Introduction to Non-Commutative Geometry

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Contents

Part I

Basics of Non-Commutative Geometry

Nigel Higson and John Roe

Part II

Applications of Non-Commutative Geometry to Topology

Jonathan Rosenberg

Chapter 1

Group Von Neumann Algebras in Topology: L^2-cohomology, Novikov-Shubin invariants

1.1 Motivation

The idea of this part of the book is to explain how non-commutative geometry, as developed in Part I, can be applied to problems in geometry and topology (in the more usual sense of those words). In this chapter, we begin by applying the first major new idea to emerge out of non-commutative geometry, namely, the concept of *continuous dimension* as developed by Murray and von Neumann.

This concept starts to come into play when we compare the spectral decomposition of the Laplacian (or more exactly, the Laplace-Beltrami operator of Riemannian geometry) in the two cases of a compact (Riemannian) manifold and a complete non-compact manifold. The comparison can be seen in the following table:

Compact manifolds	**Non-compact manifolds**
Discrete spectrum	Continuous spectrum
Finite-dimensional kernel	Infinite-dimensional kernel

Table 1.1: Spectrum of the Laplacian

So on a non-compact manifold, the dimension of the kernel of the Laplacian is not usually very interesting (it's often ∞) and knowing the eigenvalues of the

Laplacian usually does not yield much information about the operator. (For example, there may be no point spectrum at all, yet the spectral decomposition of the operator may be very rich.) In the special case where the non-compact manifold is a normal covering of a compact manifold with covering group π, we will get around these difficulties by using the group von Neumann algebra of π to measure the "size" of the infinite-dimensional kernel and the "thickness" of the continuous spectrum near 0.

1.2 An Algebraic Set-Up

Here we follow the ideas of Michael Farber [23], as further elaborated by him (in [25], [24], and [26]) and by Wolfgang Lück ([56], [56], and [57]). Let π be a discrete group. It acts on both the right and left on $L^2(\pi)$. The von Neumann algebras $\lambda(\pi)''$ and $\rho(\pi)''$ generated by the left and by the right regular representations λ and ρ are isomorphic, and $\rho(\pi)'' = \lambda(\pi)'$. These von Neumann algebras are *finite*, with a canonical (faithful finite normal) trace τ defined by

$$\tau(\lambda(g)) = \begin{cases} 1, & g = 1 \\ 0, & g \neq 1, \end{cases}$$

and similarly for ρ. Call a finite direct sum of copies of $L^2(\pi)$, with its left action of π, a *finitely generated free Hilbert π-module*, and the cut-down of such a module by a projection in the commutant $\lambda(\pi)' = \rho(\pi)''$ a *finitely generated projective Hilbert π-module*. (We keep track of the topology but forget the inner product.)

The finitely generated projective Hilbert π-modules form an additive category $\mathcal{H}(\pi)$. The morphisms are continuous linear maps commuting with the π-action. Each object A in this category has a dimension $\dim_\tau(A) \in [0, \infty)$, via

$$\dim_\tau n \cdot L^2(\pi) = n, \quad \dim_\tau eL^2(\pi) = \tau(e),$$

for each projection e in $\rho(\pi)''$ (or more generally in $M_n(\rho(\pi)'')$, to which we extend the trace the usual way, with $\tau(1_n) = n$, 1_n the identity matrix in $M_n(\rho(\pi)'')$). When π is an *ICC* (infinite conjugacy class) group, $\rho(\pi)''$ is a factor, hence a projection $e \in M_n(\rho(\pi)'')$ is determined up to unitary equivalence by its trace, and objects of $\mathcal{H}(\pi)$ are determined by their dimensions.

The category $\mathcal{H}(\pi)$ is *not abelian*, since a morphism need not have closed range, and thus there is no good notion of cokernel. It turns out, however, that there is a natural way to complete it to get an *abelian* category $\mathcal{E}(\pi)$. The finitely generated projective Hilbert π-modules are the projectives in $\mathcal{E}(\pi)$. Each element of the larger category is a direct sum of a projective and a torsion element (representing infinitesimal spectrum near 0). A torsion element is an equivalence class of pairs (A, α), where A is a projective Hilbert π-module and $\alpha = \alpha^*\colon A \to A$ is a positive π-module endomorphism of A (in other words, a positive element of the commutant of the π-action) with $\ker \alpha = 0$. (Note that this implies α has dense range, but not that it has a bounded inverse.) Two such pairs (A, α), (A', α'), are identified if we can write

$$(A, \alpha) \cong (A_1, \alpha_1) \oplus (A_2, \alpha_2), \quad (A', \alpha') \cong (A'_1, \alpha'_1) \oplus (A'_2, \alpha'_2),$$

with α_2 and α'_2 invertible and with $(A_1, \alpha_1) \cong (A'_1, \alpha'_1)$, in the sense that there is commutative diagram

$$\begin{tikzcd} A_1 \arrow[r, "\beta", "\cong"'] \arrow[d, "\alpha"] & A'_1 \arrow[d, "\alpha'"] \\ A_1 \arrow[r, "\beta", "\cong"'] & A'_1. \end{tikzcd}$$

Thus we can always "chop off" the part of α corresponding to the spectral projection for $[\varepsilon, \infty)$ $(\varepsilon > 0)$ without changing the equivalence class of the object, and only the "infinitesimal spectrum near 0" really counts. The dimension function $\dim_\tau$ extends to a map, additive on short exact sequences, from objects of $\mathcal{E}(\pi)$ to $[0, \infty)$, under which torsion objects go to 0.

To phrase things another way, the idea behind the construction of $\mathcal{E}(\pi)$ is that we want to be able to study indices of elliptic operators $D\colon A \to B$, where A and B are Hilbert π-modules and D commutes with the π-action. By usual tricks, we can assume D is a bounded operator. In the case of a compact manifold (with no π around), A and B would then be Hilbert spaces, D would be Fredholm, and the index would be defined as $\operatorname{Ind} D = \dim\ker D - \dim\operatorname{coker} D$. But in the case of a non-compact manifold, we run into the problem that D usually does not have closed range. However, if we fix $\varepsilon > 0$ and let e, f be the spectral projections for $[\varepsilon, \infty)$ for D^*D and DD^*, respectively, then the restriction of D to eA maps this projective π-module isomorphically onto fB, and $D\colon (1-e)A \to (1-f)B$ represents a formal difference of torsion elements $((1-e)A, D^*D)$ and $((1-f)B, DD^*)$ of $\mathcal{E}(\pi)$ when we pass to the limit as $\varepsilon \to 0$.

The most interesting invariant of a torsion element $\mathcal{X}$ represented by $\alpha = \alpha^*\colon A \to A$ is the rate at which

$$F_\alpha(t) = \dim_\tau(E_t A), \quad E_t = \text{spectral projection for } \alpha \text{ for } [0, t),$$

approaches 0 as $t \to 0$. Of course we need to find a way to study this that is invariant under the equivalence relation above, but it turns out (see Exercise 1.4.1) that F_α is well-defined modulo the equivalence relation

$$F \sim G \Leftrightarrow \exists C,\, \varepsilon > 0,\ G\left(\tfrac{t}{C}\right) \le F(t) \le G(tC),\ t < \varepsilon.$$

The *Novikov-Shubin capacity* of $\mathcal{X}$ is defined to be

$$c(\mathcal{X}) = \limsup_{t \to 0^+} \frac{\log t}{\log F_\alpha(t)}.$$

Note that this is well-defined modulo the equivalence relation above, since if $G\left(\frac{t}{C}\right) \le F(t) \le G(tC)$ for t sufficiently small, then

$$\begin{aligned}
\limsup_{t\to 0^+} \frac{\log t}{\log G(t)} &= \limsup_{t\to 0^+} \frac{\log t - C}{\log G(t)} = \limsup_{t\to 0^+} \frac{\log(tC/C)}{\log G(tC)} \\
= \limsup_{t\to 0^+} \frac{\log t}{\log G(tC)} &\le \limsup_{t\to 0^+} \frac{\log t}{\log F(t)} \\
&\le \limsup_{t\to 0^+} \frac{\log t}{\log G\left(\frac{t}{C}\right)} \\
&= \limsup_{t\to 0^+} \frac{\log(tC/C)}{\log G\left(\frac{t}{C}\right)} \\
&= \limsup_{t\to 0^+} \frac{\log t + C}{\log G(t)} = \limsup_{t\to 0^+} \frac{\log t}{\log G(t)}.
\end{aligned}$$

The Novikov-Shubin capacity satisfies

$$c(\mathcal{X}_1 \oplus \mathcal{X}_2) = \max(c(\mathcal{X}_1),\, c(\mathcal{X}_2))$$

and for exact sequences

$$0 \to \mathcal{X}_1 \to \mathcal{X} \to \mathcal{X}_2 \to 0,$$

$$\max(c(\mathcal{X}_1),\, c(\mathcal{X}_2)) \le c(\mathcal{X}) \le c(\mathcal{X}_1) + c(\mathcal{X}_2).$$

Many people work instead with the inverse invariant

$$\liminf_{t\to 0^+} \frac{\log F_\alpha(t)}{\log t},$$

called the *Novikov-Shubin invariant* or *Novikov-Shubin number*, but the advantage of the capacity is that "larger" torsion modules have larger capacities. If the Novikov-Shubin invariant is $\gamma > 0$, that roughly means that $\dim_\tau(E_t A) \approx t^\gamma$.

Now consider a connected CW complex X with fundamental group π and only finitely many cells in each dimension. The cellular chain complex $C_\bullet(\widetilde{X})$ of the universal cover $\widetilde{X}$ (with complex coefficients) is a chain complex of finitely generated free (left) $\mathbb{C}[\pi]$-modules. We can complete to $L^2(\pi)\otimes_\pi C_\bullet(\widetilde{X})$, a chain complex in $\mathcal{H}(\pi) \subseteq \mathcal{E}(\pi)$, and get homology, cohomology groups

$$\mathcal{H}_i(X, L^2(\pi)) \in \mathcal{E}(\pi), \quad \mathcal{H}^i(X, L^2(\pi)) \in \mathcal{E}(\pi),$$

called (extended) L^2*-homology and cohomology*, which are homotopy invariants of X. The numbers $\beta_i(X, L^2(\pi)) =$

$$\dim_\tau(\mathcal{H}_i(X), L^2(\pi)) = \dim_\tau(\mathcal{H}^i(X), L^2(\pi))$$

are called the (reduced) L^2*-Betti numbers* of X. Similarly one has *Novikov-Shubin invariants* (first introduced in [65], but analytically, using the Laplacian) defined from the spectral density of the torsion parts (though by the UCT, the torsion part of $\mathcal{H}^i(X, L^2(\pi))$ corresponds to the torsion part of $\mathcal{H}_{i-1}(X, L^2(\pi))$, so that there is some confusion in the literature about indexing).

1.3 Calculations

Theorem 1.3.1 *Suppose M is a compact connected smooth manifold with fundamental group π. Fix a Riemannian metric on M and lift it to the universal cover $\widetilde{M}$. Then the L^2-Betti numbers of M as defined above agree with the τ-dimensions of*

$$\left(L^2 \text{ closed } i\text{-forms on } \widetilde{M}\right) / \overline{d\left(L^2 \ (i-1)\text{-forms on } \widetilde{M}\right) \cap \left(L^2 \ i\text{-forms}\right)}.$$

Similarly the Novikov-Shubin invariants can be computed from the spectral density of Δ on $\widetilde{M}$ (as measured using τ).

Sketch of Proof. This is a kind of a de Rham theorem. There are two published proofs, one by Farber ([25], §7) and one by Shubin [90]. Let $\Omega^\bullet(M)$ be the de Rham complex of differential forms on M. Then a fancy form of the usual de Rham theorem says that the complexes $\Omega^\bullet(M)$ and $C^\bullet(M)$ (the latter being the cellular cochains with coefficients in $\mathbb{C}$ for some cellular decomposition) are chain homotopy equivalent. The same is therefore true for the complexes $L^2(\pi) \otimes_\pi \Omega^\bullet(M)$ and $L^2(\pi) \otimes_\pi C^\bullet(M)$. Unfortunately the first of these is a complex of Fréchet spaces, not of Hilbert π-modules, so Farber's theory doesn't directly apply to it. However, there is a trick: we can also consider the complex $\Omega^\bullet_{\text{Sobolev}}(\widetilde{M})$ of forms on $\widetilde{M}$ values in Sobolev spaces. More precisely, fix $m \geq n = \dim M$ and consider the complex

$$\Omega^\bullet_{\text{Sobolev}}(\widetilde{M})\colon \Omega^0_{(m)}(\widetilde{M}) \xrightarrow{d} \Omega^1_{(m-1)}(\widetilde{M}) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^j_{(m-j)}(\widetilde{M}) \xrightarrow{d} \cdots,$$

where $\Omega^j_{(m-j)}(\widetilde{M})$ consists of j-forms with distributional derivatives up to order $m-j$ in L^2 (with respect to the Riemannian metric on M). Then the spaces in this complex are all Hilbert spaces on which π acts by a(n infinite) multiple of the left regular representation, and the differentials are all bounded operators commuting with the action of π (since we lose one derivative with each application of d). An extension of Farber's original construction shows that the spaces in such a complex can also be viewed as sitting in an "extended" abelian category (in effect one just needs to drop the finite generation condition in the definition of $\mathcal{E}(\pi)$). Then one shows that the dense inclusion $L^2(\pi)\otimes_\pi\Omega^\bullet(M) \hookrightarrow \Omega^\bullet_{\text{Sobolev}}(\widetilde{M})$ is a chain homotopy equivalence. (The proof depends on elliptic regularity; the spectral decomposition of the Laplacian can be used to construct the inverse chain map $\Omega^\bullet_{\text{Sobolev}}(\widetilde{M}) \to L^2(\pi) \otimes_\pi \Omega^\bullet(M)$ that "smooths out" Sobolev-space-valued forms to smooth ones.) Putting everything together, we then have a chain homotopy equivalence $L^2(\pi) \otimes_\pi C^\bullet(M) \to \Omega^\bullet_{\text{Sobolev}}(\widetilde{M})$ in a suitable extended abelian category, and thus the cohomology (with values in this extended category) of the two complexes is the same. Since the L^2-Betti numbers and Novikov-Shubin invariants are obtained by looking at the projective and torsion parts of the extended cohomology, the theorem follows. □

Example 1.3.2 Let $M = S^1$ and $\pi = \mathbb{Z}$, so $\mathbb{C}[\pi] = \mathbb{C}[T, T^{-1}]$. Then $L^2(\pi)$ is identified via Fourier series with $L^2(S^1)$, and the group von Neumann algebra with $L^\infty(S^1)$, which acts on $L^2(S^1)$ by pointwise multiplication. The trace τ is then identified with the linear functional $f \mapsto \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(e^{i\theta})\,d\theta$. Take the usual cell decomposition of S^1 with one 0-cell and one 1-cell. Then the chain complexes of the universal cover become:

$$C_\bullet(\widetilde{M})\colon \mathbb{C}[T, T^{-1}] \xrightarrow{T-1} \mathbb{C}[T, T^{-1}]$$
$$C_\bullet(\widetilde{M}, L^2(\pi))\colon L^2(S^1) \xrightarrow{e^{i\theta}-1} L^2(S^1).$$

So the L^2-Betti numbers are both zero, but the Novikov-Shubin invariants are non-trivial (in fact equal to 1), corresponding to the fact that if

$$\alpha = |e^{i\theta} - 1|\colon L^2(S^1) \to L^2(S^1),$$

then $F_\alpha(t) \approx \text{const} \cdot t$ for t small.

Generalizing one aspect of this is:

Theorem 1.3.3 (Cheeger-Gromov [13]**)** *If X is an aspherical CW complex (i.e., $\pi_i(X) = 0$ for $i \neq 1$) with only finitely many cells of each dimension, and if $\pi = \pi_1(X)$ is amenable and infinite, then all L^2-Betti numbers of X vanish.*

There is a nice treatment of this theorem in [21], §4.3. The reader not familiar with amenable groups can consult [34] or [67] for the various forms of the definition, but one should know at least that finite groups and solvable groups are amenable and free groups on two or more generators (or any groups containing such a free group) are not. It is not known then (at least to the author) if for an aspherical CW complex with infinite amenable fundamental group, one of the Novikov-Shubin capacities is always positive. However, this is true in many cases for which one can do direct calculations, such as nilmanifolds modeled on stratified nilpotent Lie groups [85].

As we will see, amenability is definitely relevant here; for non-amenable groups, the L^2-Betti numbers can be non-zero.

Example 1.3.4 Let M be a compact Riemann surface of genus $g \geq 2$, $\widetilde{M}$ the hyperbolic plane, π a discrete torsion-free cocompact subgroup of $G = PSL(2, \mathbb{R})$. In this case, it's easiest to use the analytic picture, since $L^2(\widetilde{M}) \cong L^2(G/K)$, $K = SO(2)/\{\pm 1\}$. As a representation space of G, this is a direct integral of the principal series representations, and Δ corresponds to the Casimir operator, which has spectrum bounded away from 0. So $\beta_0 = 0$, and also $\beta_2 = 0$ by Poincaré duality.

Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. Then the tangent bundle of G/K is the homogeneous vector bundle induced from the representation space $\mathfrak{g}/\mathfrak{k}$ of K, and the cotangent bundle is similarly induced from $(\mathfrak{g}/\mathfrak{k})^*$. Thus the L^2 sections of $\Omega^1(\widetilde{M})$ may be identified with unitarily induced representation $\operatorname{Ind}_K^G (\mathfrak{g}/\mathfrak{k})^*$, which contains, in addition to the continuous spectrum, two discrete series representations with Casimir eigenvalue 0. Thus $\beta_1 \neq 0$. The Atiyah L^2-index theorem (to be discussed in the next chapter) implies $\beta_1 = 2(g-1)$. There are no additional Novikov-Shubin invariants, since these measure the non-zero spectrum of Δ close to 0, but the continuous spectrum of Δ is bounded away from 0.

One can also do the calculation of β_1 combinatorially. As is well known, one can construct M by making suitable identifications along the boundary of a $2g$-gon. (See Figure 1.1.) This construction gives a cell decomposition of M with one 0-cell, one 2-cell, and $2g$ 1-cells. Hence we can take $L^2(\pi) \otimes_\pi C^\bullet(M)$ to be a complex of free Hilbert π-modules of dimensions 1, $2g$, and 1. Since the extended L^2-cohomology vanishes in degrees 0 and 2, it then follows (by the Euler-Poincaré principle in the category $\mathcal{E}(\pi)$) that $\mathcal{H}^1(X, L^2(\pi))$ must be free, of dimension $2g-2$.

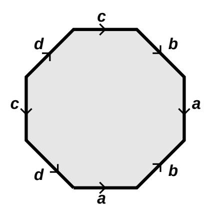

Figure 1.1: Identifications to form a closed surface (the case $g = 2$)

The vanishing of β_0 in Example 1.3.4 is not an accident. In fact, Brooks ([6] and [7]) proved the following:

Theorem 1.3.5 (Brooks [6]**)** *Let M be a compact Riemannian manifold with fundamental group π. Then 0 lies in the spectrum of the Laplacian on the universal cover $\widetilde{M}$ of M if and only if π is amenable.*

Note that this implies (if π is non-amenable) that $\beta_0(M)$ and the Novikov-Shubin capacity in dimension 0 must be 0, and in fact via Theorem 1.3.1, the de Rham theorem for extended cohomology, that the extended cohomology group $\mathcal{H}^0(M, L^2(\pi))$ must vanish in $\mathcal{E}(\pi)$.

Generalizing one aspect of Example 1.3.4 is the following result, confirming a conjecture of Singer:

Theorem 1.3.6 (Jost-Zuo [42] **and Cao-Xavier** [12]**)** *If M is a compact connected Kähler manifold of non-positive sectional curvature and complex dimension n, then all L^2-Betti numbers of M vanish, except perhaps for β_n.*

As we will see in Exercise 1.4.4 or via the Atiyah L^2-index theorem of the next chapter, this implies that

$$\beta_n = (-1)^n \chi(M),$$

where χ is the usual Euler characteristic.

Note. One should not be misled by these examples into thinking that the L^2-Betti numbers are always integers, or that most of them usually vanish. However, the *Atiyah Conjecture* asserts that they are always rational numbers. If true, this would have important implications, such as the *Zero Divisor Conjecture* that $\mathbb{Q}[G]$ has no zero divisors when G is a torsion-free group [21]. For more details on this and related matters, the reader is referred to the excellent surveys by Lück: [58] and [59].

1.4 Exercises

Exercise 1.4.1 Fill in one of the details above by showing that if $\alpha = \alpha^*: A \to A$ and $\beta = \beta^*: B \to B$ represent the same torsion element $\mathcal{X}$ of the category $\mathcal{E}(\pi)$, and if $F_\alpha(t)$ and $G_\beta(t)$ are the associated spectral growth functions, defined by applying τ to the spectral projections of α (resp., β) for $[0, t)$, then there exist C, $\varepsilon > 0$ such that

$$G_\beta\left(\tfrac{t}{C}\right) \leq F_\alpha(t) \leq G_\beta(tC),\ t < \varepsilon.$$

Exercise 1.4.2 Use Example 1.3.2 to show that the 0-th Novikov-Shubin invariant of the n-torus $T^n = \left(S^1\right)^n$ is equal to n. (In fact this is true for all the other Novikov-Shubin invariants also, since the Laplacian on p-forms simply looks like a direct sum of $\binom{n}{p}$ copies of the Laplacian on functions.)

Exercise 1.4.3 Let X be a wedge of $n \geq 2$ circles, which has fundamental group $\pi = F_n$, a free group on n generators. This space has a cell decomposition with one 0-cell and n 1-cells. Compute the L^2-Betti numbers of X directly from the chain complex $L^2(\pi) \otimes_\pi C^\bullet(\widetilde{X})$.

Exercise 1.4.4 Let X be a finite CW complex with fundamental group π. Use the additivity of $\dim_\tau$ and the Euler-Poincaré principle to show that

$$\sum_{i=0}^{\dim X} (-1)^i \beta_i(X, L^2(\pi)) = \chi(X),$$

the ordinary Euler characteristic of X.

Exercise 1.4.5 Prove the combinatorial analogue of Brooks' Theorem (1.3.5) as follows. Let X be a finite connected CW complex with fundamental group π Show that $\mathcal{H}^0(X, L^2(\pi)) = 0$ if and only if π is non-amenable, following this outline. Without loss of generality, one may assume X has exactly one 0-cell, and has 1-cells indexed by a finite generating set $g_1, \ldots, g_n$ for π. First show that $d^*d\colon L^2(\pi) \otimes_\pi C^0(\pi) \to L^2(\pi) \otimes_\pi C^0(\pi)$ can be identified with right multiplication by $\Delta = (g_1 - 1)^*(g_1 - 1) + \cdots + (g_n - 1)^*(g_n - 1)$ on $L^2(\pi)$. So the problem is to determine when 0 is in the spectrum of Δ. This happens if and only if for each $\varepsilon > 0$, there is a unit vector ξ in $L^2(\pi)$ such that $\|\rho(g_i)\xi - \xi\| < \varepsilon$ for $i = 1, \ldots, n$, where ρ denotes the right regular representation. But this means that the trivial representation is weakly contained in ρ, which is equivalent to amenability of π by Hulanciki's Theorem (see [34], Theorem 3.5.2, or [67], Theorem 4.21).

As pointed out to me by my colleague Jim Schafer, this combinatorial version of Brooks' Theorem is essentially equivalent to a classic theorem of Kesten on random walks on discrete groups [52].

Exercise 1.4.6 Deduce from the Cheeger-Gromov Theorem and Exercise 1.4.4 that if X is a finite aspherical CW complex with nontrivial amenable fundamental group, then $\chi(X) = 0$. See [81] and [86] for the history of results like this one.

Chapter 2

Von Neumann Algebra Index Theorems: Atiyah's L^2-Index Theorem and Connes' Index Theorem for Foliations

2.1 Atiyah's L^2-Index Theorem

As we saw in the last chapter, it is not always so easy to compute all of the L^2-Betti numbers of a space directly from the definition, though sometimes we can compute *some* of them. It would be nice to have constraints from which we could then determine the others. Such a constraint, and more, is provided by the following index theorem. The context, as with many index theorems, is that of linear elliptic pseudodifferential operators. The reader who doesn't know what these are exactly can think of the differential operator $d + d^*$ on a Riemannian manifold, or of the operator $\overline{\partial} + \overline{\partial}^*$ on a Kähler manifold. These special cases are fairly typical of the sorts of operators to which the theorem can be applied.

Theorem 2.1.1 (Atiyah [1]**)** *Suppose*

$$D\colon C^\infty(M, E_0) \to C^\infty(M, E_1)$$

is an elliptic pseudodifferential operator (we'll abbreviate this phrase hereafter as ψDO*), acting between sections of two vector bundles E_0 and E_1 over a closed* *manifold M, and $\widetilde{M}$ is a normal covering of M with covering group π. Let*

$$\widetilde{D}\colon C^\infty(\widetilde{M},\widetilde{E}_0)\to C^\infty(\widetilde{M},\widetilde{E}_1)$$

be the lift of D to $\widetilde{M}$. Then $\ker\widetilde{D}$ *and* $\ker\widetilde{D}^*$ *have finite τ-dimension, and*

$$\begin{aligned}&\operatorname{Ind} D(=\dim\ker D-\dim\ker D^*)\\ &=L^2\text{-}\operatorname{Ind}\widetilde{D}(=\dim_\tau\ker\widetilde{D}-\dim_\tau\ker\widetilde{D}^*).\end{aligned}$$

Sketch of Proof. For simplicity take D to be a first-order differential operator, and consider the formally self-adjoint operator

$$P=\begin{pmatrix}0 & D^*\\ D & 0\end{pmatrix}$$

acting on sections of $E=E_0\oplus E_1$. Since D is elliptic, PDE theory shows that the solution of the "heat equation" for P, $H_t=\exp(-tP^2)$, is a *smoothing operator*, an integral operator with smooth kernel, for $t>0$. And as $t\to\infty$, $H_t\to$ projection on $\ker D\oplus\ker D^*$, so that if

$$\gamma=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=\begin{cases}1 & \text{on } E_0\\ -1 & \text{on } E_1,\end{cases}$$

then γ commutes with H_t and $\operatorname{Ind} D=\lim_{t\to\infty}\operatorname{Tr}(\gamma H_t)$.

Define similarly

$$\widetilde{P}=\begin{pmatrix}0 & \widetilde{D}^*\\ \widetilde{D} & 0\end{pmatrix},\quad \widetilde{\gamma}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad \widetilde{H}=e^{-t\widetilde{P}^2},$$

acting on sections of $\widetilde{E}=\widetilde{E}_0\oplus\widetilde{E}_1$. Then $L^2\text{-}\operatorname{Ind}\widetilde{D}=\lim_{t\to\infty}\tau\left(\widetilde{\gamma}\widetilde{H}_t\right)$.

Here we extend τ to matrices over the group von Neumann algebra in the obvious way. So we just need to show that

$$\operatorname{Tr}(\gamma H_t)=\tau\left(\widetilde{\gamma}\widetilde{H}_t\right). \tag{2.1}$$

Now in fact both sides of (2.1) are constant in t, since, for instance,

$$\begin{aligned}\frac{d}{dt}\operatorname{Tr}(\gamma H_t)&=\frac{d}{dt}\operatorname{Tr}\left(\gamma e^{-tP^2}\right)=\operatorname{Tr}\frac{d}{dt}\left(\gamma e^{-tP^2}\right)=\\ &=\operatorname{Tr}\left(\frac{d}{dt}e^{-tD^*D}-\frac{d}{dt}e^{-tDD^*}\right)\\ &=\operatorname{Tr}\left(DD^*e^{-tDD^*}-D^*De^{-tD^*D}\right).\end{aligned}$$

But

$$\begin{aligned}
\operatorname{Tr}\left(DD^*e^{-tDD^*}\right) &= \operatorname{Tr}\left(\overbrace{e^{-tDD^*/2}D}\,\overbrace{D^*e^{-tDD^*/2}}\right)\\
&= \operatorname{Tr}\left(D^*e^{-tDD^*/2}e^{-tDD^*/2}D\right)\\
&= \operatorname{Tr}\left(D^*e^{-tDD^*}D\right)\\
&= \operatorname{Tr}\left(D^*\left(1 - tDD^* + \cdots\right)D\right)\\
&= \operatorname{Tr}\left(D^*D\left(1 - tD^*D + \cdots\right)\right)\\
&= \operatorname{Tr}\left(D^*De^{-tD^*D}\right).
\end{aligned}$$

So it's enough to show that

$$\lim_{t\to 0^+}\left(\operatorname{Tr}\left(\gamma H_t\right) - \tau\left(\widetilde{\gamma}\widetilde{H}_t\right)\right) = 0.$$

But for small t, the solution of the heat equation is almost local. (See [74] for further explanation.) In other words, H_t and $\widetilde{H}_t$ are given by integration against smooth kernels almost concentrated on the diagonal, and the kernel $\widetilde{k}$ for $\widetilde{H}_t$ is practically the lift of the kernel k for H_t, since, locally, M and $\widetilde{M}$ look the same. But for a π-invariant operator $\widetilde{S}$ on $\widetilde{E}$, obtained by lifting the kernel function k for a smoothing operator on M to a kernel function to $\widetilde{k}$, one can check that

$$\begin{aligned}
\tau(\widetilde{S}) &= \int_F \widetilde{k}(\widetilde{x},\widetilde{x})\,d\operatorname{vol}(\widetilde{x})\\
&= \int_M k(x,x)\,d\operatorname{vol}(x)\\
&= \operatorname{Tr}(S),
\end{aligned}$$

F a fundamental domain for the action of π on $\widetilde{M}$. So that does it. $\square$

For applications to L^2-Betti numbers, we can fix a Riemannian metric on M and take $E_0 = \bigoplus \Omega^{2i}$, $E_1 = \bigoplus \Omega^{2i+1}$, D the "Euler characteristic operator" $D = d + d^*$, so $\operatorname{Ind} D = \chi(M)$ by the Hodge Theorem, while L^2-$\operatorname{Ind}\widetilde{D}$ is the alternating sum of the L^2-Betti numbers, $\sum(-1)^i\beta_i$. Thus we obtain an analytic proof of the equality $\sum(-1)^i\beta_i = \chi(M)$, for which a combinatorial proof was given in Exercise 1.4.4.

Another application comes from taking M closed, connected, and oriented, of dimension $4k$. Then harmonic forms in the middle degree $2k$ can be split into ± 1 eigenspaces for the Hodge $*$-operator, and so the middle Betti number b_{2k} splits as $b_{2k}^+ + b_{2k}^-$. The *signature* of M can be defined to be the difference $b_{2k}^+ - b_{2k}^-$. This can be identified with the signature of the intersection pairing

$$\langle x,\, y\rangle = \langle x \cup y,\, [M]\rangle$$

on $H^{2k}(M, \mathbb{R})$, since if we represent cohomology classes x and y by closed forms φ and ψ, then $\langle x, y\rangle = \int_M \varphi \wedge \psi$, while $\int_M \varphi \wedge *\psi$ is the L^2 inner product of φ and ψ, so that the intersection pairing is positive definite on the $+1$ eigenspace of $*$ and negative definite on the -1 eigenspace of $*$.

Now as observed by Atiyah and Singer, the signature can also be computed as the index of the elliptic differential operator $D = d + d^*$ sending E_0 to E_1, where $E_0 \oplus E_1$ is a splitting of the complex differential forms defined using the Hodge $*$-operator as well as the grading by degree. More precisely, E_0 and E_1 are the ± 1 eigenspaces of the involution τ sending a complex-valued p-form ω to $i^{p(p-1)+2k} * \omega$. This formula is concocted so that the contributions of p-forms and $(4k-p)$-forms will cancel out as long as $p \neq 2k$, and so that $\tau = *$ on forms of middle degree. If we apply Theorem 2.1.1 to this D, we see that $\beta_{2k}^+ - \beta_{2k}^- = b_{2k}^+ - b_{2k}^-$, with the splitting of β_{2k} into ± 1 eigenspaces of $*$ defined similarly. (Once again, the contributions from forms of other degree cancel out.)

Example 2.1.2 Let M be a compact quotient of the unit ball $\widetilde{M}$ in $\mathbb{C}^2$. Then $\widetilde{M}$ can be identified with the homogeneous space G/K, where $G = SU(2,1)$ and K is its maximal compact subgroup $U(2)$. The signature of M must be nonzero by the "Hirzebruch proportionality principle,"[1] since G/K is the noncompact dual of the compact symmetric space $\mathbb{CP}^2$, which has signature 1. Hence the L^2-Betti number β_2 of M must be non-zero by the identity $\beta_2^+ - \beta_2^- = \operatorname{sign} M$. In this case, we have $\beta_0 = \beta_4 = 0$ by Brooks' Theorem (Theorem 1.3.5 and Exercise 1.4.5) and Poincaré duality, since the fundamental group of M is a lattice in G and is thus non-amenable. And in addition, $\beta_1 = \beta_3 = 0$ by Theorem 1.3.6, so as pointed out before, one has $\beta_2 = \chi(M)$. Together with the identities $\beta_2^+ - \beta_2^- = \operatorname{sign} M$ and $\beta_2^+ + \beta_2^- = \beta_2$, this makes it possible to compute $\beta_2^{\pm}$ exactly. (Note: for this example, vanishing of the L^2-cohomology in dimensions $\neq 2$ can also be proved using the representation theory of G, as in Example 1.3.4.)

2.2 Connes' Index Theorem for Foliations

Another important application to topology of finite von Neumann algebras is Connes' index theorem for tangentially elliptic operators on foliations with an invariant transverse measure.

2.2.1 Prerequisites

We begin by reviewing a few facts about foliations. A *foliation* $\mathcal{F}$ of a compact smooth manifold M^n is a partition of M into (not necessarily closed) connected

[1] This principle asserts that the characteristic numbers of M must be proportional to those of the compact dual symmetric space $\mathbb{CP}^2$, and thus the signature of M is nonzero since the signature of $\mathbb{CP}^2$ is nonzero. The logic behind this is that characteristic numbers are computed from integrals of universal polynomials in the curvature forms, and these forms are determined by the structure of the Lie algebra of G, hence agree for the compact and non-compact symmetric spaces except for a sign.

submanifolds L^p called *leaves*, all of some fixed dimension p and codimension $q = n - p$.[2] The leaves are required to be the integral submanifolds of some integrable subbundle of TM, which we identify with $\mathcal{F}$ itself. *Locally*, M looks like $L^p \times \mathbb{R}^q$, but it can easily happen that every leaf is dense. See Figure 2.1. When x and y lie on the same leaf, "sliding along the leaf" along a path in the leaf from x to y gives a germ of homeomorphisms from a transversal to the foliation at x to a transversal to the foliation at y, which is called the *holonomy*. This holonomy only depends on the homotopy class of the path chosen from x to y, and so defines a certain connected cover of the leaf, called the *holonomy cover*, which is trivial if the leaf is simply connected.

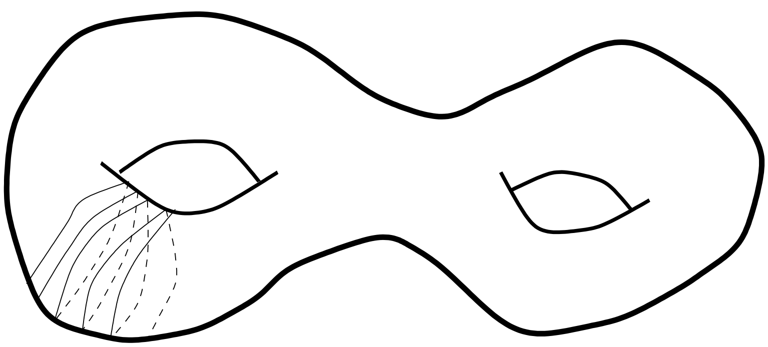

Figure 2.1: Schematic picture of a piece of a typical foliation

For purposes of Connes' index theorem we will need to do a sort of integration over the "space of leaves $M/\mathcal{F}$," even though this space may not even be T_0, let alone Hausdorff. So we will assume $(M, \mathcal{F})$ has an *invariant transverse measure* μ. This is a map $\mu\colon (T \looparrowright M) \mapsto \mu(T)$ from (immersed) q-dimensional submanifolds of M with compact closure, transverse to the leaves of $\mathcal{F}$, to the reals. It is required to satisfy countable additivity as well as the invariance property, that μ assigns the same value to every pair of transversals $T_1, T_2 \hookrightarrow M$ which are obtained from one another by a holonomy transformation. When the foliation $\mathcal{F}$ is a fibration $L^p \to M^n \to B^q$, where the base B can be identified with the space of leaves, then an invariant transverse measure μ is simply a measure on B. If the leaves are consistently oriented, then given a p-form on M, we can integrate it over the leaves, getting a function on the base B, and then integrate against μ. More generally, without any conditions on $\mathcal{F}$ except that it be orientable, an invariant transverse measure μ defines a closed *Ruelle-Sullivan current* [84] C_μ on M of dimension p. To review, a p-current is to a differential form of degree p what a distribution is to a function; it is a linear functional on (compactly supported) p-forms. The current C_μ is defined as follows: on a small open subset of M diffeomorphic to $D^p \times D^q$ (with $\mathcal{F}$ tangent to the

[2] Admittedly, there is a problem with the notation here; it seems to imply that the leaves are all diffeomorphic to one another, but this is not necessarily the case.

subsets $D^p \times \{\mathrm{pt}\}$), given a p-form ω supported in this set, one has

$$\langle C_\mu, \omega \rangle = \int \Big(\int_{D^p \times \{x\}} \omega \Big) \, d\mu(x).$$

There is a differential ∂ on currents dual to the exterior differential d on forms, and since C_μ is basically just a smeared out version of integration along the leaves, one immediately sees that $\partial C_\mu = 0$, so that C_μ defines a de Rham homology class $[C_\mu]$ in $H_p(M, \mathbb{R})$.

From the data M, $\mathcal{F}$, and μ, one can construct (see [17], [16], and [18]) a finite von Neumann algebra $A = W^*(M, \mathcal{F})$ with a trace τ coming from μ. Let's quickly review how this is constructed. When the holonomy covers of the leaves of $\mathcal{F}$ are trivial—for instance, when all leaves are simply connected—consider the graph G of the equivalence relation $\sim$ on M of "being on the same leaf." In other words, $G = \{(x, y) \in M \times M \mid x \text{ and } y \text{ on the same leaf}\}$. Note that G can be identified with a (possibly noncompact) manifold of dimension $n+p = q+2p$. The algebra A is then the completion of the convolution algebra of functions (or to be more canonical, half-densities) on G of compact support, for the action of this algebra on a suitable Hilbert space defined by μ. The construction in the general case is similar, except that we replace the graph of $\sim$ by the *holonomy groupoid* G, consisting of triples $(x, y, [\gamma])$ with x and y on the same leaf and $[\gamma]$ a class of paths from x to y all with the same holonomy.[3] (In fact usually this nicety doesn't matter much in the von Neumann algebra context since leaves for which the holonomy cover is trivial are "generic"—see [11], Theorem 2.3.12.)

Now suppose there is a differential operator D on M which only involves differentiation in directions tangent to the leaves and is elliptic when restricted to any leaf. (Examples: the Euler characteristic operator or the Dirac operator "along the leaves.") Such an operator is called *tangentially elliptic*. Since the leaves are usually not compact, we can't compute an index for the restriction of D to one leaf. But since M, the union of the leaves, is compact, it turns out one can make sense of a numerical index $\operatorname{Ind}_\tau D$ for D. In the special case where $\mathcal{F}$ has closed leaves, the foliation is a fibration $L^p \to M \xrightarrow{\text{proj}} X^q$, and μ is a probability measure on X, this reduces to $\operatorname{Ind}_\tau D = \int_X \operatorname{Ind}(D|_{L_x}) \, d\,\mu(x)$, where $L_x = \operatorname{proj}^{-1}(x)$. In general, $\operatorname{Ind}_\tau D$ is roughly the "average with respect to μ" of the L^2-index of $D|_{L_x}$, as x runs over the "space of leaves." Here we give each leaf the Riemannian structure defined by a choice of metric on the bundle $\mathcal{F}$.

Example 2.2.1 Let M_1 and M_2 be compact connected manifolds, and let π be the fundamental group of M_2. If π acts on $M_1 \times \widetilde{M_2}$ with trivial action on the first factor and the usual action on the second factor, then the quotient is $M_1 \times (\widetilde{M_2}/\pi) = M_1 \times M_2$. But suppose we take *any* action of π on M_1 and then take the diagonal action of π on $M_1 \times \widetilde{M_2}$. Then $M = (M_1 \times \widetilde{M_2})/\pi$ is compact,

[3] One has to be a little careful what one means by this when G is non-Hausdorff, but the general idea is still the same even in this case.

and projection to the second factor gives a fibration onto M_2 with fiber M_1. But M is also foliated by the images of $\{x\} \times \widetilde{M}_2$, usually non-compact. A measure μ on M_1 invariant under the action of π is an invariant transverse measure for this foliation $\mathcal{F}$. If D is the Euler characteristic operator along the leaves and all the leaves are $\cong \widetilde{M}_2$, then $\operatorname{Ind}_\tau D$ just becomes the average L^2-Euler characteristic of $\widetilde{M}$, the alternating sum of the L^2-Betti numbers, and Connes' Theorem will reduce to Atiyah's.

2.2.2 Connes' Theorem

Theorem 2.2.2 (Connes [14], [17]**)** *Let $(M, \mathcal{F})$ be a compact foliated manifold with an invariant transverse measure μ, and let $W^*(M, \mathcal{F})$ be the associated von Neumann algebra with trace τ coming from μ. Let*

$$D \colon C^\infty(M, E_0) \to C^\infty(M, E_1)$$

be elliptic along the leaves. Then the L^2 kernels of

$$P = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

on the various leaves assemble to a (graded) Hilbert $W^*(M, \mathcal{F})$-module $K_0 \oplus K_1$, and

$$\operatorname{Ind}_\tau D = \dim_\tau K_0 - \dim_\tau K_1 = \int \operatorname{Ind}_{\text{top}} \sigma(D) \, d\mu,$$

where $\sigma(D)$ denotes the symbol of D and the "topological index" $\operatorname{Ind}_{\text{top}}$ is computed from the characteristic classes of $\sigma(D)$ just as in the usual Atiyah-Singer index theorem.

We omit the proof, which is rather complicated if one puts in all the details (but see [14], [17], and [64]). However, the basic outline of the proof is similar to that for Theorem 2.1.1, except that one must replace the group von Neumann algebra by the von Neumann algebra of the measured foliation.

2.3 An Application to Uniformization

If we specialize the Connes index theorem to the Euler characteristic operator along the leaves for foliations with 2-dimensional leaves, it reduces to:

Theorem 2.3.1 (Connes) *Let $(M, \mathcal{F})$ be a compact foliated manifold with 2-dimensional leaves and $\mathcal{F}$ oriented. Then for every invariant transverse measure μ, the μ-average of the L^2-Euler characteristic of the leaves is equal to $\langle e(\mathcal{F}), [C_\mu] \rangle$, where $e(\mathcal{F}) \in H^2(M, \mathbb{Z})$ is the Euler class of the oriented 2-plane bundle associated to $\mathcal{F}$, and $[C_\mu] \in H_2(M, \mathbb{R})$ is the Ruelle-Sullivan class attached to μ.*

The result also generalizes to compact *laminations* with 2-dimensional leaves. (That means we replace M by any compact Hausdorff space X locally of the form $\mathbb{R}^2 \times T$, where T is allowed to vary.) The only difference in this case is that we have to use *tangential* de Rham theory. This variant of Connes' Theorem is explained in [64].

Corollary 2.3.2 *Suppose $(X, \mathcal{F})$ is a compact laminated space with 2-dimensional oriented leaves and a smooth Riemannian metric g. Let ω be the curvature 2-form of g. If there is an invariant transverse measure μ with $\langle[\omega], [C_\mu]\rangle > 0$, then $\mathcal{F}$ has a set of closed leaves of positive μ-measure. If there is an invariant transverse measure μ with $\langle[\omega], [C_\mu]\rangle < 0$, then $\mathcal{F}$ has a set of (conformally) hyperbolic leaves of positive μ-measure. If all the leaves are (conformally) parabolic, then $\langle[\omega], [C_\mu]\rangle = 0$ for every invariant transverse measure.*

Proof. By Chern-Weil theory, the de Rham class of $\frac{\omega}{2\pi}$ represents the Euler class of $\mathcal{F}$. So by Theorem 2.3.1, $\langle[\omega], [C_\mu]\rangle$ is the μ-average of the L^2-Euler characteristic of the leaves. The only oriented 2-manifold with positive L^2-Euler characteristic is S^2. Every hyperbolic Riemann surface has negative L^2-Euler characteristic. And every parabolic Riemann surface (one covered by $\mathbb{C}$ with the flat metric) has vanishing L^2-Euler characteristic. $\square$

This has been used in:

Theorem 2.3.3 (Ghys [33]) *Under the hypotheses of Corollary 2.3.2, if every leaf is parabolic, then $(X, \mathcal{F}, g)$ is* approximately uniformizable, *i.e., there are real-valued functions u_n (smooth on the leaves) with the curvature form of $e^{u_n} g$ tending uniformly to 0.*

Note incidentally that the reason for using the curvature *form* here, as opposed to the Gaussian curvature, is that the form, unlike the Gaussian curvature, is invariant under rescaling of the metric by a constant factor.

Sketch of Proof. The proof depends on two facts about 2-dimensional Riemannian geometry. First of all, if g is a metric on a surface, and if K is its curvature, then changing g to the conformal metric $e^u g$ changes the curvature form $K\, d\mathrm{vol}_g$ to $K' d\mathrm{vol}_{e^u g} = (K - \Delta u)\, d\mathrm{vol}_g$, where Δ is the Laplacian (normalized to be a *negative* operator). So if K is the curvature function for the lamination and $\Delta^{\mathcal{F}}$ is the leafwise Laplacian, it's enough to show that K is in the uniform closure of functions of the form $\Delta^{\mathcal{F}}(u)$. (For then if $\Delta^{\mathcal{F}}(u_n) \to K$, the curvature forms of $e^{u_n} g$ tend to 0.)

The second fact we need is that there exist *harmonic measures* ν on X, that is, measures with the property that ν annihilates all functions of the form $\Delta^{\mathcal{F}}(u)$, and that a function lies in the closure of functions of the form $\Delta^{\mathcal{F}}(u)$ if and only if it is annihilated by the harmonic measures. Indeed, by the Hahn-Banach Theorem, the uniform closure of the functions of the form $\Delta^{\mathcal{F}}(u)$ is exactly the set of functions annihilated by measures ν with $\int_X \Delta^{\mathcal{F}}(u)\, d\nu = 0$ for all leafwise smooth functions u. But on a subset of X of the form $U \times T$, where U is an open subset of a leaf, such measures consist exactly of integrals (with respect to some measure on T) of measures of the form $h(x)\, d\mathrm{vol}(x)$ on each leaf, with h a harmonic function. Thus, in the case where all the leaves are parabolic, it turns out (since there are no nonconstant positive harmonic functions on $\mathbb{C}$) that harmonic measures are just obtained by integrating the leafwise area measure with respect to an invariant transverse measure. Since $\langle K\, d\mathrm{vol}_g, [C_\mu]\rangle = 0$ for every invariant transverse measure by Corollary 2.3.2, the result follows. $\square$

Another known fact is:

Theorem 2.3.4 (Candel [10]**)** *Under the hypotheses of Corollary 2.3.2, if every leaf is hyperbolic, then* $(X, \mathcal{F}, g)$ *is* uniformizable, *i.e., there is a real-valued function* u (*smooth along the leaves*) *with* $e^u g$ *hyperbolic on each leaf.*

2.4 Exercises

Exercise 2.4.1 Let M be a compact Kähler manifold of complex dimension n. Then each Betti number $b_r(M)$ splits as $b_r(M) = \sum_{p+q=r} h^{p,q}(M)$, where the Hodge number $h^{p,q}(M)$ is the dimension of the part of the de Rham cohomology in dimension r coming from forms of type (p, q) (i.e., locally of the form $f\, dz_1 \wedge \cdots \wedge dz_p \wedge d\overline{z_{p+1}} \wedge \cdots \wedge d\overline{z_{p+q}}$). And $\sum_q (-1)^q h^{0,q}(M)$, the index of the operator $\overline{\partial} + \overline{\partial}^*$ on forms of type $(0, *)$, graded by parity of the degree, turns out to be given by the *Todd genus* $\mathrm{Td}(M)$ [41]. For example, if M is a compact Riemann surface (Kähler manifold of complex dimension 1) of genus g, then $h^{1,0}(M) = h^{0,1}(M) = g$ and $\mathrm{Td}(M) = 1 - g$. Apply the Atiyah L^2-index theorem and see what it says about the L^2 Hodge numbers (associated to the universal cover). For example, compute the L^2 Hodge numbers when M is a compact Riemann surface of genus $g \geq 1$ (see Example 1.3.4).

Exercise 2.4.2 Part of the idea for this problem comes from [2] and [3], though we have been able to simplify things considerably by restricting to the easiest special case. Suppose $G = SL(2, \mathbb{R})$ and $K = SO(2)$. Then attached to each character of K, which we can think of as being given by an integer parameter n by $e^{i\theta} \mapsto e^{in\theta}$, $e^{i\theta} \in SO(2) \cong S^1$, is a homogeneous holomorphic line bundle $\widetilde{L}_n$ over $\widetilde{M} = G/K$. Let π be a discrete cocompact subgroup of G, so that $G/K \twoheadrightarrow \pi\backslash G/K$ is the universal cover of a compact Riemann surface M of genus $g > 1$. Note that $\widetilde{L}_n$ descends in a natural way to a holomorphic line bundle L_n over M. Apply the L^2-index theorem, together with the classical Riemann-Roch Theorem for L_n, to compute the L^2-index of the $\overline{\partial}$ operator on the line bundle $\widetilde{L}_n$.

Then combine this result with a vanishing theorem to show that $\widetilde{L}_n$ has L^2 holomorphic sections (with respect to the G-invariant measure on G/K) if and only if $n \geq 2$. Here is a sketch of the proof of the vanishing theorem. Let $\mathfrak{g}$ and $\mathfrak{k}$ be the complexified Lie algebras of G and of K, respectively, and suppose the Hilbert space $\mathcal{H}_n$ of L^2 holomorphic sections of L_n is non-zero. We have a splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \oplus \overline{\mathfrak{p}}$, with $\mathfrak{p}$ corresponding to the holomorphic tangent space of G/K, and $\overline{\mathfrak{p}}$ corresponding to the antiholomorphic tangent space. Also, $\mathfrak{k}$ is a Cartan subalgebra of $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbb{C})$ and $\mathfrak{p}$, $\overline{\mathfrak{p}}$ are its root spaces. By definition, we have

$$\mathcal{H}_n = \{f \in L^2(G) : f(gk) = k^{-n} f(g),\, g \in G,\ k \in K \cong S^1,\ d\rho(\overline{\mathfrak{p}})f = 0\}.$$

Here ρ is the right regular representation, and G acts on $\mathcal{H}_n$ by the left regular representation λ. Then $\mathcal{H}_n$ carries a unitary representation of G and (because of the Cauchy integral formula) must be a "reproducing kernel" Hilbert space; that is, there must be a distinguished vector $\xi \in \mathcal{H}_n$ such that $\langle s, \xi \rangle = s(e)$ for all $s \in \mathcal{H}_n$. (Here e denotes the identity element of G, which corresponds to $0 \in \mathbb{C}$ in the unit disk model of G/K.) Since $\mathcal{H}_n \neq 0$, ξ can't vanish. It turns out that $\xi \in \mathcal{H}_n$ is a "lowest weight vector," an eigenvector for $\mathfrak{k}$ (corresponding to the character $e^{i\theta} \mapsto e^{in\theta}$ of K) that is killed by $\overline{\mathfrak{p}}$. This determines the action of $\mathfrak{g}$, hence of G, on ξ, and the vanishing theorem is deduced from the requirement that ξ lie in L^2. (See for example [53] for more details.)

Exercise 2.4.3 Let M be a compact manifold with $H^2(M, \mathbb{R}) = 0$, and suppose M admits a foliation with 2-dimensional leaves and an invariant transverse measure. Deduce from Corollary 2.3.2 that the "average L^2 Euler characteristic" of the leaves must vanish, and in particular, that the leaves cannot all be hyperbolic (uniformized by the unit disk). (Compare the combination of Theorems 12.3.1 and 12.5.1 in [11].)

Exercise 2.4.4 (Connes [17], §4**)** Let Λ_1 and Λ_2 be lattices in $\mathbb{C}$ (that is, discrete subgroups each of rank 2) and assume that $\Lambda_1 \cap \Lambda_2 = \emptyset$. Consider the 4-torus $M = (\mathbb{C}/\Lambda_1) \times (\mathbb{C}/\Lambda_2)$ and let p_j, $j = 1, 2$ be the projection of M onto $\mathbb{C}/\Lambda_j$. Fix points $z_1, z_2 \in \mathbb{C}$ and let E_1 and E_2 be the holomorphic line bundles on $(\mathbb{C}/\Lambda_j)$ attached to the divisors $-[z_1]$ and $[z_2]$, respectively. Then let $E = p_1^*(E_1) \otimes p_2^*(E_2)$. Consider the foliation $\mathcal{F}$ of M obtained by pushing down the foliation of the universal cover $\mathbb{C}^2$ by the complex planes $\{(z, w + z) : z \in \mathbb{C}\}$, $w \in \mathbb{C}$. The leaves of $\mathcal{F}$ may be identified with copies of $\mathbb{C}$. Since this foliation is linear, it has a transverse measure given by Haar measure on a 2-torus transverse to the leaves of $\mathcal{F}$. Let D be the $\overline{\partial}$ operator along the leaves, acting on E. Then on a leaf $L = \mathrm{im}\{(z, w + z) : z \in \mathbb{C}\}$, a holomorphic section of E can be identified with a meromorphic function on $\mathbb{C}$ with all its poles simple and contained in $z + \Lambda_1$ for some z and with zeros at points of Λ_2. Apply the foliation index theorem to deduce an existence result about such meromorphic functions in L^2.

Chapter 3

Group C^*-Algebras, the Mishchenko-Fomenko Index Theorem, and Applications to Topology

3.1 The Mishchenko-Fomenko Index

The last two chapters have been about applications of von Neumann algebras to topology. In this chapter, we start to talk about applications of C^*-algebras. First we recall that a (complex) commutative C^*-algebra is always of the form $C_0(Y)$, where Y is a locally compact (Hausdorff) space, so that the study of (complex) commutative C^*-algebras is equivalent to the study of locally compact spaces. Real commutative C^*-algebras are only a bit more complicated; they correspond to locally compact spaces (associated to the complexification) together with an involutive homeomorphism (associated to the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$). These were called *Real spaces* by Atiyah. We also recall that by Swan's Theorem (see for example [47], Theorem I.6.18), the sections of a vector bundle over a compact Hausdorff space X are a finitely generated projective module over $C(X)$, and conversely.

Definition 3.1.1 Let A be a C^*-algebra (over $\mathbb{R}$ or $\mathbb{C}$) with unit, and let X be a compact space. An *A-vector bundle* over X will mean a locally trivial bundle over X whose fibers are finitely generated projective (right) A-modules, with A-linear transition functions.

Example 3.1.2 If $A = \mathbb{R}$ or $\mathbb{C}$, an A-vector bundle is just a usual vector bundle. If $A = C(Y)$, an A-vector bundle over X is equivalent to an ordinary vector bundle over $X \times Y$. This is proved by the same method as Swan's Theorem, to which the statement reduces if X is just a point.

Definition 3.1.3 Let A be a C^*-algebra and let E_0, E_1 be A-vector bundles over a compact manifold M. An *A-elliptic operator*

$$D\colon C^\infty(M, E_0) \to C^\infty(M, E_1)$$

will mean an elliptic A-linear ψDO from sections of E_0 to sections of E_1. Such an operator extends to a bounded A-linear map on suitable Sobolev spaces (Hilbert A-modules) $\mathcal{H}_0$ and $\mathcal{H}_1$. One can show [62] that this map is an A-Fredholm operator, i.e., one can find a decomposition

$$\begin{gathered}\mathcal{H}_0 = \mathcal{H}_0' \oplus \mathcal{H}_0'', \quad \mathcal{H}_1 = \mathcal{H}_1' \oplus \mathcal{H}_1'', \\ \mathcal{H}_0'' \text{ and } \mathcal{H}_1'' \text{ finitely generated projective}, \\ D\colon \mathcal{H}_0' \xrightarrow{\cong} \mathcal{H}_1', \quad D\colon \mathcal{H}_0'' \to \mathcal{H}_1''.\end{gathered}$$

This means that "up to A-compact perturbation" the kernel and cokernel of D are finitely generated projective A-modules. The *Mishchenko-Fomenko index* of D (see [62]) is

$$\operatorname{Ind} D = [\mathcal{H}_0''] - [\mathcal{H}_1''],$$

computed in the group of formal differences of isomorphism classes of such modules, $K_0(A)$. (See Exercise 3.5.1 below.)

3.2 Flat C^*-Algebra Bundles and the Assembly Map

If X is a compact space and A is a C^*-algebra with unit, the group of formal differences of isomorphism classes of A-vector bundles over X is denoted $K^0(X; A)$. The following is analogous to Swan's Theorem.

Proposition 3.2.1 *If X is a compact space and A is a C^*-algebra with unit, then $K^0(X; A)$ is naturally isomorphic to $K_0(C(X) \otimes A)$.*

Sketch of Proof. Suppose E is an A-vector bundle over X. Then the space $\Gamma(X, E)$ of continuous sections of E comes with commuting actions of $C(X)$ and of A. As such, it is a module for the algebraic tensor product; we will show from the local triviality that it is in fact a module for the C^*-tensor product. Now just as in the case of ordinary vector bundles, one shows that E is complemented, i.e., that there is another A-vector bundle F such that $E \oplus F$ is a trivial bundle with fibers that are finitely generated free A-modules. Thus

$$\begin{aligned}\Gamma(X, E) \oplus \Gamma(X, F) &\cong \Gamma(X, E \oplus F) \\ &\cong \Gamma(X, X \times A^n) \cong C(X, A)^n \cong \big(C(X) \otimes A\big)^n\end{aligned}$$

for some n. Hence $\Gamma(X,E)$ is a finitely generated projective $(C(X)\otimes A)$-module. Now it's clear that the Grothendieck groups of A-vector bundles and of finitely generated projective $(C(X)\otimes A)$-modules coincide. $\square$

Definition 3.2.2 Let X be a compact space, $\widetilde{X}\to X$ a normal covering with covering group π. Let $C^*_r(\pi)$ be the reduced group C^*-algebra of π (the completion of the group ring in the operator norm for its action on $L^2(\pi)$). The *universal* $C^*_r(\pi)$*-bundle* over X is

$$\mathcal{V}_X = \widetilde{X}\times_\pi C^*_r(\pi)\to X.$$

This is clearly a $C^*_r(\pi)$-vector bundle over X. As such, by Proposition 3.2.1 it has a class $[\mathcal{V}_X]\in K^0(X;C^*_r(\pi))$, which is pulled back (via the classifying map $X\to B\pi$) from the class of

$$\mathcal{V} = E\pi\times_\pi C^*_r(\pi)\to B\pi$$

in $K^0(B\pi;C^*_r(\pi))$. Here $B\pi$ is the *classifying space* of π, a space (with the homotopy type of a CW complex) having π as its fundamental group, and with contractible universal cover $E\pi$. Such a space always exists and is unique up to homotopy equivalence. Furthermore, by obstruction theory, every normal covering with covering group π is pulled back from the "universal" π-covering $E\pi\to B\pi$. "Slant product" with $[\mathcal{V}]$ (a special case of the Kasparov product) defines the *assembly map*

$$\mathcal{A}\colon K_*(B\pi)\to K_*(C^*_r(\pi)).$$

(There is a slight abuse of notation here. $B\pi$ may not be compact, but it can always be approximated by finite CW complexes. So if there is no finite model for $B\pi$, $K_*(B\pi)$ is to be interpreted as the direct limit of $K_*(X)$ as X runs over the finite subcomplexes of $B\pi$. This direct limit is independent of the choice of a model for $B\pi$.)

Note that since the universal $C^*_r(\pi)$-bundle over X or $B\pi$ is canonically trivialized over the universal cover, it comes with a flat connection, that is, a notion of what it means for a section to be locally flat. (A locally flat section near x is one which in a small evenly covered neighborhood lifts to a constant section $U\to C^*_r(\pi)$.)

3.3 Kasparov Theory and the Index Theorem

The formalism of *Kasparov theory*(see [5] or [40]) attaches, to an elliptic operator D on a manifold M, a K-homology class $[D]\in K_*(M)$. If M is compact, the *collapse map* $c\colon M\to \mathrm{pt}$ is proper and $\operatorname{Ind} D = c_*([D])\in K_*(\mathrm{pt})$.

Now if E is an A-vector bundle over M and D is an elliptic operator over M, we can form "D with coefficients in E," an A-elliptic operator. The Mishchenko-Fomenko index of this operator is computed by pairing

$$[D]\in K_*(M)\quad\text{with}\quad [E]\in K^0(M;A).$$

In particular, if $\widetilde{M} \to M$ is a normal covering of M with covering group π, then we can form D with coefficients in $\mathcal{V}_X$, and its index is $\mathcal{A} \circ u_*([D])$, where $u\colon M \to B\pi$ is the classifying map for the covering.

Conjecture 3.3.1 (Novikov Conjecture) *The assembly map* $\mathcal{A}\colon K_*(B\pi) \to K_*(C_r^*(\pi))$ *is rationally injective for all groups* π*, and is injective for all torsion-free groups* π*.*

This is quite different from the original form of Novikov's conjecture, though it implies it. Therefore Conjecture 3.3.1 is often called the *Strong* Novikov Conjecture. We will see the exact connection with the original form of the conjecture shortly. Stronger than Conjecture 3.3.1 is the *Baum-Connes Conjecture*, which gives a conjectural calculation of $K_*(C_r^*(\pi))$.[1] When π is torsion-free, the Baum-Connes Conjecture amounts to the statement that $\mathcal{A}$ is an isomorphism. There are no known counterexamples to Conjecture 3.3.1, or for that matter to the Baum-Connes Conjecture for discrete groups (though it is known to fail for some groupoids). Conjecture 3.3.1 is known for discrete subgroups of Lie groups ([49], [48]), amenable groups [39], hyperbolic groups [50], and many other classes of groups.

3.4 Applications

1. The L^2-Index Theorem and Integrality of the Trace. The connection with Atiyah's Theorem from Chapter 2 is as follows. Suppose D is an elliptic operator on a compact manifold M, and $\widetilde{M} \to M$ is a normal covering of M with covering group π. The group C^*-algebra $C_r^*(\pi)$ embeds in the group von Neumann algebra, and the trace τ then induces a homomorphism $\tau_*\colon K_0(C_r^*(\pi)) \to \mathbb{R}$. The image under τ_* of the index of D with coefficients in $C_r^*(\pi)$ can be identified with the L^2-index of $\widetilde{D}$, the lift of D to $\widetilde{M}$. Atiyah's Theorem thus becomes the assertion that the following diagram commutes:

$$\begin{tikzcd} K_0(M) \arrow[r, "u_*"] \arrow[d, "c_*"] & K_0(B\pi) \arrow[d, "\mathcal{A}"] \\ K_0(\mathrm{pt}) = \mathbb{Z} \arrow[dr, hook] & K_0(C_r^*(\pi)) \arrow[d, "\tau_*"] \\ & \mathbb{R}. \end{tikzcd}$$

[1] The precise statement is that an assembly map $\mathcal{A}_{\mathrm{BC}}\colon K_*^\pi(\mathcal{E}\pi) \to K_*(C_r^*(\pi))$ is an isomorphism. Here $\mathcal{E}\pi$ is a contractible CW complex on which G acts properly (though not necessarily freely), and K_*^π is equivariant K-homology. When G is torsion-free, $\mathcal{E}\pi = E\pi$, $K_*^\pi(\mathcal{E}\pi) = K_*^\pi(E\pi) = K_*(B\pi)$, and $\mathcal{A}_{\mathrm{BC}} = \mathcal{A}$. In general, one has a π-equivariant map $E\pi \to \mathcal{E}\pi$, and $\mathcal{A}$ factors through $\mathcal{A}_{\mathrm{BC}}$.

This has the consequence, not obvious on its face, that τ_* takes only integral values on the image of the assembly map.[2] Thus if the assembly map is surjective, as when π is torsion-free and the Baum-Connes Conjecture holds for π, then $\tau_*\colon K_0(C^*_r(\pi)) \to \mathbb{R}$ takes only integral values. This in particular implies the *Kaplansky-Kadison Conjecture*, that $C^*_r(\pi)$ has no idempotents other than 0 or 1 [94]. The reason is that if $e = e^2 \in C^*_r(\pi)$, and if $0 \lneqq e \lneqq 1$, then e defines a class in $K_0(C^*_r(\pi))$ and $0 < \tau(e) < 1 = \tau(1)$, contradicting our integrality statement.

2. Original Version of the Novikov Conjecture. Consider the signature operator D on a closed oriented manifold M^{4k}. This is constructed (see page 18) so that $\operatorname{Ind} D$ is the *signature* of M, i.e., the signature of the form

$$\langle x, y\rangle = \langle x \cup y, [M]\rangle$$

on middle cohomology $H^{2k}(M, \mathbb{R})$. The signature is obviously an oriented homotopy invariant, since it only depends on the structure of the cohomology ring (determined by the homotopy type) and on the choice of fundamental class $[M]$ (determined by the orientation). Hirzebruch's formula says $\operatorname{sign} M = \langle \mathcal{L}(M), [M]\rangle$, where $\mathcal{L}(M)$ is a power series in the rational Pontryagin classes, the Poincaré dual of $\operatorname{Ch}[D]$. Here $\operatorname{Ch}\colon K_0(M) \to H_*(M, \mathbb{Q})$ is the Chern character, a natural transformation of homology theories (and in fact a rational isomorphism). The unusual feature of Hirzebruch's formula is that the rational Pontryagin classes, and thus the $\mathcal{L}$-class, are *not* homotopy invariants of M; only the term in $\mathcal{L}(M)$ of degree equal to the dimension of M is a homotopy invariant. For example, it is known from surgery theory how to construct "fake" complex projective spaces homotopy equivalent to $\mathbb{CP}^m$, $m \geq 3$, with wildly varying Pontryagin classes.

If $u\colon M \to B\pi$ for some discrete group π (such as the fundamental group of M), $u_*(\operatorname{Ch}[D]) \in H_*(B\pi, \mathbb{Q})$ is called a *higher signature* of M, and Novikov conjectured that, like the ordinary signature (the case $\pi = 1$), it is an oriented homotopy invariant. The conjecture follows from injectivity of the assembly map, since Kasparov ([49], §9, Theorem 2) and Mishchenko ([63], [61]) showed that $\mathcal{A} \circ u_*([D])$ is an oriented homotopy invariant. Another proof of the homotopy invariance of $\mathcal{A} \circ u_*([D])$ may be found in [46]. For much more on the background and history of the Novikov Conjecture, see [30].

3. Positive Scalar Curvature. An oriented Riemannian manifold M^n has a natural principal $SO(n)$-bundle attached to it, the (oriented) *orthonormal frame bundle*, $P \to M$. The fiber of P over any point $x \in M$ is by definition the set of oriented orthonormal bases for the tangent space T_xM, and $SO(n)$ acts simply transitively on this set. Now $SO(n)$ has a double cover $\operatorname{Spin}(n)$ (which if $n \geq 3$ is also the universal cover), and a lifting of $P \to M$ to a principal $\operatorname{Spin}(n)$-bundle $\widehat{P} \to M$ is called a *spin structure* on M. When M is connected, it is fairly easy to show that such a structure exists if and only if the second

[2] Here we are using the fact that every element of $K_0(B\pi)$ lies in the image of K_0 of *some* manifold M with a map $M \to B\pi$. This can be deduced from "Conner-Floyd type" theorems about the relationship between K-homology and bordism. Of course in the case where $B\pi$ can be chosen to be a compact manifold, this fact is obvious.

Stiefel-Whitney class $w_2(M)$ vanishes in $H^2(M,\mathbb{Z}/2)$, and that $H^1(M,\mathbb{Z}/2)$ acts simply transitively on the set of spin structures (compatible with a fixed choice of orientation) ([54], Chapter II, §2). If M^n is a closed spin manifold, then M carries a special first-order elliptic operator, the [$\mathrm{Cliff}_n(\mathbb{R})$-linear] *Dirac operator* D ([54], Chapter II, §7), with a class $[D] \in KO_n(M)$. The operator D depends on a choice of Riemannian metric, though its K-homology class is independent of the choice. Lichnerowicz [55] proved that

$$D^2 = \nabla^*\nabla + \frac{\kappa}{4}, \tag{3.1}$$

where κ is the scalar curvature of the metric. Thus if $\kappa > 0$, the spectrum of D is bounded away from 0 and $\operatorname{Ind} D = 0$ in

$$KO_n(\mathrm{pt}) = \begin{cases} \mathbb{Z}, & n \equiv 0 \bmod 4, \\ \mathbb{Z}/2, & n \equiv 1 \text{ or } 2 \bmod 8, \\ 0, & \text{otherwise.} \end{cases}$$

Gromov and Lawson [35] established the fundamental tools for proving a partial converse to this statement. Their work was completed by Stolz, who proved:

Theorem 3.4.1 (Stolz [91]**)** *If M^n is a closed simply connected spin manifold with Dirac operator class $[D] \in KO_n(M)$, and if $n \geq 5$, then M admits a metric of positive scalar curvature if and only if* $\operatorname{Ind} D = 0$ *in* $KO_n(\mathrm{pt})$.

What if M is not simply connected? Then Gromov-Lawson ([36], [37]) and Schoen-Yau ([88], [87], [89]) showed there are other obstructions coming from the fundamental group, and Gromov-Lawson suggested that the "higher index" of D is responsible. Rosenberg ([75], [76], [80]) then pointed out that the Mishchenko-Fomenko Index Theorem is an ideal tool for verifying this.

Theorem 3.4.2 (Rosenberg) *Suppose M is a closed spin manifold and $u\colon M \to B\pi$ classifies the universal cover of M. If M admits a metric of positive scalar curvature and if the* (*strong*) *Novikov Conjecture holds for π, then $u_*([D]) = 0$ in $KO_n(B\pi)$.*

Sketch of Proof. Suppose M admits a metric of positive scalar curvature. Consider the Dirac operator $D_{\mathcal{V}}$ with coefficients in the universal $C^*_r(\pi)$-bundle $\mathcal{V}_M$. As we remarked earlier, the bundle $\mathcal{V}_M$ has a natural flat connection. If we use this connection to define $D_{\mathcal{V}}$, then Lichnerowicz's identity (3.1) will still hold with $D_{\mathcal{V}}$ in place of D, since there is no contribution from curvature of the bundle. Thus $\kappa > 0$ implies $\operatorname{Ind} D_{\mathcal{V}} = \mathcal{A}(u_*([D])) = 0$. Thus if $\mathcal{A}$ is injective, we can conclude that $u_*([D]) = 0$ in $KO_n(B\pi)$. □

For some torsion-free groups, the converse is known to hold for $n \geq 5$, generalizing Theorem 3.4.1. See [80] for details.

Conjecture 3.4.3 (Gromov-Lawson) *A closed aspherical manifold cannot admit a metric of positive scalar curvature.*

Theorem 3.4.2 shows that the Strong Novikov Conjecture implies Conjecture 3.4.3, at least for spin manifolds.

For groups with torsion, the assembly map is usually not a monomorphism (see Exercise 3.5.2), so the converse of Theorem 3.4.2 is quite unlikely. However, for spin manifolds with finite fundamental group, it is possible (as conjectured in [77]) that vanishing of $\mathcal{A}(u_*([D])) = 0$ is necessary and sufficient for positive scalar curvature, at least once the dimension gets to be sufficiently large. Since not much is known about this, it is convenient to simplify the problem by "stabilizing."'

Definition 3.4.4 Fix a simply connected spin manifold J^8 of dimension 8 with $\widehat{A}$-genus 1. (Such a manifold is known to exist, and Joyce [43] constructed an explicit example with Spin(7) holonomy.) Taking a product with J does not change the KO-index of the Dirac operator. Say that a manifold M *stably admits a metric of positive scalar curvature* if there is a metric on $M \times J \times \cdots \times J$ with positive scalar curvature, for sufficiently many J factors. In support of this definition, we have:

Proposition 3.4.5 *A simply connected closed manifold M^n of dimension $n \neq 3, 4$ stably admits a metric of positive scalar curvature if and only if it actually admits a metric of positive scalar curvature.*

Sketch of Proof. We may as well assume $n \geq 5$, since if $n \leq 2$, then M is diffeomorphic to S^2 and certainly has a metric of positive scalar curvature. There are two cases to consider. If M admits a spin structure, then by Theorem 3.4.1, M admits a metric of positive scalar curvature if and only if the index of D vanishes in KO_n. But if the index is non-zero in KO_n, then M does not even stably admit a metric of positive scalar curvature, since the KO_n-index of Dirac is the same for $M \times J \times \cdots \times J$ as it is for M. If M does not admit a spin structure, then Gromov and Lawson [35] showed M always admits a metric of positive scalar curvature, and *a fortiori* it stably admits a metric of positive scalar curvature. □

For finite fundamental group, the best general result is:

Theorem 3.4.6 (Rosenberg-Stolz [79]**)** *Let M^n be a spin manifold with finite fundamental group π, with Dirac operator class $[D]$, and with classifying map $u\colon M \to B\pi$ for the universal cover. Then M stably admits a metric of positive scalar curvature if and only $\mathcal{A} \circ u_*([D]) = 0$ in $KO_n(C_r^*(\pi))$. (Of course, for π finite, $C_r^*(\pi) = \mathbb{R}[\pi]$.)*

This has been generalized by Stolz to those groups π for which the Baum-Connes assembly map $\mathcal{A}_{\mathrm{BC}}$ in KO is injective. This is a fairly large class including all discrete subgroups of Lie groups.

3.5 Exercises

Exercise 3.5.1 (Mishchenko-Fomenko) Let A be a C^*-algebra. Suppose that a bounded A-linear map $D\colon H_0 \to H_1$ between two Hilbert A-modules is

A-Fredholm, i.e., has a decomposition as in Definition 3.1.3. Show that $\operatorname{Ind} D \in K_0(A)$ is well-defined, i.e., does not depend on the choice of decomposition. On the other hand, show by example that it is not necessarily true that D has closed range, and hence it is not necessarily true that we can define $\operatorname{Ind} D$ as $[\ker D] - [\operatorname{coker} D]$.

Exercise 3.5.2 Let G be a finite group of order n. Show that

$$C_r^*(G) = \mathbb{C}G \cong \bigoplus_{\sigma \in \widehat{G}} M_{\dim(\sigma)}(\mathbb{C}), \quad \text{and} \quad K_0(C_r^*(G)) \cong \mathbb{Z}^c, \quad K_1(C_r^*(G)) = 0,$$

where $\widehat{G}$ is the set of irreducible representations of G and $c = \#(\widehat{G})$ is the number of conjugacy classes in G. (This is all for the *complex* group algebra.) Since $K_0(BG)$ is a torsion group, deduce that the assembly map $\mathcal{A} : K_*(BG) \to K_*(C_r^*(G))$ is identically zero in all degrees. (This is not necessarily the case for the assembly map on KO in degrees $1, 2, 5, 6 \mod 8$ for the *real* group ring if G is of even order—see Exercise 3.5.6 below and [79].) On the other hand, the Baum-Connes Conjecture is true for this case (for more or less trivial reasons—here $\mathcal{E}G = \mathrm{pt}$ and the definition of $K_0^G(\mathrm{pt})$ makes it coincide with $K_0(C_r^*(G))$).

Compute the trace map $\tau_* \colon K_0(C_r^*(G)) \to \mathbb{R}$ for this example, and show that it sends the generator of $K_0(C_r^*(G))$ attached to an irreducible representation σ to $\frac{\dim(\sigma)}{|G|}$. (Hint: The generator corresponds to a certain minimal idempotent in $\mathbb{C}G$. Write it down explicitly (as a linear combination of group elements), using the Schur orthogonality relations.) Deduce that $\tau_*\big(K_0(C_r^*(G))\big) = \frac{1}{|G|}\mathbb{Z}$, the rational numbers with denominator a divisor of $|G|$.

Exercise 3.5.3 Let Γ be the infinite dihedral group, the semidirect product $\mathbb{Z} \rtimes \{\pm 1\}$, where $\{\pm 1\}$ acts on $\mathbb{Z}$ by multiplication. Show by explicit calculation that $C_r^*(\Gamma)$ can be identified with the algebra

$$\big\{f \in C\big([0,1], M_2(\mathbb{C})\big) : f(0) \text{ and } f(1) \text{ are diagonal matrices}\big\}.$$

(To show this, identify $C_r^*(\Gamma)$ with the crossed product $C(S^1) \rtimes \{\pm 1\}$, where $\{\pm 1\}$ acts on $C_r^*(\mathbb{Z}) \cong C(S^1)$ by complex conjugation. The orbit space $S^1/\{\pm 1\}$ can be identified with an interval. Interior points of this interval correspond to irreducible representations of Γ of dimension 2, and over each endpoint there are two irreducible representations of Γ, each of dimension 1.

Then show that the range of the trace map $\tau_* \colon K_0(C_r^*(\Gamma)) \to \mathbb{R}$ is the half-integers $\frac{1}{2}\mathbb{Z} = \{0, \pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \cdots\}$.

This example and others like it, along with Exercise 3.5.2, led to the conjecture ([4], p. 21) that for an arbitrary group G, $\tau_*\big(K_0(C_r^*(G))\big)$ is the subgroup of $\mathbb{Q}$ generated by the numbers $\frac{1}{|H|}$, where H is a finite subgroup of G. However, this conjecture has turned out to be false ([83], [82]), even with $K_0(C_r^*(G))$ (which in general is inaccessible) replaced by the more tractable image of the Baum-Connes map $\mathcal{A}_{\mathrm{BC}}$. However, it is shown in [60] that the range of the trace on the image of the Baum-Connes map $\mathcal{A}_{\mathrm{BC}}$ is contained in the sub*ring* of $\mathbb{Q}$

generated by the reciprocals of the orders of the finite subgroups. In particular, if the Baum-Connes conjecture holds for G, then the range of the trace lies in this subring.

Exercise 3.5.4 Suppose Γ is a discrete group and π is a subgroup of Γ of finite index. Then one has a commuting diagram

$$\begin{tikzcd}
& K_0(B\pi) \arrow[r, "\mathcal{A}_\pi"] \arrow[dl, "\iota_*"'] \arrow[d, dotted] & K_0(C_r^*(\pi)) \arrow[r, "\tau_*"] \arrow[d, "\iota_*"] & \mathbb{R} \arrow[d, equal] \\
K_0(B\Gamma) \arrow[r] \arrow[rr, bend right, "\mathcal{A}_\Gamma"'] & K_0^\Gamma(\mathcal{E}\Gamma) \arrow[r, "\mathcal{A}_{\mathrm{BC}}"] & K_0(C_r^*(\Gamma)) \arrow[r, "\tau_*"] & \mathbb{R}.
\end{tikzcd}$$

Here ι_* is the map induced by the inclusion $\pi \hookrightarrow \Gamma$. But there is also a transfer map ι^* backwards from $K_0(C_r^*(\Gamma))$ to $K_0(C_r^*(\pi))$ which multiplies traces by the index $[\Gamma : \pi]$, since $C_r^*(\Gamma)$ is a free $C_r^*(\pi)$-module of rank $[\Gamma : \pi]$. Similarly, there is a compatible transfer map $\iota^* \colon K_0(B\Gamma) \to K_0(B\pi)$, and $\iota_* \circ \iota^*$ is an isomorphism on $K_0(B\Gamma)$ after inverting $[\Gamma : \pi]$. Suppose that the Baum-Connes Conjecture holds for both π and Γ, so that in this diagram, $\mathcal{A}_\pi$ and $\mathcal{A}_{\mathrm{BC}}$ are isomorphisms. Then what does this imply about integrality of the trace on $K_0(C_r^*(\Gamma))$? Compare with the conjectures discussed in Exercise 3.5.3.

Exercise 3.5.5 Suppose Γ is a discrete group and $e = e^2 \in \mathbb{C}[\Gamma]$. Show that $\tau(e)$ must lie in $\overline{\mathbb{Q}}$, the algebraic closure of $\mathbb{Q}$. Hint [9]: Consider the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ on $\mathbb{C}[\Gamma]$, as well as the positivity of τ. In fact, it is even proved in [9] that $\tau(e) \in \mathbb{Q}$, but this is much harder.

Exercise 3.5.6 Let $\pi = \mathbb{Z}/2$, a cyclic group of order 2, so that the *real* group C^*-algebra $\mathbb{R}[\pi]$ of π is isomorphic to $\mathbb{R} \oplus \mathbb{R}$ and the classifying space $B\pi = \mathbb{RP}^\infty$. Show that the assembly map $\mathcal{A} \colon KO_1(B\pi) \to KO_1(\mathbb{R}[\pi]) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ is surjective. Hints: For the summand corresponding to the trivial representation, you don't have to do any work, because of the commutative diagram

$$\begin{tikzcd}
KO_*(\mathrm{pt}) \arrow[r, "\mathcal{A}_{\{1\}}", "\cong"'] \arrow[d, tail] & KO_*(\mathbb{R}) \arrow[d, tail] \\
KO_*(B\pi) \arrow[r, "\mathcal{A}_\pi"] & KO_*(\mathbb{R} \oplus \mathbb{R}).
\end{tikzcd}$$

For the other summand, make use of the commutative diagram

$$\begin{tikzcd}
KO_*(S^1) \arrow[r, "\mathcal{A}_{\mathbb{Z}}", "\cong"'] \arrow[d] & KO_*(C_r^*(\mathbb{Z})) \arrow[d] \\
KO_*(B\pi) \arrow[r, "\mathcal{A}_\pi"] & KO_*(\mathbb{R}[\pi]),
\end{tikzcd}$$

where the vertical arrows are induced by the "reduction mod 2" map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2$.

Exercise 3.5.7 Let M^n be a smooth compact manifold and let Y be some compact space. Suppose $D\colon x \mapsto D_x$ is a continuously varying family of elliptic operators on M, parameterized by Y. Show that D defines a $C(Y)$-elliptic operator over M, and thus has a $C(Y)$-index in the sense of Mishchenko and Fomenko. (This is the same as the "families index" of Atiyah and Singer.) Also show that if $\dim\ker D_x$ and $\dim\ker D_x^*$ remain constant, so that $x \mapsto \ker D_x$ and $x \mapsto \ker D_x^*$ define vector bundles $\ker D$ and $\ker D^*$ over X, then $\operatorname{Ind} D = [\ker D] - [\ker D^*]$ in $K_0(C(Y)) \cong K^0(Y)$. (The isomorphism here is given by Swan's Theorem.)

Chapter 4

Other C^*-Algebras and Applications in Topology: Group Actions, Foliations, $\mathbb{Z}/k$-Indices, and Coarse Geometry

4.1 Crossed Products and Invariants of Group Actions

If a (locally compact) group G acts on a locally compact space X, one can form the *transformation group C^*-algebra* or *crossed product* $C^*(G, X)$ or $C_0(X) \rtimes G$. The definition is easiest to explain when G is discrete; then $C^*(G, X)$ is the universal C^*-algebra generated by a copy of $C_0(X)$ and unitaries u_g, $g \in G$, subject to the relations that

$$u_g u_h = u_{gh}, \quad u_g f u_g^* = g \cdot f \text{ for } g,\, h \in G,\, f \in C_0(X). \tag{4.1}$$

Here $g \cdot f(x) = f(g^{-1} \cdot x)$. In general, $C^*(G, X)$ is the C^*-completion of the twisted convolution algebra of $C_0(X)$-valued continuous functions of compact support on G, and its multiplier algebra still contains copies of $C_0(X)$ and of G satisfying relations (4.1). (In fact, products of an element of $C^*(G)$ and of an element of $C_0(X)$, in either order, lie in the crossed product and are dense in it.) When G acts freely and properly on X, $C^*(G, X)$ is strongly Morita equivalent to $C_0(G/X)$.[1] It thus plays the role of the algebra of functions on G/X, even

[1](Strong) Morita equivalence (see [71]) is one of the most useful equivalence relations on the class of C^*-algebras. When A and B are separable C^*-algebras, it has a simple characterization

when the latter is a "bad" space, and captures much of the equivariant topology, as we see from:

Theorem 4.1.1 (Green-Julg [44]**)** *If G is compact, there is a natural isomorphism*

$$K_*(C^*(G, X)) \cong K_G^{-*}(X).$$

There are many other results relating the structure of $C^*(G, X)$ to the topology of the transformation group (G, X); the reader interested in this topic can see the surveys [68], [69], and [66] for an introduction and references. In this chapter we will only need the rather special cases where either G is compact or else G acts locally freely (i.e., with finite isotropy groups).

Definition 4.1.2 An n-dimensional *orbifold* X is a space covered by charts each homeomorphic to $\mathbb{R}^n/G$, where G is a finite group (which may vary from chart to chart) acting linearly on $\mathbb{R}^n$, and with compatible transition functions. A *smooth orbifold* if defined similarly, but with the transition functions required to lift to be C^∞ on the open subsets of Euclidean space. The most obvious kind of example is a quotient of a manifold by a locally linear action of a finite group. But not every orbifold, not even every compact smooth orbifold, is a quotient of a manifold by a finite group action. (The simplest counterexample or "bad orbifold" is the "teardrop" X, shown in Figure 4.1. Here the bottom half of the space is a hemisphere, and the top half is the quotient of a hemisphere by a cyclic group acting by rotations around the pole. If X were of the form M/G with M a manifold and G finite, then M would have to be S^2, and we run afoul of the fact that any nontrivial orientation-preserving diffeomorphism of S^2 of finite order has to have at least two fixed points, by the Lefschetz fixed-point theorem.)

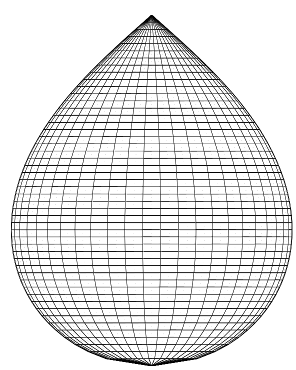

Figure 4.1: The teardrop

[8]: A and B are strongly Morita equivalent if and only if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on a separable, infinite-dimensional Hilbert space.

On a smooth orbifold X, we have a notion of Riemannian metric, which on a patch looking like $\mathbb{R}^n/G$, G finite, is simply a Riemannian metric on $\mathbb{R}^n$ invariant under the action of G. Similarly, once a Riemannian metric has been fixed, we have a notion of orthonormal frame at a point. As on a smooth manifold, these patch together to give the orthonormal frame bundle $\widetilde{X}$, and $O(n)$ acts locally freely on $\widetilde{X}$, with $\widetilde{X}/O(n)$ identifiable with X. $C^*_{\text{orb}}(X) = C^*(O(n), \widetilde{X})$ is called the *orbifold C^*-algebra* of X. This notion is due to Farsi [28]. (It depends on the orbifold structure, not just the homeomorphism class of X as a space.) Note that $C^*_{\text{orb}}(X)$ is strongly Morita equivalent to $C_0(X)$ when X is a manifold, or to $C^*(G, M)$ when X is the quotient of a manifold M by an action of a finite group G.

An elliptic operator D on a smooth orbifold X (which in each local chart $\mathbb{R}^n/G$, G a finite group, is a G-invariant elliptic operator on $\mathbb{R}^n$) defines a class $[D] \in K^{-*}(C^*_{\text{orb}}(X))$ (which we think of as $K_*^{\text{orb}}(X)$). Note that if X is actually a manifold, this is just $K_*(X)$, by Morita invariance of Kasparov theory. If X is compact, then as in the manifold case, $\operatorname{Ind} D = c_*([D]) \in K_*(\text{pt})$.

Applying the Kasparov formalism and working out all the terms, one can deduce ([27], [28], [29]) various index theorems for orbifolds, originally obtained by Kawasaki [51] by a different method.

4.2 Foliation C^*-Algebras and Applications

Definition 4.2.1 Let M^n be a compact smooth manifold, $\mathcal{F}$ a foliation of M by leaves L^p of dimension p, codimension $q = n - p$. Then one can define a C^*-algebra $C^*(M, \mathcal{F})$ encoding the structure of the foliation. (This is the C^*-completion of the convolution algebra of functions, or more canonically, half-densities, on the holonomy groupoid.) When the foliation is a fibration $L \to M \to X$, where X is a compact q-manifold, then $C^*(M, \mathcal{F})$ is strongly Morita equivalent to $C(X)$. Since K-theory is Morita invariant, this justifies thinking of $K_*(C^*(M, \mathcal{F}))$ as $K^{-*}(M/\mathcal{F})$, the *K-theory of the space of leaves.* When the foliation comes from a locally free action of a Lie group G on M, then $C^*(M, \mathcal{F})$ is just the crossed product $C^*(G, M)$.

Introducing $C^*(M, \mathcal{F})$ makes it possible to extend the Connes index theorem for foliations. If D is an operator elliptic along the leaves, then in general $\operatorname{Ind} D$ is an element of the group $K_0(C^*(M, \mathcal{F}))$. If there is an invariant transverse measure μ, then one obtains Connes' real-valued index by composing with the map

$$\int d\mu \colon K_0(C^*(M, \mathcal{F})) \to \mathbb{R}.$$

Theorem 4.2.2 (Connes-Skandalis [19]**)** *Let $(M, \mathcal{F})$ be a compact (smooth) foliated manifold and let*

$$D \colon C^\infty(M, E_0) \to C^\infty(M, E_1)$$

be elliptic along the leaves. Then $\operatorname{Ind} D \in K_0(C^*(M, \mathcal{F}))$ *agrees with a "topological index"* $\operatorname{Ind}_{\text{top}}(D)$ *computed from the characteristic classes of* $\sigma(D)$*, just as in the usual Atiyah-Singer index theorem.*

Example 4.2.3 The simplest example of this is when M splits as a product $Y \times L$, the foliation $\mathcal{F}$ is by slices $\{x\} \times L$, and D is given by a continuous family of elliptic operators D_x on L, parameterized by the points in Y, just as in Exercise 3.5.7. Then $C^*(M, \mathcal{F})$ is Morita equivalent to $C(Y)$, and $\operatorname{Ind} D$ as in the Theorem 4.2.2 is exactly the Atiyah-Singer "families index" in $K_0(C(Y)) = K^0(Y)$, which as shown in Exercise 3.5.7 can also be viewed as a case of a Mishchenko-Fomenko index.

Corollary 4.2.4 (Connes-Skandalis [19], Corollary 4.15**)** *Let* $(M, \mathcal{F})$ *be a compact foliated manifold and let* D *be the Euler characteristic operator along the leaves. Then* $\operatorname{Ind} D$ *is the class of the zeros* Z *of a generic vector field along the fibers, counting signs appropriately.*[2] (*Compare the Poincaré-Hopf Theorem, which identifies the Euler characteristic of a compact manifold with the sum of the zeros of a generic vector field, counted with appropriate signs.*)

The advantage of Theorem 4.2.2 and of Corollary 4.2.4 over Theorem 2.2.2 and its corollaries is that we don't need to assume the existence of an invariant transverse measure, which is quite a strong hypothesis. However, if such a measure μ exists, the numerical index in the situation of Corollary 4.2.4 is simply $\mu(Z)$.

Example 4.2.5 Let M be a compact Riemann surface of genus $g \geq 2$, so that its universal covering space $\widetilde{M}$ is the hyperbolic plane, and its fundamental group π is a discrete torsion-free cocompact subgroup of $G = PSL(2, \mathbb{R})$. Let $V = \widetilde{M} \times_\pi S^2$, where π acts on $S^2 = \mathbb{CP}^1$ by projective transformations (i.e., the embedding $PSL(2, \mathbb{R}) \hookrightarrow PSL(2, \mathbb{C})$); V is an S^2-bundle over M. Foliate V by the images of $\widetilde{M} \times \{x\}$. In this case there is no invariant transverse measure, since π does not leave any measure on S^2 invariant. Nevertheless, $\operatorname{Ind} D$ is nonzero in $K_0(C^*(V, \mathcal{F}))$. (It is $-2(g-1) \cdot [S^2]$, where $[S^2]$ is the push-forward of the class of $S^2 \hookrightarrow V$ [19], pp. 1173–1174.)

One case of Theorem 4.2.2 that is easier to understand is the case where the foliation $\mathcal{F}$ results from a locally free action of a simply connected solvable Lie group G on the compact manifold M. As explained before, we then have $C^*(M, \mathcal{F}) \cong C(M) \rtimes G$. However, because of the structure theory of simply connected solvable Lie groups, the crossed product by G is obtained by $\dim G$ successive crossed products by $\mathbb{R}$. However, when it comes to crossed products by $\mathbb{R}$, there is a remarkable result of Connes that can be used for computing the K-theory. For simplicity we state it only for complex K-theory, though there is a version for KO as well.

[2] Think of Z as a manifold transverse to the leaves of $\mathcal{F}$, and take the "push-forward" of the class of the trivial vector bundle over Z.

Theorem 4.2.6 (Connes' "Thom Isomorphism" [15], [70], [22]**)** *Let A be a C^*-algebra equipped with a continuous action α of $\mathbb{R}$ by automorphisms. Then there are natural isomorphisms $K_0(A) \xrightarrow{\cong} K_1(A \rtimes_\alpha \mathbb{R})$ and $K_1(A) \xrightarrow{\cong} K_0(A \rtimes_\alpha \mathbb{R})$.*

Note. Homotope the action α of $\mathbb{R}$ on A to the trivial action by considering α_t, $\alpha_t(s) = \alpha(ts)$, $0 \le t \le 1$, so $\alpha_1 = \alpha$ and α_0 is the trivial action. One way of understanding the theorem is that it says from the point of view of K-theory, $K_*(A \rtimes_{\alpha_t} \mathbb{R})$ is independent of t, and thus

$$K_*(A \rtimes_\alpha \mathbb{R}) \cong K_*(A \rtimes_{\text{trivial}} \mathbb{R}) \cong K_*(A \otimes C^*(\mathbb{R})) \cong K_*(A \otimes C_0(\mathbb{R})),$$

which can be computed easily by Bott periodicity.

Sketch of Proof. Connes' method of proof is to show that there is a unique family of maps $\phi^i_\alpha\colon K_i(A) \to K_{i+1}(A \rtimes_\alpha \mathbb{R})$, $i \in \mathbb{Z}/2$, defined for all C^*-algebras A equipped with an $\mathbb{R}$-action α, and satisfying compatibility with suspension, naturality, and reducing to the usual isomorphism $K_0(\mathbb{C}) \to K_1(\mathbb{R})$ when $A = \mathbb{C}$. Then these maps have to be isomorphisms, since Takesaki-Takai duality [93] gives an isomorphism $(A \rtimes_\alpha \mathbb{R}) \rtimes_{\widehat{\alpha}} \mathbb{R} \cong A \otimes \mathcal{K}$ (here $\widehat{\alpha}$ is the dual action of the Pontryagin dual $\widehat{\mathbb{R}} \cong \mathbb{R}$ of $\mathbb{R}$), and then by the axioms, $\phi^{i+1}_{\widehat{\alpha}} \circ \phi^i_\alpha\colon K_i(A) \to K_{i+2}(A) \cong K_i(A)$ must coincide with the Bott periodicity isomorphism. The only real problem is thus the existence and uniqueness. First Connes shows that if e is a projection in A, there is an action α' exterior equivalent to α (in other words, related to it by a 1-cocycle with values in the unitary elements of the multiplier algebra) that leaves e fixed. Since exterior equivalent actions are opposite "corners" of an action β of $\mathbb{R}$ on $M_2(A)$, by Connes' "cocycle trick," the K-theory for their crossed products is the same.[3] So if there is a map ϕ^0_α with the correct properties, $\phi^0_\alpha([e])$ is determined via the commuting diagram

$$\begin{tikzcd}
K_0(A) \arrow[r, "\phi^0_\alpha"] \arrow[d, equal] & K_1(A \rtimes_\alpha \mathbb{R}) \arrow[d, "\cong"] \\
K_0(A) \arrow[r, "\phi^0_{\alpha'}"] & K_1(A \rtimes_{\alpha'} \mathbb{R}) \\
\mathbb{Z} = K_0(\mathbb{C}) \arrow[u, "{[1] \mapsto [e]}"] \arrow[r, "\phi^0_{\text{trivial}}", "\cong"'] & K_1(C^*(\mathbb{R})) = \mathbb{Z}. \arrow[u]
\end{tikzcd}$$

Here the upward arrows at the bottom are induced by the inclusion $\mathbb{C} \cdot e \hookrightarrow A$. The axioms quickly reduce all other cases of uniqueness down to this one, so it

[3]For α and α' to be exterior equivalent means that ${\alpha'}_t(a) = u_t \alpha_t(a) u_t^*$, for some map $t \mapsto u_t$ from $\mathbb{R}$ to the unitaries of the multiplier algebra of A such that for each $a \in A$, $t \mapsto u_t a$ and $t \mapsto a u_t$ are norm-continuous. Then one can manufacture an action of $\mathbb{R}$ on $M_2(A)$, the 2×2 matrices with entries in A, by the formula

$$\beta(t) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} \alpha_t(a_{11}) & \alpha_t(a_{12}) u_t^* \\ u_t \alpha_t(a_{21}) & {\alpha'}_t(a_{22}) \end{pmatrix}.$$

remains only to prove existence. There are many arguments for this: see [70], [22], and §10.2.2 and §19.3.6 in [5]. The most elegant argument uses KK-theory, but even without this one can define ϕ_α^* to be the connecting map in the long exact K-theory sequence for the "Toeplitz extension"

$$0 \to (C_0(\mathbb{R}) \otimes A) \rtimes_{\tau\otimes\alpha} \mathbb{R} \to (C_0(\mathbb{R} \cup \{+\infty\}) \otimes A) \rtimes_{\tau\otimes\alpha} \mathbb{R} \to A \rtimes_\alpha \mathbb{R} \to 0.$$

Here τ is the translation action of $\mathbb{R}$ on $\mathbb{R} \cup \{+\infty\}$ fixing the point at infinity. But

$$(C_0(\mathbb{R}) \otimes A) \rtimes_{\tau\otimes\alpha} \mathbb{R} \cong (C_0(\mathbb{R}) \otimes A) \rtimes_{\tau\otimes\text{trivial}} \cong A \otimes \mathcal{K}$$

by Takai duality again, so the connecting map in K-theory becomes a natural map ϕ_α^* satisfying the correct axioms. □

Now we're ready to apply this to the foliation index theorem. Suppose the foliation $\mathcal{F}$ results from a locally free action of a simply connected even-dimensional solvable Lie group G on the compact manifold M. Then

$$C^*(M, \mathcal{F}) \cong C(M) \rtimes G,$$

and iterated applications of Theorem 4.2.6 set up an isomorphism

$$K_0(C^*(M, \mathcal{F})) \cong K^0(M \times \mathbb{R}^{\dim G}) \cong K^0(M),$$

the last isomorphism given by Bott periodicity. Under these isomorphisms, one can check that the index class of the leafwise Dirac operator goes first to the exterior product of the class of the trivial line bundle on M with the Bott class in $K^0(\mathbb{R}^{\dim G})$, and thus under Bott periodicity to the class of the trivial vector bundle on M.

Example 4.2.7 Let G, π, and M be as in Example 4.2.5, and consider the 2-dimensional subgroup H of G, the image in G of

$$\left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a,\, b \in \mathbb{R}, \quad a > 0 \right\} \subset SL(2, \mathbb{R}).$$

Then H acts freely on G (say on the left) and hence locally freely on $V = G/\pi$, the unit sphere bundle of M. So we have a foliation of V by orbits of H. This foliation does not have an invariant transverse measure, since such a measure would correspond to a π-invariant measure of $H\backslash G \cong S^1$, which does not exist. However, the discussion above computes the index of the leafwise Dirac operator on $(V, \mathcal{F})$ and shows it is non-zero.

4.3 C^*-Algebras and $\mathbb{Z}/k$-Index Theory

Definition 4.3.1 A $\mathbb{Z}/k$*-manifold* is a smooth compact manifold with boundary, M^n, along with an identification of ∂M with a disjoint union of k copies of a fixed manifold βM^{n-1}. It is oriented if M is oriented, the boundary components have the induced orientation, and the identifications are orientation-preserving. See Figure 4.2 for an illustration.

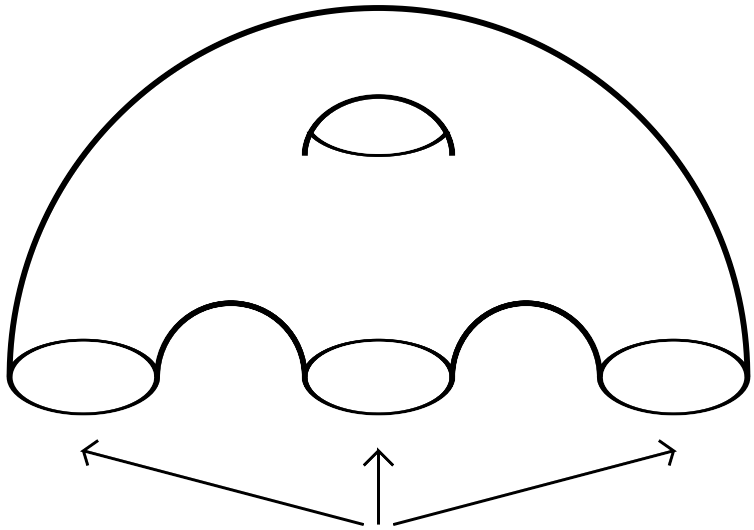

Figure 4.2: A $\mathbb{Z}/3$-manifold

One should really think of a $\mathbb{Z}/k$-manifold M as the singular space $M_\Sigma = M/\!\sim$ obtained by identifying all k of the boundary components with one another. This space is not a manifold (if $k > 2$), and does not satisfy Poincaré duality. The neighborhood of a point on βM is a cone on k copies of B^{n-1} joined along S^{n-2}, as illustrated in Figure 4.3. If M is an oriented $\mathbb{Z}/2$-manifold, then M_Σ

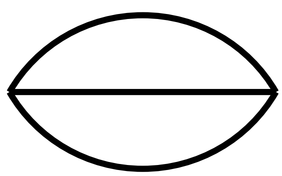

Figure 4.3: Link of a boundary point in M_Σ ($n = 2$, $k = 3$)

is a manifold, but is not orientable, because of the way the two copies of βM have been glued together. (For instance, if M is a cylinder, so $\beta M = S^1$, then M_Σ is a Klein bottle.) So an oriented $\mathbb{Z}/k$-manifold of dimension $4n$ does not have a signature in the usual sense. But it does have a signature mod k, just as a non-orientable manifold has a signature for $\mathbb{Z}/2$-cohomology. (Since $+1 = -1$ in $\mathbb{Z}/2$, the mod 2 "signature" of a non-orientable manifold is simply the middle Betti number.) The signature of a $\mathbb{Z}/k$-manifold was defined by Sullivan [92], who showed that M_Σ has a fundamental class in homology mod k, and there is a $\mathbb{Z}/k$-version of Hirzebruch's formula,

$$\operatorname{sign} M = \langle \mathcal{L}(M), [M] \rangle \in \mathbb{Z}/k.$$

This formula is a special case of an index theorem for elliptic operators on $\mathbb{Z}/k$-manifolds, due originally to Freed and Melrose [31], [32]. Other proofs were later given by Higson [38], Kaminker-Wojciechowski [45], and Zhang [97]. Higson's proof in particular made use of noncommutative C^*-algebras. The approach we will present here is due to the author [78]. For simplicity we'll deal with the "ordinary" (K_0) index of a complex elliptic operator D.

Definition 4.3.2 A $\mathbb{Z}/k$-*elliptic operator* on a $\mathbb{Z}/k$-manifold M^n, $\partial M \cong \beta M \times \mathbb{Z}/k$, will mean an elliptic operator on M (in the usual sense) whose restriction to a collar neighborhood of the boundary (diffeomorphic to $\beta M \times [0, \varepsilon) \times \mathbb{Z}/k$) is the restriction of an $\mathbb{R} \times \mathbb{Z}/k$-invariant operator on $\beta M \times \mathbb{R} \times \mathbb{Z}/k$. Thus, near the boundary, the operator is entirely determined by what happens on βM.

We want to define a $\mathbb{Z}/k$-valued index for such an operator by using the philosophy of noncommutative geometry, that says we should use a noncommutative C^*-algebra to encode the equivalence relation on M (that identifies the k copies of βM with one another), instead of working on the singular quotient space M_Σ. We begin by following a trick introduced in [38] to get rid of the complications involved with analysis near the boundary. First we attach cylinders to the boundary, replacing M by the noncompact manifold $N = M \cup_{\partial M} \partial M \times [0, \infty)$, as shown in Figure 4.4. It's important to note that an operator as in Definition 4.3.2 has a canonical extension to N, because of the translation invariance in the direction normal to the boundary.

Now we introduce the C^*-algebra $C^*(M; \mathbb{Z}/k)$ of the equivalence relation on N that is trivial on M itself and that identifies the k cylinders with one another. A simple calculation shows that

$$\begin{aligned} C^*(M; \mathbb{Z}/k) \cong \{(f, g) : f \in C(M), \quad & g \in C_0(\beta M \times [0, \infty), M_k), \\ & g|_{\beta M \times \{0\}} \text{ diagonal}, \quad f|\partial M \text{ matching } g|_{\beta M \times \{0\}}\}. \end{aligned}$$

Furthermore, just as an elliptic operator on an ordinary manifold defines a class in K-homology, a $\mathbb{Z}/k$-elliptic operator D on M, as extended canonically to N, defines a class in $K^0(C^*(M; \mathbb{Z}/k))$. (This group should be viewed as the $\mathbb{Z}/k$-manifold K-homology of M.)

Similarly we define a C^*-algebra $C^*(\mathrm{pt}; \mathbb{Z}/k)$ which is almost the same, except that M and βM are both replaced by a point. In other words,

$$C^*(\mathrm{pt}; \mathbb{Z}/k) = \{f \in C_0([0, \infty), M_k) : f(0) \text{ a multiple of } I_k\}.$$

This is simply the mapping cone of the inclusion of the scalars into $M_k(\mathbb{C})$ as multiples of the $k \times k$ identity matrix, for which the induced map on K-theory is multiplication by k on $\mathbb{Z}$, so $K^0(C^*(\mathrm{pt}; \mathbb{Z}/k)) \cong \mathbb{Z}/k$.

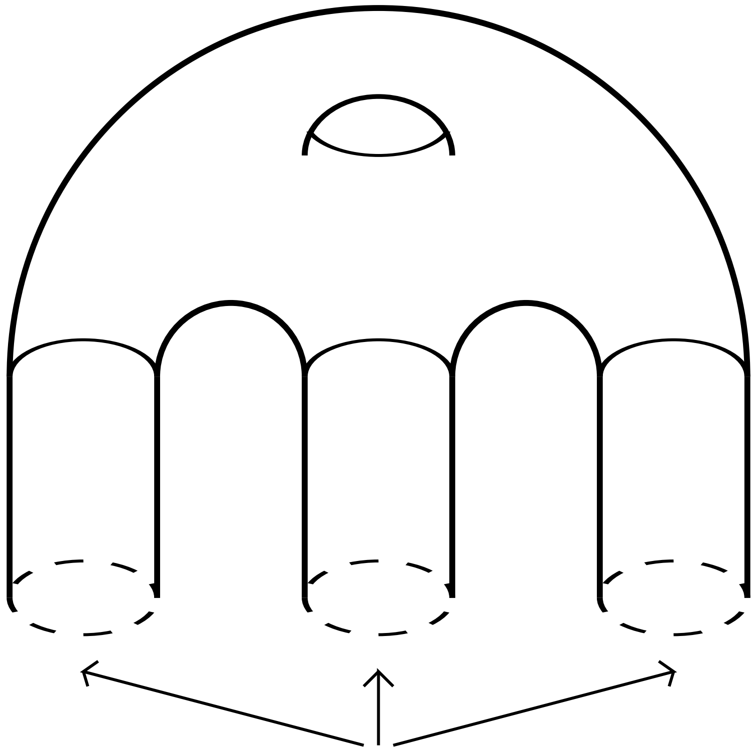

Figure 4.4: A $\mathbb{Z}/3$-manifold with infinite cylinders attached

Now the collapse map $c\colon (M, \beta M) \to (\mathrm{pt}, \mathrm{pt})$ induces a map on C^*-algebras in the other direction, $C^*(\mathrm{pt}; \mathbb{Z}/k) \hookrightarrow C^*(M; \mathbb{Z}/k)$, and hence a map of K-homology groups

$$c_*\colon K^0(C^*(M; \mathbb{Z}/k)) \to K^0(C^*(\mathrm{pt}; \mathbb{Z}/k)) \cong \mathbb{Z}/k.$$

The image of $[D]$ under this map is called the *analytic $\mathbb{Z}/k$-index* of D.

Definition 4.3.3 (the topological $\mathbb{Z}/k$-index) Let $[\sigma(D)] \in K^*(T^*M)$, the K-theory with compact supports of the cotangent bundle of M, be the class of the principal symbol of the operator. Note that $[\sigma(D)]$ is invariant under the identifications on the boundary, i.e., it comes by pullback from the quotient space T^*M_Σ (the image of T^*M with the k copies of $T^*M|_{\beta M}$ collapsed to one) under the collapse map $M \twoheadrightarrow M_\Sigma$. Following [31] we define the *topological $\mathbb{Z}/k$-index* $\mathrm{Ind}_{\mathrm{t}} D$ of D as follows. Start by choosing an embedding $\iota\colon (M, \partial M) \hookrightarrow (D^{2r}, S^{2r-1})$ of M into a ball of sufficiently large even dimension $2r$, for which ∂M embeds $\mathbb{Z}/k$-equivariantly into the boundary (if we identify S^{2r-1} with the unit sphere in $\mathbb{C}^r$, $\mathbb{Z}/k$ acting as usual by multiplication by roots of unity). We take the push-forward map on complex K-theory

$$\iota_!\colon K^0(T^*M) \to \widetilde{K}^0(T^*D^{2r}) \cong \widetilde{K}^0(D^{2r})$$

and observe that $\iota_!([\sigma(D)])$ descends to $\widetilde{K}^0(M_k^{2r}) \cong K^0(\mathrm{pt};\mathbb{Z}/k) \cong \mathbb{Z}/k$, M_k^{2r} the Moore space obtained by dividing out by the $\mathbb{Z}/k$-action on the boundary of D^{2r}, and call the image the topological index of D, $\mathrm{Ind_t}(D)$.

Theorem 4.3.4 ($\mathbb{Z}/k$-index theorem) *Let $(M, \phi\colon \partial M \xrightarrow{\cong} \beta M \times \mathbb{Z}/k)$ be a closed $\mathbb{Z}/k$-manifold, and let D be an elliptic operator on M in the sense of Definition 4.3.2. Then the analytic index of D in $K_i(\mathrm{pt};\mathbb{Z}/k)$ coincides with the topological index* $\mathrm{Ind_t}(D)$.

Sketch of Proof. The idea, based on the Kasparov-theoretic proof of the Atiyah-Singer Theorem ([5], Chapter IX, §24.5), is to write the class of D in $K^0(C^*(M;\mathbb{Z}/k))$ as a Kasparov product:

$$[D] = [\sigma(D)]\widehat{\otimes}_{C_0(T^*M_\Sigma)}\widehat{\alpha} \in K^0(C^*(M;\mathbb{Z}/k)),$$

where

$$\widehat{\alpha} \in KK\big(C_0(T^*M_\Sigma) \otimes C^*(M;\mathbb{Z}/k),\, \mathbb{C}\big)$$

is a canonical class constructed using the almost complex structure on T^*M and the Thom isomorphism, and we view $[\sigma(D)]$ as living in $K^0(T^*M_\Sigma)$.

But now by associativity of the Kasparov product, we compute that

$$\mathrm{Ind}(D) = [c^*]\widehat{\otimes}_{C^*(M;\mathbb{Z}/k)}[D] = [\sigma(D)]\widehat{\otimes}_{C_0(T^*M_\Sigma)}\Big([c^*]\widehat{\otimes}_{C^*(M;\mathbb{Z}/k)}\widehat{\alpha}\Big).$$

So we just need to identify the right-hand side of this equation with $\mathrm{Ind_t}(D)$. However, by Definition 4.3.3 $\mathrm{Ind_t}(D) = \widehat{\iota}_!([\sigma(D)])$, where

$$\widehat{\iota}_!\colon K^0(T^*M_\Sigma) \to K^0(T^*D^{2r}_\Sigma) \cong K^0(M_k^{2r})$$

is the push-forward map on K-theory. And examination of the definition of $\widehat{\iota}_!$ shows it is precisely the Kasparov product with

$$[c^*]\widehat{\otimes}_{C^*(M;\mathbb{Z}/k)}\widehat{\alpha},$$

followed by a "Poincaré duality" isomorphism $K^0(C^*(\mathrm{pt};\mathbb{Z}/k)) \xrightarrow{\cong} K_0(\mathrm{pt};\mathbb{Z}/k)$.
□

4.4 Roe C^*-Algebras and Coarse Geometry

Finally, we mention an application of C^*-algebras to the topology "at infinity" of noncompact spaces. Recall that we began Chapter 1 by talking about the differences between spectral theory of the Laplacian on compact and on noncompact manifolds. The same points would have been equally valid for arbitrary elliptic operators.

Roe had the idea of introducing certain C^*-algebras attached to a noncompact manifold, but *depending on a choice of metric*, that can be used for doing index theory "at infinity."

Definition 4.4.1 ([72], [73]) Let M be a complete Riemannian manifold (usually noncompact). Fix a suitable Hilbert space $\mathcal{H}$ (for example, $L^2(M, d\,\mathrm{vol})$) on which $C_0(M)$ acts non-degenerately, with no nonzero element of $C_0(M)$ acting by a compact operator. A bounded operator T on $\mathcal{H}$ is called *locally compact* if $\varphi T, T\varphi \in \mathcal{K}(\mathcal{H})$ for $\varphi \in C_c(M)$, *of finite propagation* if for some $R > 0$ (depending on T), $\varphi T \psi = 0$ for $\varphi, \psi \in C_c(M)$, $\mathrm{dist}(\operatorname{supp}\varphi, \operatorname{supp}\psi) > R$. Let $C^*_{\mathrm{Roe}}(M)$ be the C^*-algebra generated by the locally compact, finite propagation, operators. One can show that this algebra is (up to isomorphism) independent of the choice of $\mathcal{H}$.

Example 4.4.2 If M is compact, the finite propagation condition is always trivially satisfied, and $C^*_{\mathrm{Roe}}(M) = \mathcal{K}$, the compact operators. If $M = \mathbb{R}^n$ with the usual Euclidean metric, then $K_i(C^*_{\mathrm{Roe}}(M)) \cong \mathbb{Z}$ for $i \equiv n \mod 2$, and $K_i(C^*_{\mathrm{Roe}}(M)) = 0$ for $i \equiv n-1 \mod 2$. (See [73], p. 33 and p. 74.)

Definition 4.4.3 Let X and Y be *proper metric spaces*, that is, metric spaces in which closed bounded sets are compact. Then a map $f\colon X \to Y$ is called a *coarse map* if it is proper (the inverse image of a pre-compact set is pre-compact) and if it is *uniformly expansive*, i.e., for each $R > 0$, there exists $S > 0$ such that if $d_X(x, x') \le R$, then $d_Y(f(x), f(x')) \le S$. Note that this definition only involves the large-scale behavior of f; f need not be continuous, and we can always modify f any way we like on a compact set (as long as the image of that compact set remains bounded) without affecting this property. A *coarse equivalence* is a coarse map $f\colon X \to Y$ such that there exists a coarse map $g\colon Y \to X$ and there is a constant $K > 0$ with $d_X(x, g \circ f(x)) \le K$ and with $d_Y(y, f \circ g(y)) \le K$ for all $x \in X$ and $y \in Y$.

Example 4.4.4 The inclusion map $\mathbb{Z} \hookrightarrow \mathbb{R}$ (when $\mathbb{Z}$ and $\mathbb{R}$ are equipped with their standard metrics) is a coarse equivalence, with coarse inverse the "rounding down" map $x \mapsto \lfloor x \rfloor$. More generally, if M is a connected compact manifold with fundamental group π, and if $\widetilde{M}$ is the universal cover of M, then $\widetilde{M}$ is coarsely equivalent to $|\pi|$, the group π viewed as a metric space with respect to a word-length metric (defined by a choice of a finite generating set). The coarse equivalence is again obtained by fixing a basepoint $x_0 \in \widetilde{M}$ and a fundamental domain F for the action of π on $\widetilde{M}$, and defining $f\colon |\pi| \to \widetilde{M}$ by $g \mapsto g \cdot x_0$, $g\colon \widetilde{M} \to |\pi|$ by $x \mapsto g$ whenever $x \in g \cdot F$. (The previous example is the special case where $M = S^1$, $\widetilde{M} = \mathbb{R}$, $\pi = \mathbb{Z}$, $x_0 = 0$, and $F = [0, 1)$.)

Proposition 4.4.5 (Roe [73], Lemma 3.5**)** *A coarse equivalence $X \to Y$ induces an isomorphism $C^*_{\mathrm{Roe}}(X) \to C^*_{\mathrm{Roe}}(Y)$.*

Theorem 4.4.6 (Roe) *If M is a complete Riemannian manifold, there is a functorial "assembly map" $\mathcal{A}\colon K_*(M) \to K_*(C^*_{\mathrm{Roe}}(M))$. If D is a geometric elliptic operator on M (say the Dirac operator or the signature operator), it has a class in $K_0(M)$, and $\mathcal{A}([D])$ is its "coarse index." For noncompact spin manifolds, vanishing of $\mathcal{A}([D])$ (for the Dirac operator) is a necessary condition for there being a metric of uniformly positive scalar curvature in the quasi-isometry class of the original metric on M.*

There is a *Coarse Baum-Connes Conjecture* analogous to the usual Baum-Connes Conjecture, that the assembly map $\mathcal{A}\colon K_*(M) \to K_*(C^*_{\mathrm{Roe}}(M))$ is an isomorphism for M uniformly contractible. (The uniform contractibility assures that M has no "local topology;" without this, we certainly wouldn't expect an isomorphism, since $K_*(C^*_{\mathrm{Roe}}(M))$ only depends on the coarse equivalence class of M.)

Unfortunately, the Coarse Baum-Connes Conjecture is now known to fail in some cases. For one thing, it is known to fair for some uniformly contractible manifolds without bounded geometry [20]. That suggests that perhaps one should change the domain of the assembly map from $K_*(M)$ to its "coarsification" $KX_*(M)$ ([73], pp. 14-15), the inductive limit of the $K_*(|\mathcal{U}|)$, the nerves of coverings $\mathcal{U}$ of X by pre-compact open sets, as the coverings become coarser and coarser. As one would hope, it turns out that $\mathcal{A}\colon K_*(M) \to K_*(C^*_{\mathrm{Roe}}(M))$ factors through $KX_*(M)$, and that $K_*(M) \to KX_*(M)$ is an isomorphism when M is uniformly contractible and of bounded geometry. However, there is also an example of a manifold M of bounded geometry for which $KX_*(M) \to K_*(C^*_{\mathrm{Roe}}(M))$ is not an isomorphism [95]. But it is still conceivable (though it seems increasingly unlikely) that the Coarse Baum-Connes Conjecture holds for all uniformly contractible manifolds with bounded geometry, or at least for all universal covers of compact manifolds.

The main interest of the Coarse Baum-Connes Conjecture, aside from its aesthetic appeal as a parallel to the usual Baum-Connes Conjecture, is its connection with the usual Novikov Conjecture (Conjecture 3.3.1). One has:

Theorem 4.4.7 (Principle of descent) *The Coarse Baum-Connes Conjecture for $C^*_{\mathrm{Roe}}(|\pi|)$, π a group, but viewed as a discrete metric space, implies the Novikov Conjecture for π.*

A sketch of proof can be found in [73], Chapter 8. Theorem 4.4.7 has been applied in [96] to prove the Novikov Conjecture for any group π for which $|\pi|$ admits a uniform embedding into a Hilbert space. This covers both amenable groups and hyperbolic groups.

4.5 Exercises

Exercise 4.5.1 Consider the teardrop X shown in Figure 4.1, obtained by gluing together D^2/μ_n and D^2. (Here D^2 is the closed unit disk in $\mathbb{C}$, and μ_n is the cyclic group of n-th roots of unity, that acts on D^2 by rotations.) Compute the topology of the (oriented) orthonormal frame bundle P of X, which should be a closed 3-manifold, and describe the locally free action of $S^1 \cong SO(2)$ on P with $P/S^1 \cong X$. Show that $C^*_{\text{orb}}(X)$ is Morita equivalent to

$$A = \{f \in C(S^2, M_n(\mathbb{C})) : f(x_0) \text{ is diagonal}\},$$

where x_0 is a distinguished point on S^2, which fits into a short exact sequence

$$0 \to C_0(\mathbb{R}^2, M_n(\mathbb{C})) \to A \to \mathbb{C}^n \to 0.$$

Deduce that $K_0(C^*_{\text{orb}}(X))$ is free abelian of rank $n+1$, and that $K_1(C^*_{\text{orb}}(X)) = 0$. From this is follows by duality that $K^0(C^*_{\text{orb}}(X))$ is free abelian of rank $n+1$. Compute the class in $K^0(C^*_{\text{orb}}(X)) = K_0^{\text{orb}}(X)$ of the Euler characteristic operator D, and also its index $\operatorname{Ind} D$ in $K_0(\text{pt}) = \mathbb{Z}$.

Exercise 4.5.2 Suppose a foliation $\mathcal{F}$ results from a locally free action of a simply connected even-dimensional solvable Lie group G on a compact manifold M. Show that the index of the leafwise Euler characteristic operator is 0 in $C^*(M, \mathcal{F})$, both by an application of Corollary 4.2.4 and by a calculation using the Thom Isomorphism Theorem (Theorem 4.2.6), as was done above with the Dirac operator.

Exercise 4.5.3 Let M be a compact oriented surface of genus g and with $k > 1$ boundary components (all necessarily circles), as in Figure 4.2, which shows the case $g = 1$ and $k = 3$. Regard M as a $\mathbb{Z}/k$-manifold. Compute the $\mathbb{Z}/k$-index of the Euler characteristic operator on M.

Exercise 4.5.4 Construct complete Riemannian metrics g on $\mathbb{R}^2$ for which $K_*(C^*_{\text{orb}}(X))$, $X = (\mathbb{R}^2, g)$ is not isomorphic to $K_*(\text{pt})$, and give an example of an application to index theory on X. (Hint: The Coarse Baum-Connes Conjecture is valid for the open cone on a compact metrizable space Y. If Y is embedded in the unit sphere S^{n-1} in $\mathbb{R}^n$, the *open cone* on Y is by definition the union of the rays in $\mathbb{R}^n$ starting at the origin and passing through Y, equipped with the restriction of the Euclidean metric on $\mathbb{R}^n$.)

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Index