and a and b are non-negative integers. Proposition 4 then follows from this and computations in Section 6.

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F. Rodier
U.E.R. de mathématique et informatique
Université Paris 7
Tour 45–55, 5^e^ étage
2, Place Jussieu
75251 PARIS CEDEX 05
France

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Group C^*-algebras and topological invariants

J. Rosenberg‡

1. Introduction

The purpose of this paper is to describe a remarkable connection between representation theory, C^*-algebras, and algebraic topology. Many aspects of this relationship were discovered a number of years ago by several Soviet mathematicians: Kasparov, Miščenko, and Solov'ev. However, the subject is perhaps not as well appreciated as it should be, especially by workers in operator algebras and Lie group representations. Furthermore, new progress in the study of K-theory and extension theory of C^*-algebras makes it possible now to prove a few new results and to indicate directions for future research, especially in the operator-algebra aspects of the subject. This is the goal of this article.

Our theme will be a close relationship, for a (second countable) locally compact group G, between algebraic invariants of the group C^*-algebra $C^*(G)$ and the topology of the classifying space BG. What is remarkable is that the former depend only on the unitary representation theory of G, whereas the latter involves only notions from topology.§ The relationship may therefore be used either to say something about harmonic analysis, given topological information, or else to use analysis to prove something about topology. We shall indicate a few possible applications of both sorts.

To motivate the subject, it is necessary to recall a theorem of Atiyah, Hirzebruch, and Segal ([2, 4.8], [3]). First, however, we need some notation and definitions. For any topological group G, BG denotes the

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§This is somewhat reminiscent of the connection, described in Professor Borel's paper (Vol. I, p. 28), between the analysis of the representation of a semisimple Lie group G on $L^2(G/\Gamma)$, when Γ is a lattice subgroup of G, and the topology of G/Γ. In fact, the parallels with the present theory are probably not accidental.

classifying space of G, which is the base space of a locally trivial principal G-bundle with contractible total space EG, and is well-defined up to homotopy under very mild conditions (for standard constructions, see [24, Exercise I.9.32], [14, §§7–8], and [43, §3]). Although we shall later be interested mostly in (non-compact) groups G for which BG can be chosen compact, for compact G one must accept the possibility that BG will be infinite-dimensional and not even locally compact, and view BG as being in some sense a limit of compact spaces. If G is a compact group, $R(G)$ denotes the representation ring of G, which is the free abelian group on the irreducible (finite-dimensional complex) representations of G, with multiplication defined by the (inner) tensor product of representations.

Since K-theory will play an important role in what follows, let us review the important definitions. Recommended references on this material are [24, esp. Exercise II.6.14] and [49], although our notation does not quite agree with that of either of these authors. For A a (complex) Banach algebra with unit, $K_0(A)$ is defined to be the Grothendieck group of the category of finitely generated projective A-modules (in the algebraic sense—the topology does not enter here). Thus elements of $K_0(A)$ are formal differences $[P]-[Q]$, where P and Q are left A-modules which can be embedded as direct summands of A^n for some n, and where $([P_1]-[Q_1])+([P_2]-[Q_2])$ is defined to be $[P_1\oplus P_2]-[Q_1\oplus Q_2]$. An element $[P]-[Q]$ is zero in $K_0(A)$ if and only if $P\oplus A^n\cong Q\oplus A^n$ for some n. When A is a C^*-algebra, it is usually more convenient to regard $K_0(A)$ as consisting of formal differences of Murray–von Neumann equivalence classes of projections in $A\otimes\mathcal{K}$, where $\mathcal{K}$ denotes the compact operators on a separable infinite-dimensional Hilbert space (see, e.g. [16, §3]). The definition of K_0 is extended to Banach algebras not necessarily having a unit by setting $K_0(A)=\ker(K_0(A^+)\to K_0(\mathbb{C})\cong\mathbb{Z})$, where A^+ denotes the algebra obtained by adjoining an identity to A. This agrees with the previous definition when A has a unit, and defines a functor K_0 from Banach algebras to abelian groups.

One now defines, for any Banach algebra A, $K_n(A)=K_0(C_0(\mathbb{R}^n, A))$, where $C_0(\cdot, A)$ denotes the algebra of A-valued continuous functions vanishing at infinity. The Bott periodicity theorem sets up a natural isomorphism $K_0(A)\to K_2(A)$, so that one may view K_* as a functor from Banach algebras to $\mathbb{Z}/2$-graded abelian groups. Any short exact sequence of Banach algebras give rise to a cyclical 6-term exact sequence of K-groups. For any locally compact space Y, we let $K^*(Y)=K_*(C_0(Y))$; this defines a cohomology theory, periodic with period 2, with compact supports. We shall also need to refer to 'representable K-theory' $\mathcal{K}^*$, which is a cohomology theory, periodic with period 2, defined on the category of paracompact (not necessarily locally compact) spaces. For a compact space X, $K^*(X)$ and $\mathcal{K}^*(X)$ are naturally isomorphic. However, for a locally compact space Y, $K^*(Y)$ and $\mathcal{K}^*(Y)$ may be quite different. When Y is a direct limit of compact spaces $\{Y_\alpha\}$, $\mathcal{K}^*(Y)$ is an extension of $\varprojlim K^*(Y_\alpha)$ by a '$\varprojlim^1$ term' (which fortunately vanishes in many familiar examples)—see [3, §4], [7]. If Y is a locally compact space and A is a Banach algebra (over $\mathbb{C}$), $K^*(Y;A)$ is defined to be $K_*(C_0(Y,A))$. When Y is compact and A has a unit, $K^0(Y;A)$ coincides with the Grothendieck group of the category of vector bundles over Y with fibres that are finitely generated projective A-modules. $\mathcal{K}^*(-;A)$ is the representable functor on paracompact spaces coinciding with $K^*(-;A)$ on compact spaces.

Now Atiyah and Hirzebruch observed that for any compact Lie group G, there is a natural homomorphism $\alpha: R(G)\to\mathcal{K}^0(BG)$ defined as follows. Suppose π and σ are finite-dimensional representations of G on complex vector spaces V_π and V_σ. Then the fibre products $EG\times_G V_\pi$ and $EG\times_G V_\sigma$ define vector bundles F_π and F_σ over BG, and we let $\alpha([\pi]-[\sigma])=[F_\pi]-[F_\sigma]$, where $[F_\pi]$ denotes the class of F_π in $\mathcal{K}^0(BG)$ (as a limit of classes of vector bundles over the skeletons of BG). It was shown first in [2] in special cases, and later in [3] in general, that α is injective, that $\mathcal{K}^1(BG)=0$, and that α extends by continuity to an isomorphism $\hat{\alpha}:\hat{R}(G)\to\mathcal{K}^0(BG)$, where $\hat{R}(G)$ denotes the completion of $R(G)$ with respect to the $I(G)$-adic topology, $I(G)$ the augmentation ideal.

The connection with group C^*-algebras is now made by observing that for a compact group G, the group C^*-algebra is a c_0-direct sum (or restricted direct product) of finite-dimensional matrix algebras indexed by the irreducible representations. Since K_* commutes with C^*-direct limits, hence in particular with (c_0)-direct sums, $K_0(C^*(G))$ is a free abelian group with a basis that can be canonically identified with $\hat{G}$. Thus we have a natural isomorphism $R(G)\cong K_0(C^*(G))$ taking $[\pi]$, $\pi\in\hat{G}$, to the class of any minimal idempotent in the group L^1-algebra (or C^*-algebra) obtained from a normalized matrix coefficient of π. Furthermore, $K_1(C^*(G))=0=\mathcal{K}^1(BG)$. All of these observations suggest that for non-compact groups, $K_*(C^*(G))$ might play the role of $R(G)$, and turn out to be 'almost isomorphic' to $\mathcal{K}^*(BG)$. We shall see that this is the case in many instances. This idea is essentially due to Kasparov [25, §8]. (Caution to the reader: what we call here $K_*(A)$ is called $K^*(A)$ in [24] and [25]. Our notation is explained by the fact that our K_0 coincides with what is usually so denoted in algebraic K-theory. Kasparov's K_* is essentially what we will call Ext_*. For consistency, we should really write Ext^* since Ext defines a *contra*variant functor on C^*-algebras, but the notation Ext_* seems too well established to be changed at this point.)

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2. Some general machinery

We have seen that at least for compact groups G, there is a natural relation between $K_*(C^*(G))$, which depends only on the representation theory of G, and $\mathcal{K}^*(BG)$, which is defined strictly in terms of topology. Somewhat remarkably, this turns out to still be the case even for very non-compact and very non-commutative (non-type I) groups.

Before we take up the general situation, it might be natural to consider first two questions about the compact case. First of all, when G is compact, $R(G)$ is a ring, not just an abelian group, and $\hat{\alpha}$ is actually a ring isomorphism. One might wonder, therefore, if there is any ring structure on $K_*(C^*(G))$ making the isomorphism $R(G) \cong K_0(C^*(G))$ an isomorphism of rings and not just of abelian groups. The answer is unfortunately no, at least if one insists at looking at $C^*(G)$ as a C^*-algebra with no additional structure. Of course, one can carry the multiplication on $R(G)$ over to $K_0(C^*(G))$, but for non-compact and non-abelian groups, there is no obvious way to make $K_*(C^*(G))$ into a ring. The best one can say is that one might be able to make $K_*(C^*(G))$ into an $R(H)$-module for certain compact subgroups H of G. When G is discrete, Kasparov [25, §8, Definition 1] has proposed a definition of a ring to replace $R(G)$, but it really lives on the dual object. We shall discuss this more later.

A second problem is that one might think that the association of $C^*(G)$ to G is covariant, hence that since K_* is a covariant functor, $K_*(C^*(G))$ should be a covariant functor of G. This is perplexing since $G \mapsto R(G)$ and $G \mapsto \mathcal{K}^*(BG)$ are contravariant functors. However, this is easily resolved if one remembers that if H is a closed but not open subgroup of G, then the L^1-algebra of H embeds not in $L^1(G)$ but in the measure algebra $M(G)$. Correspondingly, one has a map from $C^*(H)$ not to $C^*(G)$ but to its multiplier algebra, so that $G \mapsto C^*(G)$ does not give a covariant functor (see [39, Introduction and Prop. 4.1]). On the contrary, on the category of compact groups, $G \mapsto K_0(C^*(G))$ can be made into a *contra*variant functor. This is due to the fact that although $C^*(G)$ may not have a unit, $K_0(C^*(G))$ is generated by classes each represented by a projection in $L^1(G)$, which must be given by a continuous function on G and so can be restricted to any closed subgroup H. When one writes out the details of this procedure, one sees that $\alpha: K_*(C^*(G)) \to \mathcal{K}^*(BG)$ defines a natural homomorphism of functors (both contravariant in G).

Extending the set-up to a non-compact group G presents a number of problems. For one thing, one must choose between three different Banach algebras which can serve as the group algebra of $G: L^1(G)$, $C^*(G)$, and the reduced C^*-algebra $C^*_{\text{red}}(G)$. When G is amenable, the last two of these coincide, and furthermore $C^*(G)$ is nuclear [18, Prop.

14], so that one has Brown–Douglas–Fillmore groups $\operatorname{Ext}_1(C^*(G)) = \operatorname{Ext}(C^*(G))$ and $\operatorname{Ext}_0(C^*(G)) = \operatorname{Ext}(C_0(\mathbb{R}, C^*(G)))$ as in [15, Ch. 2 and 6] and [26]. These are dual in some sense to $K_j(C^*(G))$, $j = 1$ and 0. When G is not amenable, there seems to be no reason why $\operatorname{Ext}(C^*(G))$ should be a group, so perhaps it would be better to take its Grothendieck group or group of invertible elements, or else to substitute the Kasparov 'K-homology functor' of [25] and [27]. Anyway, for certain special classes of groups (most of the interesting examples of which are amenable—for instance, compact groups [30, Theorem 1] or finite extensions of discrete nilpotent groups [30, Corollary to Theorem 7]), one can show that $L^1(G)$ and $C^*(G)$ have the same K-groups, using the following argument suggested by Horst Leptin.

Proposition 2.1. *Let A be a symmetric semi-simple Banach *-algebra and let B be its C^*-hull. Then the injection $A \to B$ induces isomorphisms of K-groups.*

Proof. Assume first that A has a unit. By Karoubi's density theorem [24, Exercise II.6.15], it is enough to show that $GL_n(B) \cap M_n(A) = GL_n(A)$ for all n, where $M_n(A)$ denotes $n \times n$ matrices over A and $GL_n(A)$ is its group of invertible elements. However, $M_n(A)$ is also symmetric if A is [30, p. 132], and $M_n(B)$ is its C^*-hull, so we may as well replace A by $M_n(A)$ and show that any element of A invertible in B is invertible in A. As pointed out by Leptin, this may be shown as follows. Suppose $x \in A$ is not left-invertible; then we may choose a maximal left ideal L of A containing x. By symmetry of A, the A-module A/L may be embedded in a Hilbert space $\mathcal{H}$ on which A acts by a continuous *-representation π [29], and since $\pi(x)$ is obviously not left-invertible as an operator on $\mathcal{H}$ (since $x \cdot 1 \in L$), the image of x in B is not left-invertible. The same works for right-invertibility.

If A has no unit, one just repeats the whole argument with A^+. □

There are additional difficulties in making sense of the analogue of $R(G)$ for general groups. When G is not amenable, the natural map $K_*(C^*(G)) \to K_*(C^*_{\text{red}}(G))$ need not be an isomorphism; in fact, as pointed out to the author by Alain Connes, it *cannot* be if G has property (T) of Kazhdan [28], since then the trivial representation of G defines a non-zero class in $K_0(C^*(G))$ sent to zero in $K_0(C^*_{\text{red}}(G))$. Furthermore, for arbitrary locally compact groups, $G \mapsto K_*(C^*(G))$ does not obviously give either a covariant or a contravariant functor. There are also difficulties in relating $K_*(C^*(G))$ and $\mathcal{K}^*(BG)$—since $C^*(G)$ will not have a unit when G is non-discrete, we are forced when studying $C^*(G)$ to work with what corresponds to K-theory with compact supports, but BG is only defined up to homotopy, not up to *proper* homotopy, so that we need to use representable K-theory in order to have $\mathcal{K}^*(BG)$ well-defined.

In the face of all these difficulties, it is surprising that one can say anything at all. In the next two sections, we will discuss some results concerning discrete countable groups and connected Lie groups. Therefore we only mention here in passing some general ideas. The first is a method for treating the case $G = N \rtimes H$, a semi-direct product of a compact group H and some other locally compact group N. In this case, $C^*(G)$ is a crossed product $C^*(H, C^*(N))$. In such a situation, there is a natural isomorphism $K_*(C^*(G)) \cong K_*^H(C^*(N))$, where K_*^H denotes H-equivariant K-theory (see [21] for the case N abelian and [22] for the general case—one method of proof is to first use [30, Theorem 1] and Proposition 2.1 above to get $K_*(C^*(G)) \cong K_*(L^1(H, C^*(N))))$. On the other hand, one can take in this situation $EG = EH \times EN$, with N acting only on the second factor and with H acting by the diagonal action, so that $BG = EH \times_H BN$, which is essentially what is called BN_H in [3]. Thus relating $K_*(C^*(G))$ and $\mathcal{K}^*(BG)$ in this situation is essentially equivalent to relating $K_*^H(C^*(N))$ and $\mathcal{K}_H^*(BN)$, and so one is naturally led to considering the whole problem with an equivariant action of a compact group.

A second general observation is that one knows for an arbitrary locally compact, σ-compact, finite-dimensional group G that the Čech cohomology $\check{H}^*(BG; \mathbb{Q})$ is naturally isomorphic to $H_b^*(G; \mathbb{Q})$, where H_b^* denotes group cohomology with Borel cochains as defined by C. Moore ([50]; see also [48] for a survey of related results). There is therefore a Chern character $\mathrm{ch}: \mathcal{K}^*(BG) \to H_b^*(G; \mathbb{Q})$, which is an isomorphism modulo torsion when BG can be chosen compact. Perhaps it would be best in this degree of generality to disregard BG entirely and to try instead to relate $K_*(C^*(G))$ or $\mathrm{Ext}_*(C^*(G))$ directly to topological group cohomology. This would have the advantage that one would be trying to establish an isomorphism between two objects both defined in terms of analysis on G. In particular, the discussion in [32, esp. Introduction and §9] suggests that there would be important consequences in algebraic topology of a direct (analytical) construction of Chern characters for a discrete group G, with good functorial properties:

$$\begin{cases} K_*(C^*(G)) \to H_*(G; \mathbb{Q}) \\ \qquad\text{or} \\ \mathrm{Ext}_*(C^*(G)) \to H^*(G; \mathbb{Q}). \end{cases}$$

(One may have to replace C^* by C^*_{red} here or work with real rather than rational coefficients.) So far no such construction is known except for the topological constructions to be discussed in the next section. However, the 'non-commutative Chern character' for a C^*-algebra equipped with a

Lie group action in [11] seems to be closely related, at least in the case of lattice subgroup of a Lie group.

3. Results for discrete groups

On the category of discrete (countable) groups, $\Gamma \mapsto C^*(\Gamma)$ and $\Gamma \mapsto C^*_{\text{red}}(\Gamma)$ give covariant functors (the latter of these is functorial only in a limited sense) [39, Prop. 1.2], so since $\Gamma \mapsto \mathcal{K}^*(B\Gamma)$ is a contravariant functor, it is natural to try to set up a natural transformation $\mu : \text{Ext}_*(C^*(\Gamma)) \to \mathcal{K}^*(B\Gamma)$. When Γ is finite, this is not precisely the same as the Atiyah–Hirzebruch map, although it becomes the same if one identifies $R(\Gamma)$ with its dual. When Γ is non-amenable, it may be necessary to replace C^* by C^*_{red} and possibly to replace Ext_* by its Grothendieck group or group of invertible elements. In any event, the construction of μ or of minor variants thereof is essentially the idea of [25, §8], [23, p. 24], and of [31], [32], [45], etc. In fact, granted the construction of a graded product on Ext_* (which is essentially the subject of [25], given the isomorphism for separable nuclear C^*-algebras between the Kasparov and Brown–Douglas–Fillmore functors proved in [27, §7]), [25, §8] also shows how to make μ into a ring homomorphism.

It actually seems easiest here to modify the ideas of Kasparov and Miščenko by using the Brown–Douglas–Fillmore Ext-theory in place of Fredholm representations and the Kasparov K-functor. This results in a construction first described by de la Harpe and Karoubi [23], although phrased by them in slightly different language. In what follows Γ will always denote a countable discrete group, and Ext denotes what is sometimes called 'weak Ext' (see [15, Ch. 2]). It is worth noting that since $C^*(\Gamma)$ has a one-dimensional representation (namely, the trivial representation of Γ), 'strong' and 'weak' Ext coincide for $C^*(\Gamma)$ (see [4, p. 560] and [41, §3]). This makes the construction that follows somewhat simpler, but the reader can easily make the modifications (that might be necessary if $C^*(\Gamma)$ were replaced by $C^*_{\text{red}}(\Gamma)$) needed to take the distinction between 'strong' and 'weak' Ext into account. If $\mathcal{B} = \mathcal{B}(\mathcal{H})$ denotes all bounded operators on an infinite-dimensional separable Hilbert space $\mathcal{H}$ and $\mathcal{Q} = \mathcal{B}/\mathcal{K}$ denotes the Calkin algebra, then a class in $\text{Ext}(C^*(\Gamma))$ is determined by a unital *-monomorphism $\tau : C^*(\Gamma) \to \mathcal{Q}$. We form then

$$E_\tau = \mathcal{Q} \times_\Gamma E\Gamma = \mathcal{Q} \times E\Gamma/\sim_\tau, \quad \text{where} \quad (a\tau(\gamma), b) \sim_\tau (a, \gamma \cdot b)$$
$$\text{for } \gamma \in \Gamma,\ a \in \mathcal{Q},\ b \in E\Gamma.$$

Then E_τ is the total space of a bundle over $B\Gamma$ with fibres that are copies of $\mathcal{Q}$ (in particular projective $\mathcal{Q}$-modules), and so defines a class $[E_\tau] \in \mathcal{K}^0(B\Gamma; \mathcal{Q})$.

We claim this gives a homomorphism $\mu : \text{Ext}(C^*(\Gamma)) \to \mathcal{K}^0(B\Gamma; \mathcal{Q})$.

First of all, if τ_1 and τ_2 are equivalent, which means $\tau_2(\cdot)=u\tau_1(\cdot)u^{-1}$ for some unitary $u\in\mathcal{Q}$ of index zero, then $[E_{\tau_2}]=[E_{\tau_1}]$, since the map $\phi:(a,b)\mapsto(au^{-1},b)$ for $a\in\mathcal{Q}$, $b\in E\Gamma$, commutes with the left $\mathcal{Q}$-action on $\mathcal{Q}\times E\Gamma$, is bijective and satisfies

$$\phi(a\tau_1(\gamma),b)=(au^{-1}\tau_2(\gamma),b)\sim_{\tau_2}(au^{-1},\gamma\cdot b)=\phi(a,\gamma\cdot b).$$

Secondly, μ takes 0 to 0, since if $\tau:C^*(\Gamma)\to\mathcal{Q}$ lifts to a *-homomorphism $\sigma:C^*(\Gamma)\to\mathcal{B}$, then $[E_\tau]$ lies in the image in $\mathcal{K}^0(B\Gamma;\mathcal{Q})$ of $\mathcal{K}^0(B\Gamma;\mathcal{B})$, which is zero by [24, Exercise II.6.16]. Finally, μ is a homomorphism, since if τ_1 and τ_2 are unital *-monomorphisms $C^*(\Gamma)\to\mathcal{Q}$ and $u_1u_1^*+u_2u_2^*=1$, $u_1^*u_1=u_2^*u_2=1$ in $\mathcal{Q}$, then $[\tau_1]+[\tau_2]$ is represented by $\tau:x\mapsto u_1\tau_1(x)u_1^*+u_2\tau_2(x)u_2^*$, and E_τ is equivalent to $E_{\tau_1}\oplus E_{\tau_2}$ via the map $\mathcal{Q}\times E\Gamma/\sim_\tau\to(\mathcal{Q}\oplus\mathcal{Q})\times E\Gamma/\sim_{(\tau_1,\tau_2)}$ induced by $(a,b)\mapsto(au_1,au_2,b)$.

***Proposition* 3.1.** *There are natural transformations $\mu_0:\operatorname{Ext}_0(C^*(\Gamma))\to\mathcal{K}^0(B\Gamma)$, $\mu_1:\operatorname{Ext}_1(C^*(\Gamma))\to\mathcal{K}^1(B\Gamma)$, for Γ a countable discrete group. One could also replace $C^*(\Gamma)$ by $C^*_{\text{red}}(\Gamma)$.*

Proof. The map μ_1 is just the map $\mu:\operatorname{Ext}(C^*(\Gamma))\to\mathcal{K}^0(B\Gamma,\mathcal{Q})$ constructed above, followed by the isomorphism $\mathcal{K}^0(B\Gamma;\mathcal{Q})\cong\mathcal{K}^1(B\Gamma)$ of [24, Exercise II.6.16]. Karoubi and de la Harpe give an equivalent description: given $\tau:C^*(\Gamma)\to\mathcal{Q}$, we view τ as a group homomorphism $\Gamma\to\mathcal{U}(\mathcal{Q})$ and consider the induced map on spaces $B\Gamma\to B\mathcal{U}(\mathcal{Q})$. Since $B\mathcal{U}(\mathcal{Q})$ is homotopy-equivalent to the infinite unitary group $U(\infty)$, we have associated to τ a homotopy class of maps $B\Gamma\to U(\infty)$, hence an element of $\mathcal{K}^1(B\Gamma)$. To get μ_0, replace Γ by $\Gamma\times\mathbb{Z}$ and observe that $C^*(\Gamma\times\mathbb{Z})\cong C^*(\Gamma)\otimes C(\mathbb{T})$, where $\mathbb{T}$ is the circle group, and similarly $B(\Gamma\times\mathbb{Z})\cong B\Gamma\times B\mathbb{Z}=B\Gamma\times\mathbb{T}$. Thus $\mathcal{K}^1(B(\Gamma\times\mathbb{Z}))\cong\mathcal{K}^0(B\Gamma)\oplus\mathcal{K}^1(B\Gamma)$ and $\operatorname{Ext}(C^*(\Gamma\times\mathbb{Z}))\cong\operatorname{Ext}(C^*(\Gamma))\oplus\operatorname{Ext}_0(C^*(\Gamma))$. We define μ_1 to be the composite $\operatorname{Ext}_0(C^*(\Gamma))\to\operatorname{Ext}(C^*(\Gamma\times\mathbb{Z}))\to\mathcal{K}^1(B(\Gamma\times\mathbb{Z}))\to\mathcal{K}^0(B\Gamma)$. Everything of course works with C^*_{red} also. Note that the definitions of μ for Γ and for $\Gamma\times\mathbb{Z}$ are properly consistent.

It remains to check naturality. If we are given a homomorphism $\psi:\Gamma_1\to\Gamma_2$ inducing $\psi_*:C^*(\Gamma_1)\to C^*(\Gamma_2)$, then for $\tau:C^*(\Gamma_2)\to\mathcal{Q}$, clearly $E_{\tau\circ\psi_*}\cong\psi^*(E_\tau)$. The same argument with $\psi\times\mathrm{id}:\Gamma_1\times\mathbb{Z}\to\Gamma_2\times\mathbb{Z}$ establishes the claim. □

We are ready now for the main results about the map μ of (3.1). In the cases treated, they sharpen the results of [25, §8, Thm. 2], [45], and [32, Thm. 1], and so presumably could be used as discussed in [25, §8] and in [33] in proving cases of the 'Novikov conjecture' on higher signatures of non-simply connected manifolds. Even if we leave aside the topological content of the results, the proofs seem to have some independent interest since they illustrate applications of [36], [21], and [10]. In particular, the exact sequences of [36] can be seen to be analogues of a Gysin sequence in a non-commutative setting.

We call a solvable group Γ *poly-(infinite cyclic)* if it has a composition series in which the successive quotients are free abelian. For such a Γ, $B\Gamma$ can be chosen compact, and hence with slight abuse of notation, we shall write $K^*(B\Gamma)$ in place of $\mathcal{K}^*(B\Gamma)$. Any polycyclic group contains such a subgroup Γ of finite index, and these are essentially the same as the groups that can be embedded discretely in solvable connected Lie groups (for precise statements, see [51, Prop. 4.1 and Prop. 4.4]).

Lemma 3.2. *The map μ_1 is an isomorphism for $\Gamma = \mathbb{Z}$ (or equivalently, μ_0 is an isomorphism when $\Gamma = \{1\}$).*

Proof. We have $C^*(\mathbb{Z}) \cong C(\mathbb{T})$, $B\mathbb{Z} \cong \mathbb{T}$. Thus $\operatorname{Ext}(C^*(\mathbb{Z}))$ and $K^1(B\mathbb{Z})$ are both infinite cyclic, so it is enough to check that μ sends a generator to a generator. Now $\operatorname{Ext}(C^*(\mathbb{Z}))$ is generated by the homomorphism from $\mathbb{Z}$ to $GL(\mathcal{Q})$ taking $1 \in \mathbb{Z}$ to a unitary u of index 1. The continuous sections of the corresponding $\mathcal{Q}$-bundle over $\mathbb{T}$ may be identified with continuous functions $\mathbb{R} \to \mathcal{Q}$ such that $f(x+1) = f(x)u$ for $x \in \mathbb{R}$. Therefore, the bundle (as a bundle of left $\mathcal{Q}$-modules) is not trivial, since otherwise there would have to exist a section f as above taking values everywhere in $GL(\mathcal{Q})$, which is impossible since the Fredholm index of $f(x)$ would have to increase by 1 each time x is increased by 1. One sees from the same argument that the bundle generates $K^0(\mathbb{T}; \mathcal{Q})$. □

Theorem 3.3. *Let Γ be any poly-(infinite cyclic) group. Then the maps $\mu_j : \operatorname{Ext}_j(C^*(\Gamma)) \to K^j(B\Gamma)$ are isomorphisms.*

Proof. This is proved by induction on the length of a composition series for Γ in which all quotients are infinite cyclic. Lemma 3.2 starts the induction. Therefore we may assume $\Gamma = \Gamma_1 \rtimes \mathbb{Z}$ and the theorem is already known for Γ_1. Let β denote the automorphism of Γ_1 induced by a generator of $\mathbb{Z}$. Then we may take $B\Gamma = B\Gamma_1 \times_{\mathbb{Z}} \mathbb{R}$, where we divide $B\Gamma_1 \times \mathbb{R}$ by the $\mathbb{Z}$-action by powers of $(x, t) \to (\beta_* x, t+1)$. In particular, we have a fibration

$$\begin{array}{ccc} B\Gamma_1 & \to & B\Gamma \\ & & \downarrow{\scriptstyle\alpha} \\ & & \mathbb{T}. \end{array}$$

Thus we may compute $K^*(B\Gamma)$ via a sort of Gysin sequence. More exactly, let Y and Z be the images of $B\Gamma_1 \times (0,1)$ and $B\Gamma_1 \times \{0\}$ in $B\Gamma$. Then since Y is open in $B\Gamma$ and $B\Gamma \backslash Y = Z \cong B\Gamma_1$, we get a long exact K-theory sequence by which we can compute $K^*(B\Gamma)$:

$$\cdots \to K^{i+1}(B\Gamma_1) \xrightarrow{\delta} K^i(B\Gamma_1 \times (0,1)) \to$$
$$K^i(B\Gamma) \to K^i(B\Gamma_1) \xrightarrow{\delta} K^{i+1}(B\Gamma_1 \times (0,1)) \to \cdots$$

which when explicated gives ($i: B\Gamma_1 \cong Z \to B\Gamma$ denoting the inclusion)

$$\cdots \longrightarrow K^{i+1}(B\Gamma_1) \xrightarrow{1-\beta^*} K^{i+1}(B\Gamma_1) \xrightarrow{\psi} K^i(B\Gamma) \xrightarrow{i^*}$$
$$K^i(B\Gamma_1) \xrightarrow{1-\beta^*} K^i(B\Gamma_1) \longrightarrow \cdots.$$

On the other hand, $C^*(\Gamma)$ is a crossed product $C^*(\mathbb{Z}, C^*(\Gamma_1))$ and is nuclear since Γ is solvable. Thus we may compute $\mathrm{Ext}_*(C^*(\Gamma))$ using the long exact sequence of [36, Theorem 3.5]. Here we have rewritten the sequence slightly to conform to present notation, and have not needed to check the quasi-diagonality hypothesis because of the improved homotopy-invariance results of [27]. Putting the two exact sequences together (using the map μ for both Γ and Γ_1) we get the following diagram with exact rows:

$$\begin{array}{ccccccc}
\longrightarrow \mathrm{Ext}_{j+1}(C^*(\Gamma_1)) & \xrightarrow{1-\beta^*} & \mathrm{Ext}_{j+1}(C^*(\Gamma_1)) & \xrightarrow{\delta} & \mathrm{Ext}_j(C^*(\Gamma)) & \xrightarrow{i^*} & \\
\downarrow{\scriptstyle \mu_{\Gamma_1}} & & \downarrow{\scriptstyle \mu_{\Gamma_1}} & & \downarrow{\scriptstyle \mu_{\Gamma}} & & \\
\cdots \longrightarrow K^{i+1}(B\Gamma_1) & \xrightarrow{1-\beta^*} & K^{i+1}(B\Gamma_1) & \xrightarrow{\psi} & K^i(B\Gamma) & \xrightarrow{i^*} &
\end{array}$$

$$\begin{array}{ccc}
\mathrm{Ext}_j(C^*(\Gamma_1)) & \xrightarrow{1-\beta^*} & \mathrm{Ext}_j(C^*(\Gamma_1)) \longrightarrow \cdots \\
\downarrow{\scriptstyle \mu_{\Gamma_1}} & & \downarrow{\scriptstyle \mu_{\Gamma_1}} \\
K^i(B\Gamma_1) & \xrightarrow{1-\beta^*} & K^i(B\Gamma_1) \longrightarrow \cdots.
\end{array}$$

Here $i^*: \mathrm{Ext}_j(C^*(\Gamma)) \to \mathrm{Ext}_j(C^*(\Gamma_1))$ is induced from the natural inclusion $C^*(\Gamma_1) \to C^*(\Gamma)$; note that the square involving the two maps labelled i^* commutes by the naturality assertion in Proposition 3.1. Similarly the two squares involving maps labelled $1-\beta^*$ commute. Finally, the square involving the maps δ and ψ also commutes, as can be checked from a description of the connecting map of the Pimsner–Voiculescu sequence. Since μ_{Γ_1} is an isomorphism by inductive hypothesis, μ_Γ must be an isomorphism, by the Five-Lemma. □

Theorem 3.4. *If Γ is a free group on countably many generators, $\mu : \operatorname{Ext}(C^*(\Gamma)) \to \mathcal{K}^1(B\Gamma)$ is an isomorphism.*

Proof. This is essentially a reformulation of results of L. Brown, who observed [4, p. 563] that if Γ is free on n generators (n finite), then even though $C^*(\Gamma)$ is non-nuclear for $n > 1$, $\operatorname{Ext}(C^*(\Gamma))$ is a group and in fact is isomorphic to $\mathbb{Z}^n$ via the usual index invariant. (An element of $\operatorname{Ext}(C^*(\Gamma))$ is given by a choice of n unitaries in $\mathcal{Q}$; it lifts if and only if all of these have index zero.) Since $B\Gamma$ is just a wedge of n circles, this is equivalent to our statement (via the argument of Lemma 3.2). If $n = \infty$, $B\Gamma$ is a wedge of countably many circles and $\mathcal{K}^1(B\Gamma)$ is an infinite product of copies of $\mathbb{Z}$. On the other hand, $\operatorname{Ext}(C^*(\Gamma))$ is isomorphic to this same product by the index map. □

It would be nice to know about Ext_0 for the C^*-algebra of a free group or about the Ext-semigroup of the reduced C^*-algebra (which is simple by [38]), but it does not even seem to be known if these are groups. Our knowledge about the case of other non-amenable groups is in most cases even more limited. However, the following generalization of Theorem 3.4 is known (due to work of Brown and of Karoubi–de la Harpe).

Theorem 3.5. *Let Γ be a free product of countable groups Γ_1 and Γ_2 and assume $\operatorname{Ext}(C^*(\Gamma_j))$ is a group for $j = 1, 2$. Then there is an isomorphism $\operatorname{Ext}(C^*(\Gamma)) \cong \operatorname{Ext}(C^*(\Gamma_1)) \oplus \operatorname{Ext}(C^*(\Gamma_2))$ compatible (via μ_1) with the isomorphism $\mathcal{K}^1(B\Gamma) \cong \mathcal{K}^1(B\Gamma_1) \oplus \mathcal{K}^1(B\Gamma_2)$.*

Proof. This follows immediately from the theorem of [5] about $\operatorname{Ext}(A *_{\mathbb{C}} B)$, in the case $A = C^*(\Gamma_1)$, $B = C^*(\Gamma_2)$, $C = \mathbb{C}$. According to Brown's theorem, the kernel of the natural map $\operatorname{Ext}(C^*(\Gamma)) \xrightarrow{\gamma} \operatorname{Ext}(C^*(\Gamma_1)) \oplus \operatorname{Ext}(C^*(\Gamma_2))$ will be isomorphic to the cokernel of a map '$\operatorname{Ext}'_0(A) \oplus$ '$\operatorname{Ext}'_0(B) \to \operatorname{Ext}_0(C)$ which is easily seen to be surjective, and the cokernel of γ will be zero since $\operatorname{Ext}(\mathbb{C}) = 0$. In this situation, we may take $B\Gamma = B\Gamma_1 \vee B\Gamma_2$, and it is clear that

$$\begin{array}{ccc} \operatorname{Ext}(C^*(\Gamma)) & \xrightarrow{\gamma} & \operatorname{Ext}(C^*(\Gamma_1)) \oplus \operatorname{Ext}(C^*(\Gamma_2)) \\ \downarrow{\scriptstyle \mu_1^{\Gamma}} & & \downarrow{\scriptstyle (\mu_1^{\Gamma_1}, \mu_1^{\Gamma_2})} \\ \mathcal{K}^1(B\Gamma) & \xrightarrow{\cong} & \mathcal{K}^1(B\Gamma_1) \oplus \mathcal{K}^1(B\Gamma_2) \end{array}$$

commutes. For another version of the result, see [23]. □

It is worth mentioning at this point that composition of μ with the usual Chern character $\operatorname{ch} : \mathcal{K}^*(B\Gamma) \to H^*(B\Gamma, \mathbb{Q}) = H^*(\Gamma, \mathbb{Q})$ yields the Chern character map $\mathrm{ch}: \mathrm{Ext}_*(C^*(\Gamma)) \to H^*(\Gamma, \mathbb{Q})$ mentioned earlier. This can be described explicitly as follows. View an element of $\mathrm{Ext}\,(C^*(\Gamma))$ as being a class of group homomorphisms $\tau: \Gamma \to \mathcal{U}(\mathcal{Q})$. Now $\mathcal{U}(\mathcal{Q})$ is a topological group in the norm topology, so τ induces a map $\tau_*: B\Gamma \to B\mathcal{U}(\mathcal{Q})$ (well-defined up to homotopy). As a consequence of Bott periodicity, $\mathcal{U}(\mathcal{Q})$ has the homotopy type of $\mathbb{Z} \times BU(\infty)$ and $B\mathcal{U}(\mathcal{Q})$ has the homotopy type of $U(\infty) = \varinjlim U(n)$. Then $\mu([\tau])$ is the class of τ_* as sitting in $[B\Gamma, \Omega BU(\infty)] \cong [SB\Gamma, BU(\infty)]$. Then $\mathrm{ch}\,([\tau])$ is defined by the usual formula $\sum_{n=1}^{\infty} 1/n!\,(\tau_*)^*(Q_n(c_1, c_2, \ldots))$ (24, V.3.22], where c_i is the universal Chern class in $H^{2i}(BU(\infty), \mathbb{Z})$ and Q_n is the nth Newton polynomial. The sum is of course to be computed in $H^{\mathrm{even}}(SB\Gamma, \mathbb{Q}) \cong H^{\mathrm{odd}}(B\Gamma, \mathbb{Q})$. Alternatively, one can compute $\mathrm{ch}\,([\tau])$ directly from the 'Chern classes' of τ, which are the elements of $H^{\mathrm{odd}}(B\Gamma, \mathbb{Z})$ obtained by pulling back via $\tau_*: B\Gamma \to B\mathcal{U}(\mathcal{Q})$ the canonical generators of $H^*(U(\infty), \mathbb{Z})$ (which is an exterior algebra on generators in odd degrees). The 'Chern classes' of course vanish if $[\tau]$ is trivial in $\mathrm{Ext}\,(C^*(\Gamma))$, since then τ_* is null-homotopic.

We conclude this section by mentioning now a number of results about $K_*(C^*(\Gamma))$ for discrete groups. As we mentioned earlier, for reasons of functoriality one should not expect these to be isomorphic or 'almost isomorphic' to $\mathcal{K}^*(B\Gamma)$ in general. Instead, $K_*(C^*(\Gamma))$ should be related to *K-homology* of $B\Gamma$ via Kasparov's map β [25, §8].† For instance, let $\Gamma = \mathbb{Z} \rtimes \mathbb{Z}$, where the semidirect product is defined by the action of $\mathbb{Z}$ on $\mathbb{Z}$ by multiplication by powers of -1. Then $B\Gamma$ may be taken to be a Klein bottle, and by Theorem 3.3, we have $\mathrm{Ext}_1\,(C^*(\Gamma)) \cong K^1(B\Gamma) \cong H^1(B\Gamma, \mathbb{Z}) \cong \mathrm{Hom}\,(\Gamma, \mathbb{Z}) \cong \mathbb{Z}$, $\mathrm{Ext}_0\,(C^*(\Gamma)) \cong K^0(B\Gamma) \cong H^0(B\Gamma, \mathbb{Z}) \oplus H^2(B\Gamma, \mathbb{Z}) \cong \mathbb{Z} \oplus (\mathbb{Z}/2)$, while $K_1(C^*(\Gamma)) \cong K_1(B\Gamma) \cong H_1(B\Gamma, \mathbb{Z}) \cong \Gamma/[\Gamma, \Gamma] \cong \mathbb{Z} \oplus (\mathbb{Z}/2)$, $K_0(C^*(\Gamma)) \cong K_0(B\Gamma) \cong H_0(B\Gamma, \mathbb{Z}) \oplus H_2(B\Gamma, \mathbb{Z}) \cong \mathbb{Z}$ (by calculation with the Pimsner–Voiculescu sequence for K-theory instead of Ext, or else via Theorem 3.3 and the universal coefficient theorem for Ext, proved in [6] or [42]).

Nevertheless, there are a number of cases when one can relate $K_*(C^*(\Gamma))$ to $\mathcal{K}^*(B\Gamma)$. This happens particularly when both are torsion-free, for instance if Γ is finite. Recall in fact that for a finite group, the Atiyah–Hirzebruch map α can be thought of as an injection of $K_*(C^*(\Gamma))$ into $\mathcal{K}^*(B\Gamma)$, which becomes an isomorphism upon completion. There does not seem to be a good analogue of α for general discrete groups, but

† The definition of β is complicated but the existence of such a map is easy to motivate. In fact, there is an obvious inclusion of Γ into $\mathcal{U}(C^*(\Gamma))$ (or the 'stable' unitary group $\mathcal{U}^s(C^*(\Gamma)) = \varinjlim \mathcal{U}(C^*(\Gamma) \otimes M_n)$), which induces a map of spaces $B\Gamma \to B\mathcal{U}^s(C^*(\Gamma))$. The induced map on fundamental groups is the natural map $\Gamma = \pi_1(B\Gamma) \to \pi_1(B\mathcal{U}^s(C^*(\Gamma))) = K_1(C^*(\Gamma))$ [25, II.6.14.d], and there is also a map induced in K-homology from $K_n(B\Gamma)$ to $K_n(B\mathcal{U}^s(C^*(\Gamma)))$. The latter is at least closely related to $K_n(C^*(\Gamma))$.

for certain of the groups treated in 3.3 and 3.4, there are good substitutes.

Theorem 3.6. *Let Γ be a discrete cocompact subgroup of a connected, simply connected solvable Lie group G. (In this case we may identify $B\Gamma$ with the compact solvmanifold G/Γ.) Then there is an isomorphism $K_*(C^*(\Gamma)) \to \mathcal{K}^*(B\Gamma)$ of degree equal to the dimension of G.*

Proof. By Rieffel's version of the Mackey imprimitivity theorem [39, Theorem 7.18], $C^*(\Gamma)$ is strongly Morita equivalent to the transformation group algebra $C^*(G, G/\Gamma)$, and in fact by [20], the latter is isomorphic to $C^*(\Gamma) \otimes \mathcal{K}$. Thus $K_*(C^*(\Gamma)) \cong K_*(C^*(G, G/\Gamma))$ (without shift of degree). Then by [10, §V, Corollary 7], there is an isomorphism $K_*(C^*(G, G/\Gamma)) \to K^{*+\dim G}(G/\Gamma)$. □

Remark 3.7. If Γ is poly-(infinite cyclic), one may compute $H^*(\Gamma, \mathbb{Z})$ from the Hochschild–Serre spectral sequence, starting from the case of a free abelian group. This shows among other things that $B\Gamma$ has Euler–Poincaré characteristic zero (a classical fact), so that $K^0(B\Gamma) \otimes \mathbb{Q} \cong K^1(B\Gamma) \otimes \mathbb{Q}$ (as abstract groups). It follows from 3.3 and the universal coefficient theorem for Ext (as stated in [6] and [42]) that the torsion-free parts of Ext_0, Ext_1, K_0, and K_1 for $C^*(\Gamma)$ are all isomorphic. Again by the universal coefficient theorem, the torsion part of K_0 (resp. K_1) is isomorphic to the torsion part of Ext_1 (resp. Ext_0). As we have already seen by example, the torsion may not vanish and may be different in K_0 and K_1. But if the rank of Γ is even and Γ satisfies the condition of 3.6, then 3.3 and 3.6 together imply that all the K- and Ext-groups of $C^*(\Gamma)$ coincide, a fact which may be attributed to Poincaré duality for the manifold G/Γ. □

It is perhaps worth mentioning a possible application of Theorem 3.6 to harmonic analysis on discrete groups. If Γ is finitely generated, torsion-free, and nilpotent, then it satisfies the condition of Theorem 3.6 for a unique nilpotent Lie group G (by a famous theorem of Malcev), and so the K-groups of $C^*(\Gamma)$ are easily computed. As indicated in the remarks preceding Proposition 2.1, $L^1(\Gamma)$ is symmetric and so its K-groups coincide with those of $C^*(\Gamma)$. Putting these facts together, we see that the Betti numbers of $B\Gamma$ determine the number of 'independent' solutions to the convolution equation $f * f = f$, for f a matrix-valued L^1-function on the group (since such fs correspond to projective $L^1(\Gamma)$-modules). Such use of K-theory to study idempotent measures on groups was first proposed in [49, §§11–12]. The same idea is valid for any group with symmetric L^1-algebra, for instance for the groups of the following theorem.

Theorem 3.8. *Let Γ be a semidirect product $\Gamma_1 \rtimes \mathbb{Z}^n$, where Γ_1 is a finite group (acting on $\mathbb{Z}^n$). Then a suitable completion of $K_*(C^*(\Gamma))$ is isomorphic to $\mathcal{H}^*(B\Gamma)$: via a generalized Atiyah–Hirzebruch–Segal map.*

Proof. As indicated earlier, [21] gives an isomorphism $K_*(C^*(\Gamma)) \cong K_*^{\Gamma_1}(C^*(\mathbb{Z}^n)) \cong K_{\Gamma_1}^*(\mathbb{T}^n)$. This is an $R(\Gamma_1)$-module, and so may be completed in the $I(\Gamma_1)$-adic topology. Furthermore, $B\Gamma$ is just $(\mathbb{T}^n)_{\Gamma_1}$, so this theorem follows immediately from [3, Prop. 4.2]. □

Theorem 3.9 (Cuntz [13]). *If Γ is a free group on countably many generators, the inclusion $\Gamma \to \mathcal{U}(C^*(\Gamma))$ induces an isomorphism $\Gamma/[\Gamma, \Gamma] \to K_1(C^*(\Gamma))$, and $K_0(C^*(\Gamma))$ is infinite cyclic with generator the class of the identity.*

Theorem 3.10 (Pimsner–Voiculescu [37]). *The conclusions of Theorem 3.9 remain valid if $C^*(\Gamma)$ is replaced by $C^*_{\text{red}}(\Gamma)$.*

***Remark* 3.11.** Theorem 3.9 (proved in [13] for the case of Γ free on two generators, although the general case is clearly the same) may be interpreted as giving a natural isomorphism from $K_*(B\Gamma)$ to $K_*(C^*(\Gamma))$ for free groups, which is of course what one would expect on the basis of [25, §8]. This improves and 'explains' an earlier theorem of Cohen [9] (see [8] for major simplifications in the proof) stating that the C^*-algebra of a free group contains no non-trivial idempotents. The same is now known for the reduced C^*-algebra by Theorem 3.10. □

It would be nice to have calculations of K-groups for C^*-algebras or reduced C^*-algebras of other non-amenable groups. About the only result in this direction is the following, which bears the same relation to 3.9 as 3.5 to 3.4.

Theorem 3.12 (Cuntz [13]). *If Γ is the free product of groups Γ_1 and Γ_2 and $\psi_j : C^*(\Gamma_j) \to \mathbb{C}$ is the homomorphism induced by the trivial representation, then the K-groups of $C^*(\Gamma)$ are naturally isomorphic to those of $\{(a, b) \mid a \in C^*(\Gamma_1),\ b \in C^*(\Gamma_2),\ \psi_1(a) = \psi_2(b)\}$.*

One group to which 3.5 and 3.12 apply is $\Gamma = PSL(2, \mathbb{Z}) \cong (\mathbb{Z}/2) * (\mathbb{Z}/3)$. Thus we see that $\text{Ext}\,(C^*(PSL(2, \mathbb{Z}))) = 0$, $K_1(C^*(PSL(2, \mathbb{Z}))) = 0$, and $K_0(C^*(PSL(2, \mathbb{Z})))$ is free abelian of rank 4.† Theorems similar to those of

† The marked difference between this and the cohomology of $B\Gamma$, which is periodic and has torsion, is easily understood in terms of the Atiyah–Hirzebruch theorem. $B\Gamma$ is an infinite complex, and the Chern character is far from being an isomorphism. However, $\mathcal{H}^*(B\Gamma)$ is, as expected, torsion-free and concentrated in even degree.

[5] and [13] but valid for more general amalgamated free products would make it possible to treat the fundamental groups of compact surfaces. However, entirely different methods would be needed to treat groups like $PSL(3, \mathbb{Z})$. There is hope that one might be able to find a substitute for the results of [10] and [17] valid for reduced crossed products by semisimple Lie groups, in which case the method of 3.6 could be applied to lattice subgroups of more general Lie groups. (The author thanks A. Connes for this idea.)

4. Results for connected Lie groups

When G is a connected Lie group, unless G is compact there is no obvious map connecting the K- and Ext-groups of $C^*(G)$ (or $C^*_{\text{red}}(G)$) with the topological K-groups of BG. Nevertheless, there are striking similarities between these that seem to be due to more than just coincidence. In fact, one seems to be led to the following conjecture, which the author believes should be attributed to Connes:

***Conjecture* 4.1.** For G a connected Lie group with maximal compact subgroup H (for obvious reasons we are trying to avoid over-using the letter K), there are isomorphisms $K_i(C^*_{\text{red}}(G)) \to K_H^{i+\dim(G/H)}(pt)$ (at least as abstract groups),† where as usual K_H^* denotes the H-equivariant K-theory of [44] and pt is a one-point space. (There is an analogous conjecture about $\text{Ext}_i\,(C^*_{\text{red}}(G))$.)

The connection with our theme should be clear, since G and H are homotopy-equivalent and thus BG can be identified with BH. Furthermore, we know $\mathcal{K}^*(BH) \cong R(H) \cong K_H^*(pt)$ by the Atiyah–Hirzebruch–Segal Theorem, so that this would yield an isomorphism of a suitable completion of $K_*(C^*_{\text{red}}(G))$ with $\mathcal{K}^*(BG)$ (of degree the dimension of G/H).

Let us now discuss some of the evidence for the conjecture and some of its implications. The strongest evidence comes from the case of solvable groups.

***Theorem* 4.2** (Connes [10, §V, *Corollary* 7], Kasparov [27, p. 574]). *Let G be a connected, simply connected solvable Lie group. Then* $K_i(C^*(G)) \cong K^{i+\dim G}(pt)$.

***Theorem* 4.3** (Fack–Skandalis [17, Corollary 2], Kasparov [27, p. 574]). *Let G be a connected, simply connected solvable Lie group. Then* $\text{Ext}_i\,(C^*(G)) \cong K_{i+\dim G}(pt)$.

† Note that we are not claiming that $K_*(C^*_{\text{red}}(G))$ necessarily carries an $R(H)$-module structure, or even if it does, that it is necessarily free as an $R(H)$-module.

Theorem 4.4. *Let G be a connected Lie group with maximal compact subgroup H, and assume G is either* (a) *nilpotent or* (b) *a motion group. Then $K_i(C^*(G)) \cong K_H^{i+\dim(G/H)}(pt)$ (as abelian groups).*

Proof. (a) If G is nilpotent, then H is central, and so G has a series of closed subgroups $G \supset G_1 \supset \cdots \supset G_n = H$ with $G_{j+1} \lhd G_j$ and $G_j/G_{j+1} \cong \mathbb{R}$. Thus $C^*(G_j)$ must be a crossed product $C^*(\mathbb{R}, C^*(G_{j+1}))$. It suffices now to apply the 'Thom isomorphism' theorem of [10] n times. (b) If G is a motion group, i.e. $G = H \rtimes V$ with V a vector group, then $C^*(G) \cong C^*(H, C^*(V)) = C^*(H, \hat{V})$ and, by [21], $K_*(C^*(G)) \cong K_H^*(\hat{V})$ (as $R(H)$-modules). The result now follows from the Thom isomorphism theorem in equivariant K-theory, applied either to $\hat{V}$ (if V is even-dimensional) or to $\hat{V} \times \mathbb{R}$ (if V is odd-dimensional), as explained in [12, §4]. It should be pointed out, however, that although $K_H^*(\hat{V})$ is a free $R(H)$-module when $\hat{V}$ carries an H-invariant spin structure, this may be false in general. For instance, suppose $H = SO(4)$ and $\hat{V} = \mathbb{R}^4$ (with the obvious action). The map $H \to SO(\hat{V})$ does not lift to a map $H \to \mathrm{Spin}(\hat{V})$, so by [12, Proposition 4.3], $K_H^0(\hat{V})$ is isomorphic as an $R(H)$-module to the *odd* part of $R(\mathrm{Spin}(4)) = R(SU(2)) \otimes R(SU(2))$, which is not free as an $R(H)$-module. □

Unfortunately, the argument given for Theorem 4.4 does not seem to work for general non-simply connected solvable Lie groups. Nevertheless, it seems very likely that Conjecture 4.1 is at least true for all connected amenable Lie groups. Our evidence in the semisimple case is much sketchier, but at least we know the following.

Theorem 4.5 [41, §7]. *Conjecture* 4.1 *is valid for all connected semisimple Lie groups of real-rank one.*

The primary reason for interest in Conjecture 4.1, as pointed out in [12], comes from the 'theory of the discrete series'. Suppose G is a connected Lie group with compact centre. Then G may or may not have square-integrable representations, i.e., factor representations which occur as direct summands of the regular representation. If, however, such representations exist, then they define open points in the primitive ideal space of $C^*_{\mathrm{red}}(G)$, and if G is unimodular, these points are also closed [19, Theorem 1 and Corollaries]. The most familiar examples where such representations exist are compact groups, $SL(2, \mathbb{R})$, the Heisenberg group (the non-trivial central extension of $\mathbb{R}^2$ by $\mathbb{T}$), and the '$ax+b$' group ($\mathbb{R} \rtimes \mathbb{R}_+^\times$). Of these, the latter is non-unimodular and has two square-integrable irreducible representations each of which weakly contains all the one-dimensional representations; the other examples are unimodular.

In the case of semisimple groups (and by extension, for other unimodular groups with a type I regular representation), the square-integrable irreducible representations are commonly referred to as the 'discrete series', and are the basic building blocks in Harish-Chandra's theory of harmonic analysis on G. An important problem (now essentially solved, but not without considerable effort) in harmonic analysis on Lie groups is to determine which groups have square-integrable representations and to parameterize and construct these representations when they exist. The relevance of Conjecture 4.1 to this problem is apparent from the following result of Philip Green, which he kindly communicated to the author.

Theorem 4.6 [P. Green, unpublished]. *Let G be a connected unimodular Lie group. Then each square-integrable irreducible representation of G contributes a summand of $\mathbb{Z}$ to $K_0(C^*_{\text{red}}(G))$, and each non-type* I *square-integrable factor representation contributes to both $K_0(C^*_{\text{red}}(G))$ and to $K_1(C^*_{\text{red}}(G))$.*

Corollary 4.7. *Assume G is connected, unimodular, and satisfies Conjecture* 4.1. *Then all square-integrable factor representations of G are type* I, *and such representations can exist only if G/H is even-dimensional.*

Proof (Sketch). Let G be a connected unimodular Lie group, π a square-integrable factor representation of G with kernel $J \in \text{Prim}\,(C^*_{\text{red}}(G))$. By [40, Theorem 2.13], π must be traceable. Since by [19] J is isolated for the hull-kernel topology on $\text{Prim}\,(C^*_{\text{red}}(G))$, we see that $C^*_{\text{red}}(G) \cong \text{im}\,(\pi) \oplus J$ (C^*-algebra direct sum), and thus $K_i(C^*_{\text{red}}(G)) \cong K_i(\text{im}\,(\pi)) \oplus K_i(J)$. If π is type I, then $\text{im}\,(\pi) \cong \mathcal{K}$ since π is traceable, and we get a contribution of $\mathbb{Z}$ to $K_0(C^*_{\text{red}}(G))$, 0 to K_1. If π is not type I, then using the machinery of [18], [19], and [36], one can show that both $K_0(\text{im}\,(\pi))$ and $K_1(\text{im}\,(\pi))$ are non-zero. Thus this case is ruled out if $K_*(C^*_{\text{red}}(G))$ is concentrated in one degree. Furthermore, if 4.1 holds, then K_0 can only be non-zero when G/H is even-dimensional. □

Finally, we should mention that a proof of 4.1, assuming it involved construction of an explicit isomorphism of K-groups, would not only prove 4.7, but also give geometric realization and exhaustion theorems for the discrete series, since each discrete series representation contributes a free generator to $K_0(C^*_{\text{red}}(G))$. To illustrate how this works, consider the case of a nilpotent Lie group G with compact centre Z and discrete series, for which 4.1 is proved via 4.4. By the argument of 4.7, G/Z is even-dimensional, and the iterated Thom isomorphism of 4.4, together with Bott periodicity, gives an explicit isomorphism

$\alpha : K_0(C^*(Z)) \cong R(Z) \to K_0(C^*(G))$. This means each character of Z gives rise to a generator of $K_0(C^*(G))$. For a generic character, this will correspond to the unique discrete series representation with this central character [35], and α will in principle give a method for constructing a minimal idempotent in $C^*(G)$ generating this representation. (Surjectivity of α proves the uniqueness of the discrete series representation with a fixed central character.) Exceptional characters of Z (for instance, the trivial character unless G is abelian) will be sent under α to elements of $K_0(C^*(G))$ coming from non-discrete series. Similarly, in the case of $SL(2, \mathbb{R})$, all but one of the free generators of $K_0(C^*_{\text{red}}(G))$ come from the discrete series. The remaining generator is associated with the odd principal series and the two 'limits of discrete series'. We should mention that not only are no counter-examples to the conclusion of 4.7 known, but in fact [1] gives a (quite complicated, since it requires the Harish-Chandra machine) proof of necessity of evenness of $\dim(G/H)$ for unimodular Lie groups with discrete series.

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J. ROSENBERG
Department of Mathematics
University of Maryland
College Park, MD 20742
USA

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