Section 3 (with covering group $\mathbb{Z}_2$), based on the $\mathbb{Z}_2$-valued obstruction of Hitchin [15].

Meanwhile, I have been able to improve the results of [13] on the simply-connected spin case, and to rework the method of [30] using real KK-theory so as to prove one direction of Conjecture 2.1 below for a large class of fundamental groups. These results will appear in a separate publication.

Finally, the dimension restriction in Proposition 2.3 and in Theorem 2.5 may be simplified to "$n \geq 5$," using the following argument suggested by Shmuel Weinberger. If $n \geq 4$ and π is any finitely presented group, one may construct a spin manifold M^n with $\pi_1(M) = \pi$ by starting with a connected sum of copies of $S^1 \times S^{n-1}$ and by doing surgeries (preserving the spin structure) to build the correct relations into the fundamental group. Since $S^1 \times S^{n-1}$ has positive scalar curvature and all the surgeries needed are in codimensions n and $n - 1$, M has positive scalar curvature by [13] or [32]. This M may be substituted for the $V^k \times S^{n-k}$ in 2.3.

1. Positive scalar curvature and the index theory of the Dirac operator

Throughout this article, we shall be interested in the following geometric problem. Given a manifold M^n, which we shall always take to be orientable, smooth, compact, connected, and without boundary, when can one choose a Riemannian metric g on M such that the associated scalar curvature function κ on M is everywhere positive? More generally, what functions κ can arise from metrics on M?

When the dimension n of M is 2, information about this question is immediately provided by the Gauss-Bonnet Theorem, since in this case $\kappa/2$ coincides with the Gaussian curvature. Since $\int \kappa \, d\,\mathrm{vol} = 4\pi\chi$, where χ is the Euler characteristic, we see that $\kappa > 0$ is impossible unless M is a sphere, and that $\kappa < 0$ is impossible unless M has genus > 1. Also note that every orientable closed surface except for the sphere is aspherical, i.e., has contractible universal covering. A quick summary of what Gauss-Bonnet says about positive scalar curvature is thus: no closed aspherical 2-manifold admits a metric with $\kappa > 0$.

In dimension $n \geq 3$, Aubin ([4], §3) pointed out that the situation is fundamentally different: any closed manifold with $n \geq 3$ admits a metric with $\kappa < 0$. In fact, Kazdan and Warner ([21], Theorem 1.1) showed that given M^n with $n \geq 3$ and any smooth real-valued function on M that is negative somewhere, one can realize this function as the scalar curvature κ for some Riemannian metric on M. In particular, there can be no result like the Gauss-Bonnet Theorem relating the integral of κ to the topology of M. Nevertheless, we know that $\kappa > 0$ is not always possible (a fact which for $n = 3$ seems to be relevant to relativistic cosmology). Indeed, there seem to be topological obstructions to positive scalar curvature of two very different sorts: some that apply even in the simply connected case, and others that depend on the size of the fundamental group. We shall discuss several of these obstructions and try to relate them to the common framework of the index of the Dirac operator.

The oldest, and in a sense the most basic, result saying that certain manifolds of dimension > 2 do not admit a metric with $\kappa > 0$ is due to Lichnerowicz [25]. He showed that if M^n (as always closed,

connected, and orientable) satisfies $w_2(M) = 0$ (where $w_2(M) \in H^2(M, \mathbb{Z}_2)$ denotes the second Stiefel-Whitney class) and $n \equiv 0 \pmod 4$, and if M admits a metric of positive scalar curvature, then one must have $\hat{A}(M) = 0$. Here

$$\hat{A}(M) = \langle \hat{\mathcal{A}}(M), [M] \rangle,$$

and

$$\hat{\mathcal{A}}(M) = 1 - \frac{p_1}{24} - \frac{1}{32.45}\left(p_2 - \frac{7}{4} p_1^2\right) - \dots$$

is the "total $\hat{A}$-class", a certain polynomial in the rational Pontrjagin classes $p_j \in H^{4j}(M, \mathbb{Q})$ of the stable tangent bundle of M, and $[M] \in H_n(M, \mathbb{Z})$ is the fundamental homology class defined by the choice of an orientation. This may sound complicated but amounts to a definite restriction. For instance, a "K3 surface" K^4 (a smooth algebraic hypersurface in $\mathbb{CP}^3$ defined by an equation of degree 4) cannot admit positive scalar curvature since $w_2(K) = 0$ and $\hat{A}(K) = 2$. It is also worth pointing out that the invariants w_2 and $\hat{A}$ which come into the theorem depend only on the homeomorphism type of M, not on the differentiable structure. This is because the Stiefel-Whitney classes depend only on the homotopy type of M (by the Wu formulae - see [33], p. 100) and (by a deep theorem of Novikov [29]) the p_j's are topological invariants. (In the case of a 4-manifolds, p_1 (and so $\hat{A}$) is a homotopy invariant, but this fails to be true for manifolds of dimension ≥ 6.) Also one may replace the condition $\kappa > 0$ in the theorem by $\kappa \geq 0$, $\kappa \not\equiv 0$, since as pointed out in [21], Proposition 3.8, a metric with $\kappa \geq 0$ and $\kappa \not\equiv 0$ can be modified to achieve $\kappa > 0$ everywhere.

Lichnerowicz's theorem was later extended by Hitchin in perhaps a surprising way. Hitchin ([15], §4.2) found additional obstructions to positive scalar curvature for closed manifolds M^n with $w_2 = 0$, this time for $n \equiv 1$ or $2 \pmod 8$. Unlike the obstruction of Lichnerowicz, these do not just depend on the homeomorphism class of M, since they are non-zero for some exotic spheres of dimension 9 but zero for others. However, one can formulate the Hitchin and Lichnerowicz theorems in a unified way as follows: suppose M^n is an oriented closed manifold with $w_2(M) = 0$. Then M admits a spin structure (see [26]), i.e., a lifting of the oriented frame bundle, which is a principal bundle for the structure group $GL(n,\mathbb{R})^+$, to a principal bundle for the double cover of GL^+. Choose such a spin structure, which is unique if M is simply connected (or if $H^1(M,\mathbb{Z}_2) = 0$). By definition, M together with its spin structure s determines a class in the spin bordism group Ω_n^{spin}, and then a class $[M,s]$ in the real K-homology group $KO_n(pt)$. This K-homology class is either the image of the spin bordism class of (M,s) under the natural transformation $\hat{\alpha} : \Omega_*^{spin} \to KO_*$ of [27], §3, or else is to be thought of in terms of the geometric generator/relation presentation of KO-homology discussed in [5], p. 168, where we take for the real vector bundle E over M just the trivial one-dimensional bundle. The class $[M,s]$ may be viewed as a "generalized index" of the Dirac operator on M, as defined using the spin structure s. In fact, $KO_n(pt) \cong \mathbb{Z}$ for $n \equiv 0 \pmod 4$, $\cong \mathbb{Z}_2$ for $n \equiv 1$ or $2 \pmod 8$, and $= 0$ for $n \equiv 3,5,6$, or $7 \pmod 8$. When $n \equiv 0 \pmod 4$, $[M,s]$ (if $n \equiv 0 \pmod 8$) or $2[M,s]$ (if $n \equiv 4 \pmod 8$) may be identified with $\hat{A}(M)$, which was computed in [3] to be the index of the Dirac operator on M taking "positive" to "negative" "half-spinors". When $n \equiv 1$ or $2 \pmod 8$, $[M,s]$ may again be viewed as a mod 2 index of the (real) Dirac operator. Then the Hitchin and Lichnerowicz theorems may be formulated as

Theorem 1.1: Let M^n be a closed, connected, oriented manifold with $w_2(M) = 0$ and with a Riemannian metric for which $\kappa > 0$. Then for any choice of a spin structure s on M, $[M,s] = 0$ in $KO_n(pt)$.

The proofs in all cases involve Lichnerowicz's observation that if D is the Dirac operator on M defined by s and the Riemannian metric (an elliptic first-order self-adjoint differential operator), $D^2 = \nabla^*\nabla + \frac{\kappa}{4}$, where $\nabla^*\nabla$ is positive and self-adjoint. Thus if $\kappa > 0$, D^2 is strictly positive, and so all index invariants associated to D will vanish.

When M is simply connected, s is unique and we may write simply $[M] \in KO_n$. In this case, Gromov and Lawson conjecture in [13] that the condition $[M] = 0$ is also sufficient for M to admit positive scalar curvature. The status of this conjecture will be discussed in §2 below.

Now consider the case of non-simply connected manifolds. It seems that the bigger the fundamental group of M, the harder it is to achieve a metric of positive scalar curvature on the manifold. In fact, Gromov and Lawson have suggested the following, which meshes well with the corollary of Gauss-Bonnet in dimension 2:

Conjecture 1.2: No closed aspherical manifold (of any dimension) admits a metric of positive scalar curvature.

Our evidence for this is spotty and has accumulated piecemeal. This conjecture was proved first by Schoen and Yau for the 3-torus [31], then by the same authors for the n-torus and certain other manifolds with $n \leq 7$ [32], then by Gromov and Lawson for a large class of aspherical closed manifolds, including compact manifolds of non-positive sectional curvature (such as locally symmetric spaces of non-compact type) and compact solvmanifolds (which of course include tori of all dimensions) - [12] and [14]. A feature of the Gromov-Lawson approach is that it gives homotopy-type obstructions: a manifold homotopy-equivalent to a solvmanifold can't admit positive scalar curvature, whether or not it's diffeomorphic (or even homeomorphic) to a solvmanifold.

It is at this point that one notices a relationship with the so-called "Novikov Conjecture" in differential topology. This conjecture, or rather class of conjectures, exists in various forms (see in particular [11] and [17]), all of which say roughly that the bigger the fundamental group of a closed manifold, the more the homotopy type determines the structure of the manifold. In the case of closed aspherical manifolds, it is possible that the fundamental group actually determines the manifold up to homeomorphism or at least stable homeomorphism (see, e.g., [10] and [11]). Here we say M_1 and M_2 are <u>stably homeomorphic</u> if $M_1 \times \mathbb{R}^k$ and $M_2 \times \mathbb{R}^k$ are homeomorphic for a suitable (usually small) value of k. Stable homeomorphism is sometimes easier to work with than homeomorphism. For instance, although for $n \geq 5$ not all contractible n-manifolds are homeomorphic to $\mathbb{R}^n$, they are all <u>stably</u> homeomorphic to $\mathbb{R}^n$.

To formulate things more precisely, we need one additional ingredient. Let M be a closed, connected, oriented n-manifold with fundamental group π. Then the universal cover $\tilde{M}$ of M is a principal π-bundle over M, hence is determined by a classifying map $f : M \to B\pi$ which is an isomorphism on fundamental groups. Here $B\pi$ is an Eilenberg-MacLane space with $\pi_1(B\pi) \cong \pi$ and $\pi_j(B\pi) = 0$ for $j > 0$, and $B\pi$ and f are well-defined up to homotopy. (Of course, the homotopy class of f depends on the choice of a specific isomorphism $\pi_1(M) \to \pi$.) The usual formulation of the Novikov Conjecture is the statement that the "higher signatures"

$$L_a(M) = \langle \mathbb{L}(M) \cup f^*(a), [M] \rangle,$$

where $a \in H^*(B\pi,\mathbb{Q}) \cong H^*(\pi,\mathbb{Q})$ (group cohomology) and

$$\mathbb{L}(M) = 1 + \frac{p_1}{3} + \frac{1}{45}(7p_2 - p_1^2) + \cdots$$

is the "total Hirzebruch L-class", are oriented homotopy invariants (of manifolds with fundamental group π).

Gromov and Lawson define analogously the "higher $\hat{A}$-genera"

$$\hat{A}_a(M) = \langle \hat{A}(M) \cup f^*(a), [M] \rangle, \ a \in H^*(B\pi,\mathbb{Q}),$$

and conjecture that for closed oriented manifolds M with $w_2(M) = 0$, M cannot admit a metric of positive scalar curvature unless all of these vanish. They also suggest that the condition on $w_2(M)$ can be weakened to $w_2(\tilde{M}) = 0$, which of course is automatic if M is aspherical. This would then imply Conjecture 1.2 about closed aspherical manifolds, since for oriented such M^n, one can always find $a \in H^n(B\pi,\mathbb{Q})$ with $\hat{A}_a(M) = 1$. Many cases of this conjecture are proved in [12] and [14].

However, I would like to briefly describe a method introduced in [30] for obtaining similar results using the C^*-algebraic index theory [28] of Miščenko and Fomenko. Assume M^n has been given a spin structure s. Then in place of the Dirac operator D of (M,s), which was used in the Lichnerowicz argument, we may use D_E, the Dirac operator with coefficients in a bundle E. The identity $D^2 = \nabla^*\nabla + \frac{\kappa}{4}$ must generally be adjusted by the curvature Ω_E of E, but will continue to hold without modification if E has a flat connection. As is well known, this is possible precisely when E is the pull-back of a flat bundle on $B\pi$, via the map f. So a slight modification of [25] yields the fact that if M admits a metric of positive scalar curvature, then $\hat{A}_a(M) = 0$ for every $a \in H^*(B\pi,\mathbb{Q})$ which is a characteristic class of a flat vector bundle over $B\pi$. This by itself is little help, since the rational characteristic classes of ordinary flat vector bundles mut be trivial by Chern-Weil theory (which relates these characteristic classes to the curvature). However, examination of the proof shows that the argument still works when E is a flat A-vector bundle in the sense of [28], where A is a C^*-algebra with unit, provided that we use an appropriate index theory for D_E. The index of D_E (when M is even-dimensional) will take its values in $K_0(A)$ rather than in $K_0(\mathbb{C}) \cong \mathbb{Z}$. In applications, we always take $A = C^*(\pi)$ or $C_r^*(\pi)$ and E = the "universal" flat A-bundle over M, $\tilde{M} \times_{\pi} A$. This A-bundle is of course pulled back from the flat A-bundle $\mathcal{V} = E\pi \times_{\pi} A$ over $B\pi$ (where $E\pi$ is the universal cover of $B\pi$, a contractible space on which π acts freely). The advantage of this is precisely that Chern-Weil theory breaks down for A-vector bundles, so that the flat bundle E may have non-zero rational characteristic classes. (This is due, roughly speaking, to the fact that the structure group for an A-vector bundle is infinite-dimensional and non-compact.)

In fact, Kasparov has used the A-bundle $\mathcal{V}$ to define a homomorphism

$$\beta : RK_*(B\pi) \to K_*(A).$$

Here RK_* denotes complex K-homology as extended to infinite CW-complexes. If $B\pi$ can be chosen compact, this map is easy to define. The bundle $\mathcal{V}$ has a class $[\mathcal{V}]$ in

$$K^0(B\pi; A) \cong K_0(C(B\pi) \otimes A) = KK(\mathbb{C}, C(B\pi) \otimes A),$$

and β is the Kasparov product $[\mathcal{V}] \underset{C(B\pi)}{\otimes} \cdot$ as defined in [18] (see also [7], [9]). When $B\pi$ is an infinite complex, one defines β first this way on finite skeletons of $B\pi$, then passes to the limit.

With $A = C^*(\pi)$ or $C_r^*(\pi)$ and $E = f^*(\mathcal{V})$, the index of D_E (for M even-dimensional) in $K_0(A)$ is just $\beta([M,s,f]_c)$, where $[M,s,f]_c$ denotes the image of the bordism class of $(M,s) \xrightarrow{f} B\pi$ in the complex (hence the "c") K-homology group $RK_0(B\pi)$, as defined in [5]. The case of odd-dimensional M is reduced to the even-dimensional case by taking a product with S^1; the net effect is to have $[M,s,f]_c \in RK_1(B\pi)$ and to have an index of D_E defined in $K_1(A)$. The conclusion of this analysis is the following result (a combination of Theorem 2.11 and Theorem 3.3) of [30]:

Theorem 1.3: Let (M^n,s) be a closed, connected spin manifold admitting a metric of positive scalar curvature, and let $f : M \to B\pi$ be the classifying map for its universal covering. If the Kasparov map $\beta : RK_*(B\pi) \to K_*(C^*(\pi))$ is injective, then $[M,s,f]_c = 0$ in $RK_n(B\pi)$. If β is injective modulo torsion, then the higher $\hat{A}$-genera $\hat{A}_a(M)$ vanish for all $a \in H^*(B\pi, \mathbb{Q})$.

Kasparov has shown [19] that injectivity of β modulo torsion implies the Novikov Conjecture, and holds if π can be embedded discretely in a connected Lie group. In fact, Mišč̌enko had claimed this earlier if π is the fundamental group of a complete Riemannian manifold of non-positive sectional curvatures. The proof seemed to contain a gap when this manifold was non-compact (see [16]), but evidently this gap can be filled by using Kasparov's machinery (see sketches of the argument in [9] and [20]), so that one also has:

Theorem 1.4: If π is the fundamental group of a complete Riemannian manifold of non-positive sectional curvatures, then $\beta : RK_*(B\pi) \to K_*(C_r^*(\pi))$ is injective.

Some additional cases may be handled by [30], Theorem 2.6:

Theorem 1.5: If π is a countable solvable group having a composition series in which the composition factors are torsion-free abelian (but not necessarily finitely generated), then $\beta : RK_*(B\pi) \to K_*(C^*(\pi))$ is an isomorphism.

It is clear from the example of finite groups that β cannot be injective for arbitrary groups (with torsion). However, if a group π contains a subgroup π_1 of finite index for which β is injective, then β for π is at least an injection modulo torsion ([30], Proposition 2.7), which suffices for vanishing of the higher $\hat{A}$-genera. It is conceivable (although perhaps overly optimistic) that β is injective whenever π is finitely presented and torsion-free, and always injective modulo torsion when π is finitely presented. Whether or not this is true, it is rather striking that the classes of groups for which good results about the Novikov Conjecture or the positive scalar curvature problem have been obtained largely coincide with the classes of groups for which one can prove injectivity of β modulo torsion (which we called in [30] the "Strong Novikov Conjecture").

2. Towards a conjecture of Gromov and Lawson

Mikhael Gromov and Blaine Lawson have proposed a neat way of organizing all the results on topological obstructions to positive scalar curvature on closed manifolds, at least if we restrict attention to spin manifolds. Here is their conjecture:

Conjecture 2.1. Let M^n be a closed, connected manifold with $w_2(M) = 0$, and let $f : M \to B\pi$ be the classifying map for the universal covering of M. Then (if π is suitable and at least for n sufficiently large) M admits a metric of positive scalar curvature if and only if the following topological condition holds: for any spin structure s on M, one has $[M,s,f] = 0$ in $KO_n(B\pi)$, where $[M,s,f]$ is the image in real K-homology of the spin bordism class of $(M,s) \xrightarrow{f} B\pi$.

When M is simply connected, $B\pi$ is a single point and s is unique. The conjecture thus reduces in this case to the conjecture of [13] that a simply connected spin manifold admits positive scalar curvature exactly when the "Hitchin obstruction" $[M] \in KO_n(pt)$ vanishes. In this case, substantial results were obtained in [13], Corollary B, under the restriction $n \geq 5$. Of course, the conjecture is also clearly true for simply connected closed 1-manifolds (because there aren't any) and 2-manifolds. It would be true for simply connected 3-manifolds if one knew the Poincaré Conjecture. As for dimension 4, any simply connected spin 4-manifold with vanishing Lichnerowicz obstruction must have signature 0 and so by the Freedman classification theorem ([34], Theorem 1.5) be homeomorphic to S^4 or to a connected sum of $S^2 \times S^2$'s. If the homeomorphism were a diffeomorphism, the manifold would admit positive scalar curvature by [13], Theorem A, so the conjecture would again be correct. However, to the best of my knowledge, it is not yet known if the differentiable structure on S^4 (or a connected sum of $S^2 \times S^2$'s) is unique.

The non-simply connected case is of course much harder. As in the simply connected case, the available methods of proof for attacking the conjecture again impose mild restrictions on n, which might turn out to be unnecessary. However, we will see that the conjecture cannot possibly hold for all groups π, so that any "suitable" π should at least be torsion-free. An extra annoyance of the non-simply connected case is that M usually admits more than one spin structure, and $[M,s,f]$ may depend on s. For instance, if $M = B\pi = S^1$, $KO_1(S^1) \cong \widetilde{KO}_1(S^1) \oplus KO_1(pt) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. In this case, M has two spin structures, each with the same image in $\widetilde{KO}_1(S^1)$, but with distinct images in $KO_1(pt) \cong \mathbb{Z}_2$. However, if $[M,s,f]$ vanishes for <u>one</u> choice of the spin structure s, then it vanishes for all s. To see this, recall from [26] that $H^1(M,\mathbb{Z}_2)$, which may be identified with the group of real line bundles over M (the group operation being the tensor product of line bundles), operates transitively on the set of spin structures on M. But $H^1(M,\mathbb{Z}_2) \cong H^1(B\pi,\mathbb{Z}_2)$ sits inside the group of invertible elements in the ring $RKO^0(B\pi)$, which acts naturally on $RKO_*(B\pi)$. One easily sees that if s and s' are two different spin structures on M, then $[M,s,f]$ and $[M,s',f]$ must lie in the same orbit for this group of invertible RKO^0-elements; hence one vanishes if and only if the other does as well.

Before going on to a more detailed discussion of Conjecture 2.1, it is perhaps worthwhile to say a word about manifolds with $w_2(M) \neq 0$. In [13], Corollary C, it was shown that such a manifold M^n always admits a metric of positive scalar curvature if M is simply connected and $n \geq 5$. (Here $n \leq 3$ is impossible anyway, since every homotopy 3-sphere is certainly parallelizable, and the case $n = 4$ comes down to the active problem of classifying up to diffeomorphism the smooth, simply connected 4-manifolds having odd intersection form. By work of Freedman ([34], Theorem 1.5) and Donaldson [8], all such manifolds are homeomorphic to connected sums of $\mathbb{CP}^2$'s and $-\mathbb{CP}^2$'s. If the homeomorphism could always be made a diffeomorphism, there would always be positive-scalar-curvature metrics by [13], Theorem A.) Note that by [14], p. 186, it is not true that M always admits a metric of positive scalar curvature if $w_2(\tilde{M}) \neq 0$, where $\tilde{M}$ is the universal cover of M. For the case $w_2(\tilde{M}) = 0$, some partial results were obtained in [30], §3B.

Let us return now to Conjecture 2.1. As far as the simply connected case is concerned, one direction is given by [15], so to prove the conjecture for $n \geq 5$, it would suffice by [13], Theorem B, to show that if M^n

is simply connected and spin, and if $[M] = 0$ in $KO_n(pt)$, then M is spin cobordant to a known spin manifold of positive scalar curvature. Actually we can improve on [13], Corollary B, in low dimensions. As pointed out already in [13], $\Omega_n^{spin} = 0$ for $n = 5,6$ or 7, so the conjecture is certainly valid in these dimensions. Next, Ω_8^{spin} is free abelian of rank 2, with generators $\mathbb{HP}^2$ (which has signature 1 and $\hat{A}$-genus 0) and the Milnor-Kervaire almost-parallelizable manifold M_0^8 with $\hat{A}$-genus 1 [26]. Thus if $[M^8] = 0$ in $KO_8(pt)$, M is spin cobordant to S^8 or to a connected sum of $\mathbb{HP}^2$'s or $-\mathbb{HP}^2$'s, and so admits positive scalar curvature by [13], Theorem A. The last case which is easy to treat is dimension 9, since $\Omega_9^{spin} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, with generators $\mathbb{HP}^2 \times \alpha$ and $M_0^8 \times \alpha$, where α is S^1 with the unusual spin structure (see [33], p. 339). Note that $[M_0^8 \times \alpha]$ generates $KO_9(pt) \cong \mathbb{Z}_2$, and $[\mathbb{HP}^2 \times \alpha] = 0$. Thus if $[M^9] = 0$ in $KO_9(pt)$, M is spin cobordant to S^9 or to $\mathbb{HP}^2 \times \alpha$, which as a manifold is $\mathbb{HP}^2 \times S^1$ and obviously admits positive scalar curvature. Hence M admits positive scalar curvature by [13], Theorem B.

Aside from case-by-case checking of generators of Ω_*^{spin}, which seems impractical in large dimensions, we can think of one other strategy for trying to prove the conjecture in the simply connected case. The idea would be that although generators for the kernel of the map $\Omega_*^{spin} \to KO_*(pt)$ are quite complicated, there seems to be a better description of KO_* as a quotient of bordism classes of pairs (M,E), where M is a spin manifold and E a real vector bundle over M, by an operation of "vector bundle modification" (see [5], p. 168). Thus one might try to prove the conjecture by first verifying that "vector bundle modification" indeed generates the kernel of the map from cobordism classes of pairs

(M^n,E) to $KO_n(pt)$, and then trying to generalize the whole problem to such pairs. The idea would be to determine when one could choose a Riemannian metric on M, together with an orthogonal structure on E, such that $\frac{\kappa}{4} + \mathfrak{R}_E > 0$ (in the notation of [14]). This condition would again imply vanishing of the index of D_E, so that one might try to show that $[M^n,E] = 0$ in $KO_n(pt)$ would be a necessary and sufficient condition. For this it would suffice to show bordism invariance and invariance under vector bundle modification (although this might not be easy). Then the positive scalar curvature problem would reduce to the special case of E the one-dimensional trivial bundle.

We return now to the non-simply connected spin case. Before giving some results on specific groups π, it is necessary to mimic [13] and show that, at least under mild restrictions, the question of whether or not M admits positive scalar curvature only depends on the spin bordism class of $(M,s,f : M \to B\pi)$. For this part of the analysis it is not necessary to assume anything about the group π.

Theorem 2.2. Let X_1^n, X_2^n be closed, oriented, connected n-manifolds with $\pi_1(X_1) \cong \pi_1(X_2) \cong \pi$, with $n \geq 5$, and with $w_2(X_i) = 0$ $(i = 1,2)$. Let s_i be a spin structure on X_i, and let $f_i : X_i \to B\pi$ be the classifying map for the universal covering of X_i. Finally, assume X_2 has a metric of positive scalar curvature and that $(X_i,s_i) \xrightarrow{f_i} B\pi$ $(i = 1,2)$ lie in the same bordism class in $\Omega_n^{spin}(B\pi)$. Then X_1 also admits a metric of positive scalar curvature.

Proof. We proceed as in [13], proof of Theorem B, except that the results of Smale used there (preliminaries to the proof of the h-cobordism Theorem) must be replaced by the corresponding steps in the proof of the s-cobordism Theorem of Barden, Mazur, and Stallings.

Let W^{n+1} be a compact spin manifold with $\partial W = X_1 - X_2$, where X_1 and X_2 are as in the theorem, and where there is a map $W \to B\pi$ restricting on the two boundary components to the maps $f_i : X_i \to B\pi$. Then we have a commutative diagram

$$\begin{array}{ccccc} \pi_1(X_1) & \xrightarrow{i_*} & \pi_1(W) & \xleftarrow{i_*} & \pi_1(X_2) \\ & \searrow^{\cong} & \downarrow^{\cong} & \swarrow^{\cong} & \\ & & \pi & & \end{array}$$

and thus a split exact sequence

$$1 \longrightarrow \Pi \longrightarrow \pi_1(W) \leftrightarrows \pi \longrightarrow 1$$

for some group Π. The group Π is generated by embedded copies of S^1 in W which don't meet the boundary, and we may remove these by surgeries preserving the spin structure. Hence, without loss of generality, we may assume (W,X_1) and (W,X_2) are 1-connected. Then we have an exact homotopy sequence

$$\pi_2(X_1) \xrightarrow{i_*} \pi_2(W) \longrightarrow \pi_2(W,X_1) \longrightarrow 0.$$

Since $\dim W \geq 6$ (in fact ≥ 5 would suffice here), a set of elements of $\pi_2(W)$ which generate $\pi_2(W,X_1) = \pi_2(W)/i_*\pi_2(X_1)$ can be represented by smoothly embedded 2-spheres which do not meet the boundary. Since W is a spin manifold, these 2-spheres have stably trivial, hence trivial, normal bundles, and can be removed by surgeries preserving the spin structure and fundamental group. Thus, again without loss of generality, we may assume (W,X_1) is 2-connected. Choose a handle decomposition of the cobordism W, and proceed to eliminate the 0-, 1-, and 2-handles by [23], Lemma 1. (This doesn't require that W be an h-cobordism, only that $\pi_1(X_1) \cong \pi_1(W) \cong \pi_1(X_2)$ and that $\pi_2(W,X_1) = 0$. This step in the proof requires $\dim W \geq 6$.) Turning the handle decomposition upside down, we observe that X_1 is obtained from X_2 by attaching handles of index $\leq n-2$, i.e., by performing surgeries on embedded spheres of codimension ≥ 3. Then by [13], Theorem A, or by [32], Corollary 4 to Theorem 4, X_1 admits a metric of positive scalar curvature.

Proposition 2.3: Let (X^n, s) be a closed, connected spin manifold with $n \geq 5$, and let $f: X \to B\pi$ be the classifying map for its universal covering, where $\pi = \pi_1(X)$. If $(X,s) \xrightarrow{f} B\pi$ represents the trivial element of $\Omega_n^{spin}(B\pi)$, and if there exists a closed, connected spin manifold V^k with $k \leq n-2$ (this is automatic if $n \geq 6$), then X admits a metric of positive scalar curvature.

Proof: As is well known, for any finitely presented group π there is a closed stably parallelizable manifold V^4 with $\pi_1(V^4) \cong \pi$ ([22], p. 109), so the existence of the requisite V is automatic if $n \geq 6$. Anyway, given any V^k, there is always a metric of positive scalar curvature on $V^k \times S^{n-k}$, provided $n - k \geq 2$, namely, the Riemannian product of any metric on V with the constant-curvature metric on a Euclidean sphere of very small radius. Furthermore, if V is a spin manifold, then $V^k \times S^{n-k}$ is clearly the boundary of the spin manifold $V^k \times D^{n-k+1}$ with the same fundamental group. Hence we may apply Theorem 2.2 with $X_1 = X$, $X_2 = V^k \times S^{n-k}$.

Remark 2.4: Although X may admit more than one spin structure s, the condition that $(X,s) \xrightarrow{f} B\pi$ represent the trivial element of $\Omega_n^{spin}(B\pi)$ is independent of s. Indeed, suppose $(W,\tilde{s})$ is a spin manifold with $\partial(W,\tilde{s}) = (X,s)$, and suppose $g: W \to B\pi$ extends f. If s' is any other spin structure on X associated to the same orientation of X as determined by s, then s' is obtained by modifying s by an element

$a \in H^1(X,\mathbb{Z}_2) = \mathrm{Hom}(\pi, \mathbb{Z}_2)$ [26]. But then if $b = a \circ g_* \in H^1(W,\mathbb{Z}_2)$, the action of b on $\tilde{s}$ defines a spin structure $\tilde{s}'$ on W restricting to s' on X. Hence $(X,s') \xrightarrow{f} B\pi$ also represents the trivial element of $\Omega_n^{spin}(B\pi)$. Similarly, to reverse the orientation on X, we may reverse the orientation on W.

Theorem 2.5: Let π be any finitely presented group such that there exists a closed, connected spin k-manifold V^k with $\pi_1(V) \cong \pi$, and let $n \geq \max(5, k+2)$. (Recall that $n \geq 6$ will always do.) Then there exists a subgroup $P_n(\pi)$ of $\Omega_n^{spin}(B\pi)$ with the following property: for any closed, connected n-manifold M^n with $w_1(M) = w_2(M) = 0$ and $\pi_1(M) \cong \pi$, then M admits a metric of positive scalar curvature if and only if the spin bordism class of $(M,s) \xrightarrow{f} B\pi$ lies in $P_n(\pi)$. (Here $f: M \to B\pi$ is the classifying map for the universal cover of M and s is any spin structure on M.)

Proof: Let $P_n(\pi)$ be the set of classes in $\Omega_n^{spin}(B\pi)$ of triples (M,s,f), where (M,s) is a closed, connected spin n-manifold admitting a metric of positive scalar curvature, and where $f: M \to B\pi$ is a classifying map for the universal covering. If we can show that $P_n(\pi)$ is a group, then the conclusion will follow from Theorem 2.2.

To begin with, $P_n(\pi)$ contains the 0-element of $\Omega_n^{spin}(B\pi)$ by Proposition 2.3. Furthermore, it is clear that $P_n(\pi)$ is closed under inversion (reversal of spin structure). So we must show that $P_n(\pi)$ is closed under addition. This is non-trivial becuase of our restriction to connected manifolds M; the addition operation in Ω_n^{spin} comes from the disjoint sum of manifolds, not the connected sum.

Thus suppose $(M_i, s_i) \xrightarrow{f_i} B\pi$ represent classes in $P_n(\pi)$, where M_i is a connected spin n-manifold with fundamental group π and positive scalar curvature (i = 1,2). Choose generators $\alpha_1,\dots,\alpha_k$ for π, and represent these by disjoint embedded oriented circles $C_1^i,\dots,C_k^i$ in M_i. Since M_i is oriented, C_j^i has trivial normal bundle, and hence a tubular neighborhood N_j^i diffeomorphic to $S^1 \times D^{n-1}$. We form a new manifold M from $(M_1 \setminus \bigcup_j N_j^1) \cup (M_2 \setminus \bigcup_j N_j^2)$ by gluing ∂N_j^1 to ∂N_j^2 so as to match $s_1|\partial N_j^1$ with $s_2|\partial N_j^2$. (This may require, for some values of j, changing a preliminary choice for the surgery by a generator of $\pi_1(SO(n-1)) \cong \mathbb{Z}_2$; see [24], §6.) Then M will be a spin manifold, and by repeated application of Van Kampen's Theorem, $\pi_1(M)$ will be isomorphic to the amalgamated free product of two copies of π in which corresponding copies of α_j are identified, which is just π again. The class of M (together with its spin structure and classifying map) in $\Omega_n^{spin}(B\pi)$ will be the sum of the classes of the (M_i, s_i, f_i), and since M is obtained from M_1 and M_2 by surgery along embedded spheres of codimension $n - 1 \geq 4$, M admits positive scalar curvature by [13], Theorem A, or [32], Theorem 4. This shows that $P_n(\pi)$ is a group.

Remark 2.6: We needed Proposition 2.3 in the above proof only to show that $P_n(\pi)$ is non-empty. Actually, under the hypotheses of 2.5, there are "lots" of spin n-manifolds with fundamental group π, in the sense that any element of $\Omega_n^{spin}(B\pi)$ can be realized by a triple (M,s,f) for which $f_*: \pi_1(M) \to \pi_1(B\pi) = \pi$ is an isomorphism. Then whether or not M admits positive scalar curvature is determined by the image of the given bordism class in $\Omega_n^{spin}(B\pi)/P_n(\pi)$. The proof of this fact is a similar surgery argument. Given $(N,s') \xrightarrow{g} B\pi$ representing a spin bordism class, glue a copy of $V^k \times S^{n-k}$ (notation of 2.3) onto N as in the above proof to obtain $(N',s'') \xrightarrow{h} B\pi$ in the same spin bordism class, but with $h_*: \pi_1(N') \to \pi_1(B\pi) = \pi$ surjective. Then kill the kernel of h_* by additional surgeries to obtain (M,s).

Now we are ready for some results regarding Conjecture 2.1 for specific groups. We begin with the following observation:

Proposition 2.7: Under the hypotheses and with the notation of Theorem 2.5, $P_n(\pi)$ is contained in the kernel of the composition $\Omega_n^{spin}(B\pi) \to \Omega_n^{spin}(pt) \xrightarrow{\hat{\alpha}} KO_n(pt)$. Furthermore, if the Kasparov map $\beta: RK_*(B\pi) \to K_*(C_r^*(\pi))$ is injective, then $P_n(\pi)$ is also contained in the kernel of the composition

$$\Omega_n^{spin}(B\pi) \xrightarrow{\hat{\alpha}} RKO_n(B\pi) \xrightarrow{c} RK_n(B\pi),$$

where c is the "complexification map" from real to complex K-homology.

Corollary 2.8: If $\beta: RK_*(B\pi) \to K_*(C_r^*(\pi))$ is injective (i.e., SNC3 holds for π, in the notation of [30]) and if $RKO_*(B\pi)$ has no 2-primary torsion, then one direction of Conjecture 2.1 holds for spin manifolds with $\pi_1(M) \cong \pi$: viz., $P_n(\pi) \subseteq \ker(\hat{\alpha}: \Omega_n^{spin}(B\pi) \to RKO_n(B\pi))$ for all $n \geq 6$ (and also for n = 5 if there exists a spin 5-manifold with positive scalar curvature and fundamental group π).

Proof of Proposition 2.7: The first statement follows from Hitchin's result, Theorem 1.1 above. The second statement follows from Theorem 1.3 above.

Proof of Corollary 2.8: This follows from 2.7 and the fact that $c: KO \to K$ is an injection modulo 2-torsion.

Theorem 2.9: Conjecture 2.1 holds for spin manifolds M^n with $\pi_1(M) \cong \mathbb{Z}$, provided $5 \leq n \leq 9$.

Proof: We may take $B\mathbb{Z} = S^1$, so $RKO_n(B\mathbb{Z}) \cong KO_n(pt) \oplus KO_{n-1}(pt)$. One direction follows from 1.5 and 2.8, since $KO_{n-1}(pt)$ is torsion-free for $5 \leq n \leq 9$. For the other direction, we must show that if (M^n, s) is closed connected spin manifold with $\pi_1(M) \cong \mathbb{Z}$ and with $[M,s,f] = 0$ in $KO_n(S^1)$ ($n = 5,6,7,8$, or 9), then some representative for the bordism class of M in $\Omega_n^{spin}(S^1)$ admits positive scalar curvature. Since $\Omega_6^{spin}(S^1) = 0$ and $\Omega_7^{spin}(S^1) = 0$, the cases $n = 6$ and 7 are trivial. Since $\Omega_5^{spin}(S^1) = \mathbb{Z}$ with generator $K^4 \times S^1$, which has image of infinite order in $K_5(S^1)$, the case $n = 5$ is also trivial. Next, $\Omega_8^{spin}(S^1) \cong \Omega_8^{spin}(pt)$, and a generator for the kernel of the map to $KO_8(pt)$ is $\mathbb{H}P^2$. The manifold $\mathbb{H}P^2 \# (S^1 \times S^7)$ is then a manifold in the same spin bordism class, admitting positive scalar curvature and having infinite cyclic fundamental group. Finally, $\Omega_9^{spin}(S^1) \cong \Omega_9^{spin}(pt) \oplus \Omega_8^{spin}(pt)$ and generators for the kernel of the map $\hat{\alpha} : \Omega_9^{spin}(S^1) \to KO_9(S^1) \cong \mathbb{Z}_2 \oplus \mathbb{Z}$ are $\mathbb{H}P^2 \times \alpha$ (recall α is S^1 with the unusual spin structure) and $\mathbb{H}P^2 \times S^1$. These are identical as manifolds (they differ only in spin structure) and admit positive scalar curvature, as required.

<u>Theorem 2.10</u>: Conjecture 2.1 holds for spin manifolds M^n with $\pi_1(M) \cong F_k$ (the free group on k generators), provided $5 \leq n \leq 9$.

<u>Proof</u>. The argument for this is almost exactly the same as the proof of 2.9, once we replace S^1 with $S^1 \vee \overset{k}{\cdots} \vee S^1$ and $S^1 \times S^7$ with a connected sum of copies of $S^1 \times S^7$. The requisite C^*-algebraic result follows from [30], Proposition 2.10 or else from Kasparov's Theorem, 1.4 above.

<u>Theorem 2.11</u>: Conjecture 2.1 holds for spin manifolds M^n with $\pi_1(M) \cong \pi_1(S_g)$, S_g an oriented surface of genus $g \geq 1$, provided $5 \leq n \leq 9$.

<u>Proof</u>: This is again a situation where Kasparov's Theorem applies, since S_g has nonpositive curvature. Since $\tilde{H}_*(S_g)$ is concentrated in degrees 1 and 2, the Atiyah-Hirzebruch spectral sequences (see [1], §7) for calculation of $\Omega_*^{spin}(S_g)$, $KO_*(S_g)$, and $K_*(S_g)$ collapse, and one can see that $\widetilde{KO}_n(S_g)$ is torsion-free for $5 \leq n \leq 9$, so the argument of 2.8 applies in one direction. As for the other direction, the kernel of $\hat{\alpha} : \Omega_n^{spin}(S_g) \to KO_n(S_g)$ for $5 \leq n \leq 9$ comes from the kernel of $\Omega_n^{spin}(pt) \to KO_n(pt)$ except when $n = 9$, so in all other cases we can argue as before. The kernel of $\widetilde{\Omega_9^{spin}}(S_g) \to \widetilde{KO}_9(S_g)$ is free abelian of rank $2g$, and is generated by $f_i : \mathbb{H}P^2 \times S^1 \to S_g$, $1 \leq i \leq 2g$, where f_i is projection of $\mathbb{H}P^2 \times S^1$ onto S^1, followed by the i-th generator of $H_1(S_g)$. Each such generator obviously corresponds to a manifold of positive scalar curvature obtained by taking the "connected sum along a circle" of $\mathbb{H}P^2 \times S^1$ and $S^7 \times S_g$ using f_i. This completes the proof.

Similar calculations can be done for many other fundamental groups for which SNC 3 applies. However, the situation is somewhat different when π has torsion. In fact we have the following result.

<u>Theorem 2.12</u>: In general, Conjecture 2.1 fails if π is a cyclic group. For instance, every 5-dimensional closed spin manifold M^5 with $|\pi_1(M)| = 3$ admits a metric of positive scalar curvature, even though $\hat{\alpha}(\Omega_5^{spin}(B\mathbb{Z}_3)) \subseteq RKO_5(B\mathbb{Z}_3)$ has order 9.

For $n \geq 5$ and π a cyclic group $\mathbb{Z}_q$ of odd order q, so that Theorem 2.5 applies, we have the estimate

$$|\tilde{\Omega}_n^{spin}(B\mathbb{Z}_q)/\tilde{\Omega}_n^{spin}(B\mathbb{Z}_q) \cap P_n(\mathbb{Z}_q)| \leq q \cdot rk_{\mathbb{Z}}(\Omega_{n-1}),$$

which can be improved to $\leq q$ if Conjecture 2.1 holds for <u>simply connected</u> manifolds. (For comparison, $|\hat{\alpha}(\tilde{\Omega}_n^{spin}(B\mathbb{Z}_q))| = q^2$, q for $n = 5$ or

7, and $|\hat{\alpha}(\tilde{\Omega}_9^{spin}(B\mathbb{Z}_q))| = q^3$. As indicated earlier, Conjecture 2.1 does hold for simply connected manifolds of dimension ≤ 9, except possibly in dimensions 3 and 4, which is enough to guarantee that

$$|\Omega_5^{spin}(B\mathbb{Z}_q)/P_5(\mathbb{Z}_q)| \leq q,$$
$$|\tilde{\Omega}_9^{spin}(B\mathbb{Z}_q)/\tilde{\Omega}_9^{spin}(B\mathbb{Z}_q) \cap P_9(\mathbb{Z}_q)| \leq q.)$$

<u>Proof</u>: The natural transformations of homology theories $\hat{\alpha}: \Omega_*^{spin} \to RKO_*$ and $\Omega_*^{spin} \to \Omega_*$ ("forget the spin structure") induce maps of Atiyah-Hirzebruch spectral sequences ([1], §7)

$$\begin{array}{ccc} \tilde{H}_*(B\mathbb{Z}_q, KO_*) & \Rightarrow & \widetilde{RKO}_*(B\mathbb{Z}_q) \\ \uparrow & & \uparrow \\ \tilde{H}_*(B\mathbb{Z}_q, \Omega_*^{spin}) & \Rightarrow & \tilde{\Omega}_*^{spin}(B\mathbb{Z}_q) \\ \downarrow & & \downarrow \\ \tilde{H}_*(B\mathbb{Z}_q, \Omega_*) & \Rightarrow & \tilde{\Omega}_*(B\mathbb{Z}_q) \end{array}$$

When q is odd, since KO_*, Ω_*^{spin}, and Ω_* have only 2-torsion [33] and $H_*(\mathbb{Z}_q, \mathbb{Z}_2) = 0$, the only non-zero E^2-terms in these spectral sequencies are the terms $E^2_{2s+1, 4t}$, $s \geq 0$ (and also $t \geq 0$ for Ω_* and Ω_*^{spin}). Hence all the spectral sequences collapse and $E^2 = E^\infty$. Thus we can easily write down generators for $\tilde{\Omega}_*^{spin}(B\mathbb{Z}_q)$, namely the manifolds $L^{2s+1} \times M^{4t}$ ($s > 0$, $t \geq 0$), where M runs over (torsion-free) generators of Ω_{4t}^{spin}, which we may take to be simply connected, and where L^{2s+1} is a circle for $s = 0$ and a lens space $S^{2s+1}/\mathbb{Z}_q$ for $s \geq 1$. (The map $L \times M \to B\mathbb{Z}_q$ is the obvious one factoring through L and generating $H_{2s+1}(B\mathbb{Z}_q, \mathbb{Z}) \cong \mathbb{Z}_q$.) Note that these manifolds have obvious metrics of positive scalar curvature when $s \geq 1$ (a Riemannian product of a metric of large constant curvature on L with any metric on M) or when $s = 0$ and M has positive scalar curvature (conjecturally, whenever $[M] = 0$ in $KO_{4t}(pt) \cong \mathbb{Z}$). Also note that since $\Omega_*^{spin} \otimes \mathbb{Z}_q \to \Omega_* \otimes \mathbb{Z}_q$ is an isomorphism, the maps

$$\tilde{H}_*(\mathbb{Z}_q, \Omega_*^{spin}) \to \tilde{H}_*(B\mathbb{Z}_q, \Omega_*)$$

are isomorphisms, and by repeated applications of the 5-lemma, the map $\tilde{\Omega}_*^{spin}(B\mathbb{Z}_q) \to \tilde{\Omega}_*(B\mathbb{Z}_q)$ is an isomorphism. This is useful since the detailed structure of $\Omega_*(B\mathbb{Z}_q)$ is computed in [6], Ch. VII, VIII, and IX.

Now consider some special cases. The map $\Omega_4^{spin} \to KO_4$ is an isomorphism and $\Omega_8^{spin} \to KO_8$ is split surjective, so additional applications of the 5-lemma show that

$$\tilde{\Omega}_5^{spin}(B\mathbb{Z}_q) \to \widetilde{RKO}_5(B\mathbb{Z}_q),$$
$$\tilde{\Omega}_7^{spin}(B\mathbb{Z}_q) \to \widetilde{RKO}_7(B\mathbb{Z}_q), \quad \text{and}$$
$$\tilde{\Omega}_9^{spin}(B\mathbb{Z}_q) \to \widetilde{RKO}_9(B\mathbb{Z}_q)$$

have images of order q^2, q, and q^3, respectively. In the first two cases, we can even remove the tildes since $\Omega_5^{spin} = \Omega_7^{spin} = 0$. However, the only generators for $\Omega_5^{spin}(B\mathbb{Z}_q)$ and $\tilde{\Omega}_9^{spin}(B\mathbb{Z}_q)$ which don't obviously have positive scalar curvature are

$$S^1 \times K^4 \to B\mathbb{Z}_q$$

and

$$S^1 \times M_0^8 \to B\mathbb{Z}_q.$$

This gives the indicated estimates. If $q = 3$, we can do even better since by [6], Theorem 36.1, $\Omega_5(B\mathbb{Z}_3)$ (and thus $\tilde{\Omega}_5^{spin}(B\mathbb{Z}_3)$) is cyclic of order 9, with generator $L^5 \to B\mathbb{Z}_3$ that admits positive scalar curvature. Hence, by Theorem 2.5, every spin 5-manifold M^5 with $\pi_1(M) \cong \mathbb{Z}_3$ admits a metric of positive scalar curvature. This completes the proof for these special cases. In general, we have seen that

$$\tilde{\Omega}_n^{spin}(B\mathbb{Z}_q)/\tilde{\Omega}_n^{spin}(B\mathbb{Z}_q) \cap P_n(\mathbb{Z}_q)$$

is non-zero only for $n \equiv 1 \pmod 4$, and is generated by manifolds $S^1 \times M^{n-1}$, where M^{n-1} runs over generators for $(\Omega^{spin}_{n-1}/P_{n-1}(1)) \otimes \mathbb{Z}_q$. This gives the estimates in the remaining cases.

To conclude our discussion, we briefly say a few words about positive scalar curvature on non-spin, non-simply connected manifolds. Following the argument of [13], Theorem C, one obtains the following analogue of Theorem 2.2 in the non-spin case:

Theorem 2.13: Let X_1^n, X_2^n be closed, oriented, connected n-manifolds with $\pi_1(X_1) \cong \pi_2(X_2) = \pi$, with $n \geq 5$, and with $w_2(\tilde{X}_1) \neq 0$. Let $f_i : X_i \to B\pi$ be the classifying map for the universal covering of X_i. Assume X_2 has a metric of positive scalar curvature and that $X_i \xrightarrow{f_i} B\pi$ lie in the same bordism class in $\Omega_n(B\pi)$. Then X_1 admits a metric of positive scalar curvature.

Proof: One repeats the argument in the proof of Theorem 2.2, but substituting the idea of the proof of [13], Theorem C: since

$$\begin{array}{ccc} \pi_2(X_1) & \xrightarrow{i_*} & \pi_2(W) \\ & {\scriptstyle w_2}\searrow \quad \swarrow{\scriptstyle w_2} & \\ & \mathbb{Z}_2 & \end{array}$$

commutes and $w_2(\tilde{X}_1) \neq 0$, one may reduce to the case where $\pi_2(W, X_1)$ is generated by elements of $\pi_2(W)$ in the kernel of w_2, hence to the case where $\pi_2(W, X_1) = 0$. The argument is concluded as before.

One can prove a similar result for non-orientable manifolds, but this case is more complicated. It is not enough to look at non-oriented bordism; instead one must look at $\mathbb{Z}_2$-equivariant bordism of the oriented double cover, and one must distinguish the cases where this does and does not have a spin structure. We hope to discuss this case in a future publi-

cation. Meanwhile, we conclude with two simple applications of Theorem 2.13. Of course, certain other cases could be treated similarly.

Theorem 2.14: Let M^n be any connected closed manifold with dimension $n \geq 5$, with $w_2(M) \neq 0$, and with fundamental group $\pi_1(M) \cong \mathbb{Z}_q$ cyclic of odd order q. Then M admits a metric of positive scalar curvature.

Proof: Since $H^1(M, \mathbb{Z}_2) = 0$, M is automatically orientable. So by Theorem 2.13, it is enough to check that each generator of $\Omega_n(B\mathbb{Z}_q)$ is represented by a manifold with positive scalar curvature with fundamental group $\mathbb{Z}_q$. As in Theorem 2.12, the bordism spectral sequence shows it is enough to look at the bordism classes of $S^1 \times N^{4t} \to B\mathbb{Z}_q$ and of $L^{2s+1} \times N^{4t} \to B\mathbb{Z}_q$, where L is a lens space and N is one of the torsion-free generators of Ω_{4t}. By [13], Corollary C, we may take N to have positive scalar curvature (this doesn't cover the cases $t = 0$ or $t = 1$, but of course when $t = 0$, we're left just with L, and when $t = 1$, we may take $N = \mathbb{C}\,\mathbb{P}^2$). This finishes the proof except for the case of $S^1 \times N$, where it is necessary first to make a surgery to reduce the fundamental group to $\mathbb{Z}_q$. By [13], Theorem A, this doesn't destroy the positive-scalar-curvature property.

Theorem 2.15: Let M^n be any connected closed orientable manifold with dimension $n \geq 5$, with $w_2(\tilde{M}) \neq 0$, and with infinite cyclic fundamental group. Then M admits a metric of positive scalar curvature.

Proof: $\Omega_n(B\mathbb{Z})$ is generated by the bordism classes of N^n and of $S^1 \times N^{n-1}$, where N runs over generators of Ω_*. Since N has sufficiently high dimension, we may take N to be simply connected, either $= \mathbb{C}\,\mathbb{P}^2$ or one of the manifolds covered by [13], Corollary C. In the case of $S^1 \times N^{n-1}$, we already have a manifold with infinite cyclic fundamental group and with positive scalar curvature. In the other case, adjust the fundamental group by taking a connected sum with $S^1 \times S^{n-1}$. As before, we are done by Theorem 2.13.

3. Behavior under finite coverings and the transfer map in spin bordism

In this final section, we apply the results of §2 to study the following problem: given closed manifolds M_1^n and M_2^n and a finite covering $p : M_1 \to M_2$, what does existence or non-existence of a metric of positive scalar curvature on one manifold say about the other? One fact is obvious: any Riemannian metric on M_2 can be lifted to M_1, so if M_2 has a metric of positive scalar curvature, then so does M_1. However, the situation going the other way is not at all clear. The vague guess that a metric of positive scalar curvature on M_1 can be "averaged" in some way and then pushed down to M_2 is essentially correct in many cases, but not in all. In fact, we shall produce an example in which M_1 has a metric of positive scalar curvature and M_2 does not.

To analyze the situation in greater detail, we shall restrict attention to the case where M_2 is a connected spin manifold, say with spin structure s_2. Pulling back s_2 by p then defines a spin structure s_1 on M_1. Of course, M_1 may not be connected, but the only interesting case is when it is. Let $f_i : M_i \to B\pi_i$ be the classifying map for the universal covering of M_i; we may choose these so that the diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{f_1} & B\pi_1 \\ \downarrow p & & \downarrow q \\ M_2 & \xrightarrow{f_2} & B\pi_2 \end{array}$$

commutes and $p = f_2^* q$, where $q : B\pi_1 \to B\pi_2$ is also a finite covering.

As we have seen in §2, the question of when M_1 admits a metric of positive scalar curvature seems to involve only the spin bordism and KO-homology classes of $(M_i, s_i) \xrightarrow{f_i} B\pi_i$, at least if $n \geq 5$. Now the generalized homology theories KO_*, K_*, Ω_*^{spin}, etc., come equipped with <u>transfer maps</u> (see [2], Ch. 4 and [6], §20). In our particular case, these may be simply defined geometrically so that

$$q^! [M_2, s_2, f_2] = [M_1, s_1, f_1],$$

just as one defines $q^!$ in ordinary homology by lifting a singular simplex $\Delta^n \to B\pi_2$ to the pull-back $q^{-1}(\Delta^n) = \Delta^n \times (\text{finite set}) \to B\pi_1$. If Conjecture 2.1 were to hold in our situation, our question would reduce to the description of the kernel of the transfer map $q^! : RKO_n(B\pi_2) \to RKO_n(B\pi_1)$. To summarize, if this kernel were zero, then M_2 would have a metric of positive scalar curvature whenever M_1 did. If the kernel were not zero, it would classify examples where M_1 admits positive scalar curvature and M_2 does not. Even if Conjecture 2.1 didn't hold, we could (in situations where Theorem 2.5 applied to both π_1 and π_2) examine instead the kernel of the composite

$$\gamma : \Omega_n^{spin}(B\pi_2) \xrightarrow{q^!} \Omega_n^{spin}(B\pi_1) \longrightarrow \Omega_n^{spin}(B\pi_1)/P_n(\pi_1);$$

the analogous obstruction group is then $\ker\gamma/P_n(\pi_2)$. ($P_n(\pi_2) \subseteq \ker\gamma$ because we can lift positive-scalar-curvature metrics.)

<u>Theorem 3.1</u>: There exists an example of a regular 3-fold covering $p : M_1 \to M_2$, where M_1 and M_2 are closed 5-manifolds, M_1 admits a metric of positive scalar curvature, and M_2 does not.

<u>Proof</u>: Let π_2 be the semidirect product $\mathbb{Z}^2 \rtimes \mathbb{Z}$, where the generator of $\mathbb{Z}$ acts on $\mathbb{Z}_2$ by the matrix

$$A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \in SL(2,\mathbb{Z}).$$

Note that the eigenvalues of A are cube roots of 1, so that $A^3 = I$ and $\det(A - I) = 3$. One can compute the homology of π_2 from the Hochschild-Serre spectral sequence with E^2-term $H_*(\mathbb{Z}, H_*(\mathbb{Z}^2, \mathbb{Z}))$, where $H_*(\mathbb{Z}^2,\mathbb{Z})$ is just $\bigwedge_* \mathbb{Z}^2$ and $\mathbb{Z}$ acts by exterior powers of A. Thus

$$H_k(\pi_2,\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } k = 0,2,3, \\ \mathbb{Z} \oplus \mathbb{Z}_3 & \text{if } k = 1, \\ 0 & \text{if } k > 3 \end{cases} .$$

It is clear that π_2 is a discrete cocompact subgroup of a Lie group of the form $\mathbb{R}^2 \rtimes \mathbb{R}$, hence we may choose for $B\pi_2$ a closed solvmanifold V^3, which of course is parallelizable (and hence admits a spin structure). Note that Theorem 1.5 (or 1.4) applies to $C^*(\pi_2)$. We may compute $\widetilde{KO}_*(V)$ from the Atiyah-Hirzebruch spectral sequence with E^2-terms $\tilde{H}_*(V, KO_*(pt))$, and in particular, one sees that $\widetilde{KO}_5(V)$ must contain a summand $= \mathbb{Z}_3$, coming from $H_1(V, KO_4(pt))$. Also note that a generator for this summand may be realized by the spin bordism class of a spin manifold M_2^5 mapping to V; by Remark 2.6, we may assume without loss of generality that $M_2^5 \to V$ induces an isomorphism on fundamental groups.

Now let $\pi_1 = \mathbb{Z}^3$, $B\pi_1 = T^3$. Since $A^3 = I$, $\mathbb{Z}^2 \rtimes 3\mathbb{Z}$ is a normal subgroup of π_2 isomorphic to π_1, and we evidently have a triple covering map $q : T^3 \to V^3$ induced by the homomorphism $\pi_2 \to \pi_2/\pi_1 \cong \mathbb{Z}_3$. Let M_1 be the corresponding triple cover of M_2. Since $M_2 \to V$ defines an odd torsion class in $KO_5(V)$ and hence also in $K_5(V)$, M_2 cannot admit a metric of positive scalar curvature, by Theorem 1.3. On the other hand, we claim M_1 does have a metric of positive scalar curvature. In fact, $M_2 \to V^3$ also defines an odd torsion class in $\Omega_5^{spin}(V)$, so the class

of $M_1 \to T^3$ in $\Omega_5^{spin}(T^3)$, being the image of this class under $q^!$, is also an odd torsion class. However, from the Atiyah-Hirzebruch spectral sequence with E^2-terms $H_*(T^3, \Omega_*^{spin})$, $\Omega_5^{spin}(T^3)$ can contain no odd torsion. So the spin bordism class of $M_1 \to T^3$ is trivial and M_1 admits a metric of positive scalar curvature by Theorem 2.5.

Note that the above proof depended on our being able to find a (torsion-free) group π_2 for which $RK_*(B\pi_2) \to K_*(C^*(\pi_2))$ is injective, yet for which π_2 has odd torsion in its integral homology. If we had started with π_2 finite and π_1 trivial, the situation would be quite different. For instance, we may rewrite part of Theorem 2.12 as follows:

<u>Theorem 3.2</u>: Let M_2^n be a closed spin manifold with fundamental group $\mathbb{Z}_q$ cyclic of odd order q, and let M_1 be its universal covering. If $6 \leq n$ and $n \not\equiv 0,1 \pmod 4$, or if $n = 8$, or if $n = 5$ and $q = 3$, then M_2 admits a metric of positive scalar curvature if and only if M_1 does.

<u>Proof</u>: By Theorem 2.12, under the given hypotheses, $P_n(\mathbb{Z}_q) \cap \tilde{\Omega}_n^{spin}(B\mathbb{Z}_q) = \tilde{\Omega}_n^{spin}(B\mathbb{Z}_q)$. Hence the only obstruction to positive scalar curvature on M_2 comes from the spin bordism class of M_2 in $\Omega_n^{spin}(pt)$. Now as elements of $\Omega_n^{spin}(pt)$, $[M_1] = q[M_2]$, because M_2 is spin bordant to a simply connected manifold, which under the transfer map will go to q disjoint copies of itself. So if $\Omega_n^{spin}/P_n(1)$ has no q-torsion, which is certainly the case if $n = 8$ or $n \not\equiv 0 \pmod 4$, $[M_1] \in P_n(1)$ if and only if $[M_2] \in P_n(1)$. The condition $n \not\equiv 0 \pmod 4$ could be removed if we knew Conjecture 2.1 for simply connected manifolds.

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Jonathan Rosenberg
Department of Mathematics
University of Maryland
College Park, MD 20742

H SUZUKI

Modular cohomology class from the viewpoint of characteristic class

1. INTRODUCTION

In the geometrical study of foliations topologists are familiar with secondary characteristic classes such as Godbillon-Vey classes (see. e.g., [1]) and leaf invariants (cf. [6]). They are closely related, and I will refer to the latter in the present note. In operator theory, remarkable progress has been made using primary characteristic classes through K-theory. But I am not aware of any works on secondary classes related to operator theory so far. This note is an attempt to link secondary characteristic classes to operator theory.

Let M be a Hausdorff C^∞-manifold with a countable open base and $\mathfrak{F}$ a C^∞-foliation of codimension q on M. Sometimes we denote $\mathfrak{F}$ by $(M,\mathfrak{F})$. Let p be the dimension of leaf of $\mathfrak{F}$ so that $n = p + q$ is the dimension of M. We denote by $E(h_1,h_3,\dots,h_r)$ for $r = 2[(q+1)/2] - 1$ the graded exterior algebra generated by $h_1,h_3,\dots,h_r$ where $\deg h_i = 2i - 1$. Let L be a leaf of $\mathfrak{F}$. Then a graded algebra map

$$\alpha_{\mathfrak{F},L;M} : E(h_1,h_3,\dots,h_r) \to H^*_{DR}(L)$$

depending only on $\mathfrak{F}$ and L in M, is determined by virtue of Bott vanishing [1]. $h_i(\mathfrak{F},L) = \alpha_{\mathfrak{F},L;M}(h_i)$ is the i-th leaf invariant of $\mathfrak{F}$ with respect to L. $h_i(\mathfrak{F},L)$ is natural with respect to transverse maps of foliated manifolds and hence it is regarded as a secondary characteristic class. B. Reinhart [6] and R. Goldman [3] constructed these invariants for the foliations with trivial and nontrivial transverse vector bundles.