A Mini-Course on Applications of Non-Commutative Geometry to Topology

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Outline:

I. Group Von Neumann Algebras in Topology

II. Von Neumann Algebra Index Theorems

III. Group C^*-Algebras in Topology

IV. Other C^*-Algebras in Topology

I. Group Von Neumann Algebras in Topology: L^2-cohomology, Novikov-Shubin invariants

I.1. Motivation

Spectrum of the Laplacian:

Compact manifolds	**Non-compact manifolds**
Discrete spectrum	Continuous spectrum
Finite-dimensional kernel	Infinite-dimensional kernel

In the special case where the non-compact manifold is a normal covering of a compact manifold with covering group π, we use the group von Neumann algebra of π to measure the "size" of the infinite-dimensional kernel and the "thickness" of the continuous spectrum near 0.

I.2. An Algebraic Set-Up (following M. Farber, GAFA **6** (1996), 628–665)

Let π be a discrete group. It acts on both the right and left on $L^2(\pi)$. The von Neumann algebras $\lambda(\pi)''$ and $\rho(\pi)''$ generated by the left and by the right regular representations λ and ρ are isomorphic, and $\rho(\pi)'' = \lambda(\pi)'$. These von Neumann algebras are *finite*, with a canonical (faithful finite normal) trace τ defined by

$$\tau(\lambda(g)) = \begin{cases} 1, & g = 1 \\ 0, & g \neq 1, \end{cases}$$

and similarly for ρ. Call a finite direct sum of copies of $L^2(\pi)$ a *finitely generated free Hilbert* π*-module*, and the cut-down of such a module by a projection in the commutant a *finitely generated projective Hilbert* π*-module*. (We keep track of the topology but forget the inner product.)

The finitely generated projective Hilbert π-modules form an additive category $\mathcal{H}(\pi)$. The morphisms are continuous linear maps commuting with the π-action. Each object A in this category has a dimension $\dim_\tau(A) \in [0, \infty)$, via

$$\dim_\tau n \cdot L^2(\pi) = n, \quad \dim_\tau eL^2(\pi) = \tau(e).$$

When π is an *ICC* (infinite conjugacy class) group, $\lambda(\pi)''$ is a factor and objects of $\mathcal{H}(\pi)$ are determined by their dimensions.

The category $\mathcal{H}(\pi)$ is *not abelian*, since a morphism need not have closed range. It turns out there is a natural way to complete it to get an *abelian* category $\mathcal{E}(\pi)$. The finitely generated projective Hilbert π-modules are the projectives in $\mathcal{E}(\pi)$. Each element of the larger category is a direct sum of a projective and a torsion element (representing infinitesimal spectrum near 0). A torsion element is defined by a positive operator $\alpha = \alpha^*\colon A \to A$ with $\ker \alpha = 0$ and $\dim_\tau A$ arbitrarily small.

The most interesting invariant of a torsion element $\mathcal{X}$ represented by $\alpha = \alpha^*\colon A \to A$ is the rate at which

$$F_\alpha(t) = \dim_\tau(E_t A), \quad \alpha = \int_0^\infty t\, dE_t$$

approaches 0 as $t \to 0$. This is well-defined modulo the equivalence relation

$$F \sim G \Leftrightarrow \exists C, \varepsilon > 0,\ G\Big(\frac{t}{C}\Big) \le F(t) \le G(tC),\ t < \varepsilon.$$

The *Novikov-Shubin capacity* of $\mathcal{X}$ is

$$c(\mathcal{X}) = \limsup_{t \to 0+} \frac{\log t}{\log F_\alpha(t)};$$

it satisfies

$$c(\mathcal{X}_1 \oplus \mathcal{X}_2) = \max(c(\mathcal{X}_1), c(\mathcal{X}_2))$$

and for exact sequences

$$0 \to \mathcal{X}_1 \to \mathcal{X} \to \mathcal{X}_2 \to 0,$$

$$\max(c(\mathcal{X}_1), c(\mathcal{X}_2)) \le c(\mathcal{X}) \le c(\mathcal{X}_1) + c(\mathcal{X}_2).$$

Now consider a connected CW complex X with fundamental group π and only finitely many cells in each dimension. The cellular chain complex $C_*(\widetilde{X})$ of the universal cover $\widetilde{X}$ is a chain complex of finitely generated free (left) $\mathbb{C}[\pi]$-modules. We can complete to $L^2(\pi) \otimes_\pi C_*(\widetilde{X})$, a chain complex in $\mathcal{H}(\pi) \subseteq \mathcal{E}(\pi)$, and get homology, cohomology groups

$$\mathcal{H}_i(X, L^2(\pi)) \in \mathcal{E}(\pi), \mathcal{H}^i(X, L^2(\pi)) \in \mathcal{E}(\pi),$$

called (extended) L^2*-homology and cohomology*, which are homotopy invariants of X. The numbers $\beta_i(X, L^2(\pi)) =$

$$\dim_\tau(\mathcal{H}_i(X), L^2(\pi)) = \dim_\tau(\mathcal{H}^i(X), L^2(\pi))$$

are called the (reduced) L^2*-Betti numbers* of X. Similarly one has *Novikov-Shubin invariants* defined from the spectral density of the torsion parts (though by the UCT, the torsion part of $\mathcal{H}^i(X, L^2(\pi))$ corresponds to the torsion part of $\mathcal{H}_{i-1}(X, L^2(\pi))$).

I.3. Calculations

Theorem 1 *Suppose M is a compact connected manifold with fundamental group π. Then the L^2-Betti numbers of M as defined above agree with the τ-dimensions of*

$$\left(L^2 \text{ closed } i\text{-forms on } \widetilde{M}\right) / \overline{d\left(L^2\ (i-1)\text{-forms on } \widetilde{M}\right) \cap \left(L^2\ i\text{-forms}\right)}.$$

Similarly the Novikov-Shubin invariants can be computed from the spectral density of Δ on $\widetilde{M}$ (as measured using τ).

Example 2 $M = S^1$, $\pi = \mathbb{Z}$, $\mathbb{C}[\pi] = \mathbb{C}[T, T^{-1}]$. $L^2(\pi)$ identified via Fourier series with $L^2(S^1)$, group von Neumann algebra with $L^\infty(S^1)$, τ with $f \mapsto \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} f(e^{i\theta})\, d\theta$,

$$C_*(\widetilde{M})\colon \mathbb{C}[T, T^{-1}] \xrightarrow{T-1} \mathbb{C}[T, T^{-1}]$$

$$C_*(\widetilde{M}, L^2(\pi))\colon L^2(S^1) \xrightarrow{e^{i\theta}-1} L^2(S^1).$$

So the L^2-Betti numbers are both zero, but the Novikov-Shubin invariants are non-trivial, corresponding to the fact that if

$$\alpha = |e^{i\theta} - 1| \colon L^2(S^1) \to L^2(S^1),$$

then $F_\alpha(t) \approx t$ for t small.

Generalizing one aspect of this is:

Theorem 3 (Cheeger-Gromov) *If X is an aspherical CW complex (i.e., $\pi_i(X) = 0$ for $i \neq 1$) with only finitely many cells of each dimension, and if $\pi = \pi_1(X)$ is amenable and infinite, then all L^2-Betti numbers of X vanish.*

However, under the hypotheses of Theorem 3, Brooks proved that 0 lies in the spectrum of Δ on 0-forms. It is not known then (at least to the author) if one of the Novikov-Shubin capacities is always positive.

Example 4 M a compact Riemann surface of genus $g \geq 2$, $\widetilde{M}$ the hyperbolic plane, π a discrete torsion-free cocompact subgroup of $G = PSL(2, \mathbb{R})$. In this case, it's easiest to use the analytic picture, since $L^2(\widetilde{M}) \cong L^2(G/K)$, $K = SO(2)/\{\pm 1\}$. As a representation space of G, this is a direct integral of the principal series representations, and Δ corresponds to the Casimir operator, which has spectrum bounded away from 0. So $\beta_0 = 0$, and also $\beta_2 = 0$ by Poincaré duality.

Now the L^2 sections of $\Omega^1(\widetilde{M})$ may be identified with $\mathrm{Ind}_K^G(\mathfrak{g}/\mathfrak{k})^*$, which contains, in addition to the continuous spectrum, two discrete series representations with Casimir eigenvalue 0. Thus $\beta_1 \neq 0$. The Atiyah L^2-index theorem implies $\beta_1 = 2(g-1)$. There are no additional N-S invariants, since these measure the nonzero spectrum of Δ close to 0, but the continuous spectrum of Δ is bounded away from 0.

Generalizing one aspect of Example 4 is:

Theorem 5 (Jost-Zuo, conj. by Singer) *If M is a compact connected Kähler manifold of non-positive sectional curvature and complex dimension n, then all L^2-Betti numbers of M vanish, except perhaps for β_n.*

As we will see in the next lecture, the Atiyah L^2-index theorem then implies that

$$\beta_n = (-1)^n \chi(M),$$

where χ is the usual Euler characteristic.

Note. One should not be misled by these examples into thinking that the L^2-Betti numbers are always integers, or that most of them usually vanish.

II. Von Neumann Algebra Index Theorems: Atiyah's L^2-Index Theorem and Connes' Index Theorem for Foliations

II.1. Atiyah's L^2-Index Theorem

As we saw in the last lecture, it is not always so easy to compute all of the L^2-Betti numbers of a space directly from the definition, though sometimes we can compute *some* of them. It would be nice to have constraints from which we could then determine the others. Such a constraint, and more, is provided by:

Theorem 6 (Atiyah) *Suppose*

$$D\colon C^\infty(M, E_0) \to C^\infty(M, E_1)$$

is an elliptic pseudodifferential operator (ψDO) (*acting between sections of two vector bundles* E_0 *and* E_1) *on a closed manifold* M, *and* $\widetilde{M}$ *is a normal covering of* M *with covering group* π. *Let*

$$\widetilde{D}\colon C^\infty(\widetilde{M}, \tilde{E}_0) \to C^\infty(\widetilde{M}, \tilde{E}_1)$$

be the lift of D *to* $\widetilde{M}$. *Then*

$$\begin{aligned} &\operatorname{Ind} D (= \dim\ker D - \dim\ker D^*) \\ = L^2\text{-}&\operatorname{Ind} \widetilde{D} (= \dim_\tau \ker \widetilde{D} - \dim_\tau \ker \widetilde{D}^*). \end{aligned}$$

For applications to L^2-Betti numbers, we can take $E_0 = \bigoplus \Omega^{2i}$, $E_1 = \bigoplus \Omega^{2i+1}$, D the "Euler characteristic operator" $D = d + d^*$, so $\operatorname{Ind} D = \chi(M)$ by the Hodge Theorem, while L^2-$\operatorname{Ind} \widetilde{D}$ is the alternating sum of the L^2-Betti numbers, $\sum (-1)^i \beta_i$.

Sketch of Proof. For simplicity take D to be a first-order differential operator, and consider the formally self-adjoint operator

$$P = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

acting on sections of $E = E_0 \oplus E_1$. Since D is elliptic, PDE theory shows that the solution of the "heat equation" for P, $H_t = \exp(-tP^2)$, is a *smoothing operator*, an integral operator with smooth kernel, for $t > 0$. And as $t \to \infty$, $H_t \to$ projection on $\ker D \oplus \ker D^*$, so that if

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{cases} 1 & \text{on } E_0 \\ -1 & \text{on } E_1, \end{cases}$$

then $\operatorname{Ind} D = \lim_{t\to\infty} \operatorname{Tr}(\gamma H_t)$.

Define similarly

$$\widetilde{P} = \begin{pmatrix} 0 & \widetilde{D}^* \\ \widetilde{D} & 0 \end{pmatrix}, \quad \widetilde{\gamma} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \widetilde{H} = e^{-t\widetilde{P}^2},$$

acting on sections of $\widetilde{E} = \widetilde{E}_0 \oplus \widetilde{E}_1$. Then L^2- $\operatorname{Ind} \widetilde{D} = \lim_{t\to\infty} \tau\left(\widetilde{\gamma}\widetilde{H}_t\right)$.

Here we extend τ to matrices over the group von Neumann algebra in the obvious way. So we just need to show that

$$\mathrm{Tr}\,(\gamma H_t) = \tau\left(\widetilde{\gamma}\widetilde{H_t}\right). \tag{1}$$

Now in fact both sides of (1) are constant in t, since, for instance,

$$\begin{aligned}
\frac{d}{dt}\mathrm{Tr}\,(\gamma H_t) &= \frac{d}{dt}\mathrm{Tr}\left(\gamma e^{-tP^2}\right)\\
&= \frac{d}{dt}\left(\mathrm{Tr}\, e^{-tD^*D} - \mathrm{Tr}\, e^{-tDD^*}\right)\\
&= \mathrm{Tr}\left(DD^*e^{-tDD^*} - D^*De^{-tD^*D}\right).
\end{aligned}$$

But

$$\begin{aligned}
\mathrm{Tr}\left(DD^*e^{-tDD^*}\right) &= \mathrm{Tr}\left(\overbrace{e^{-tDD^*/2}D}\,\overbrace{D^*e^{-tDD^*/2}}\right)\\
&= \mathrm{Tr}\left(D^*e^{-tDD^*/2}e^{-tDD^*/2}D\right)\\
&= \mathrm{Tr}\left(D^*e^{-tDD^*}D\right)\\
&= \mathrm{Tr}\left(D^*\left(1 - tDD^* + \cdots\right)D\right)\\
&= \mathrm{Tr}\left(D^*D\left(1 - tD^*D + \cdots\right)\right)\\
&= \mathrm{Tr}\left(D^*De^{-tD^*D}\right).
\end{aligned}$$

So it's enough to show that

$$\lim_{t\to 0^+}\left(\mathrm{Tr}\,(\gamma H_t) - \tau\left(\tilde{\gamma}\widetilde{H_t}\right)\right) = 0.$$

But for small t, the solution of the heat equation is almost local. In other words, H_t and $\widetilde{H_t}$ are given by integration against smooth kernels almost concentrated on the diagonal, and the kernel $\widetilde{k}$ for $\widetilde{H_t}$ is practically the lift of the kernel k for H_t, since, locally, M and $\widetilde{M}$ look the same. But for a π-invariant operator $\widetilde{S}$ on $\widetilde{E}$, obtained by lifting the kernel function k for a smoothing operator on M to a kernel function to $\widetilde{k}$, one can check that

$$\begin{aligned}\tau(\widetilde{S}) &= \int_F \widetilde{k}(\tilde{x},\tilde{x})\,d\,\mathrm{vol}(\tilde{x})\\ &= \int_M k(x,x)\,d\,\mathrm{vol}(x)\\ &= \mathrm{Tr}(S),\end{aligned}$$

F a fundamental domain for the action of π on $\widetilde{M}$. So that does it. □

II.2. Connes' Index Theorem for Foliations

Another important application to topology of finite von Neumann algebras is Connes' index theorem for tangentially elliptic operators on foliations with an invariant transverse measure.

Setup. M^n a compact smooth manifold, $\mathcal{F}$ a foliation of M by leaves L^p of dimension p, codimension $q = n-p$. $\mathcal{F}$ itself can be identified with an integrable subbundle of TM. *Locally*, M looks like $L^p \times \mathbb{R}^q$, but it can easily happen that every leaf is dense. Not every such foliation comes with an *invariant transverse measure* μ, but when μ exists, it gives a way to integrate over the "space of leaves $M/\mathcal{F}$" even though this space may not even be T_0. More precisely, from M, $\mathcal{F}$, and μ, one can construct a finite von Neumann algebra $W^*(M, \mathcal{F})$ with a trace τ coming from μ. This is (except for holonomy) the von Neumann algebra defined by M and the equivalence relation $\sim$ of "being on the same leaf."

Now suppose there is a differential operator D on M which only involves differentiation in directions tangent to the leaves and is elliptic when restricted to any leaf. (Examples: the Euler characteristic operator or the Dirac operator "along the leaves.") Since the leaves are usually not compact, we can't compute an index for the restriction of D to one leaf. But since M, the union of the leaves, is compact, it turns out one can make sense of a numerical index $\text{Ind}_\tau D$ for D. In the special case where $\mathcal{F}$ has closed leaves, the foliation is a fibration $L^p \to M \xrightarrow{\text{proj}} X^q$, and μ is a probability measure on X, this reduces to $\text{Ind}_\tau D = \int_X \text{Ind}\left(D|_{L_x}\right) d\,\mu(x)$, where $L_x = \text{proj}^{-1}(x)$. In general, $\text{Ind}_\tau D$ is roughly the "average with respect to μ" of the L^2-index of $D|_{L_x}$, as x runs over the "space of leaves." Here we give each leaf the Riemannian structure defined by a choice of metric on the bundle $\mathcal{F}$.

Example 7 Let M_1 and M_2 be compact connected manifolds, and let π be the fundamental group of M_2. If π acts on $M_1 \times \widetilde{M}_2$ with trivial action on the first factor and the usual action on the second factor, then the quotient is $M_1 \times (\widetilde{M}_2/\pi) = M_1 \times M_2$. But suppose we take *any* action of π on M_1 and then take the diagonal action of π on $M_1 \times \widetilde{M}_2$. Then $M = (M_1 \times \widetilde{M}_2)/\pi$ is compact and projection to the second factor gives a fibration onto M_2 with fiber M_1. But M is also foliated by the images of $\{x\} \times \widetilde{M}_2$, usually non-compact. A measure μ on M_1 invariant under the action of π is an invariant transverse measure for this foliation $\mathcal{F}$. If D is the Euler characteristic operator along the leaves and all the leaves are $\cong \widetilde{M}_2$, then $\mathrm{Ind}_\tau D$ just becomes the average L^2-Euler characteristic of $\widetilde{M}$, and Connes' Theorem will reduce to Atiyah's.

Theorem 8 (Connes) *Let $(M, \mathcal{F})$ be a compact foliated manifold with an invariant transverse measure μ, and let $W^*(M, \mathcal{F})$ be the associated von Neumann algebra with trace τ coming from μ. Let*

$$D\colon C^\infty(M, E_0) \to C^\infty(M, E_1)$$

be elliptic along the leaves. Then the L^2 kernels of

$$P = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

on the various leaves assemble to a (graded) Hilbert $W^*(M, \mathcal{F})$-module $K_0 \oplus K_1$, and

$$\mathrm{Ind}_\tau D = \dim_\tau K_0 - \dim_\tau K_1 = \int \mathrm{Ind}_{\mathrm{top}}\, \sigma(D)\, d\,\mu,$$

where $\sigma(D)$ denotes the symbol of D and the "topological index" $\mathrm{Ind}_{\mathrm{top}}$ is computed from the characteristic classes of $\sigma(D)$ just as in the usual Atiyah-Singer index theorem.

II.3. An Application

If we specialize the Connes index theorem to the Euler characteristic operator along the leaves for foliations with 2-dimensional leaves, it reduces to:

Theorem 9 (Connes) *Let* $(M, \mathcal{F})$ *be a compact foliated manifold* (*oriented and transversally oriented*) *with* 2*-dimensional leaves. Then for every invariant transverse measure* μ*, the* μ*-average of the* L^2*-Euler characteristic of the leaves is equal to* $\langle e(\mathcal{F}), C_\mu \rangle$*, where* $e(\mathcal{F}) \in H^2(M, \mathbb{Z})$ *is the Euler class of the* 2*-plane bundle associated to* $\mathcal{F}$*, and* $C_\mu \in H_2(M, \mathbb{R})$ *is the "Ruelle-Sullivan class" attached to* μ*.*

The result also generalizes to compact *laminations* with 2-dimensional leaves. (That means we replace M by any compact Hausdorff space X locally of the form $\mathbb{R}^2 \times T$, where T is allowed to vary.) The only difference in this case is that we have to use *tangential* de Rham theory.

Corollary 10 *Suppose* $(X, \mathcal{F})$ *is a compact laminated space with 2-dimensional oriented leaves and a smooth Riemannian metric* g*. Let* ω *be the curvature 2-form of* g*. If there is an invariant transverse measure* μ *with* $\langle[\omega], C_\mu\rangle > 0$*, then* $\mathcal{F}$ *has a set of closed leaves of positive* μ*-measure. If there is an invariant transverse measure* μ *with* $\langle[\omega], C_\mu\rangle < 0$*, then* $\mathcal{F}$ *has a set of (conformally) hyperbolic leaves of positive* μ*-measure. If all the leaves are (conformally) parabolic, then* $\langle[\omega], C_\mu\rangle = 0$ *for every invariant transverse measure.*

Proof. The only oriented 2-manifold with positive L^2-Euler characteristic is S^2. Every hyperbolic Riemann surface has negative L^2-Euler characteristic. And every parabolic Riemann surface (one covered by $\mathbb{C}$ with the flat metric) has vanishing L^2-Euler characteristic. □

This has been used in:

Theorem 11 (Ghys) *Under the hypotheses of Corollary 10, if every leaf is parabolic, then* $(X, \mathcal{F}, g)$ *is* approximately uniformizable, *i.e., there are real-valued functions* u_n *(smooth on the leaves) with the curvature form of* $e^{u_n}g$ *tending uniformly to* 0.

Another known fact is:

Theorem 12 (Candel) *Under the hypotheses of Corollary 10, if every leaf is hyperbolic, then* $(X, \mathcal{F}, g)$ *is* uniformizable, *i.e., there is a real-valued function* u *(smooth along the leaves) with* $e^u g$ *hyperbolic on each leaf.*

III. Group C^*-Algebras, the Mishchenko-Fomenko Index Theorem, and Applications to Topology

III.1. The Mishchenko-Fomenko Index

Definition 13 Let A be a C^*-algebra (over $\mathbb{R}$ or $\mathbb{C}$) with unit, and let X be a compact space. An A-*vector bundle* over X will mean a locally trivial bundle over X whose fibers are finitely generated projective (right) A-modules, with A-linear transition functions.

Example 14 If $A = \mathbb{R}$ or $\mathbb{C}$, an A-vector bundle is a usual vector bundle. If $A = C(Y)$, an A-vector bundle over X is equivalent to an ordinary vector bundle over $X \times Y$.

Definition 15 Let A be a C^*-algebra and let E_0, E_1 be A-vector bundles over a compact manifold M. An *A-elliptic operator*

$$D\colon C^\infty(M, E_0) \to C^\infty(M, E_1)$$

will mean an elliptic A-linear ψDO from sections of E_0 to sections of E_1. Such an operator extends to a bounded A-linear map on suitable Sobolev spaces (Hilbert A-modules) $\mathcal{H}_0$ and $\mathcal{H}_1$. One can find a decomposition

$$\mathcal{H}_0 = \mathcal{H}_0' \oplus \mathcal{H}_0'', \quad \mathcal{H}_1 = \mathcal{H}_1' \oplus \mathcal{H}_1'',$$
$$\mathcal{H}_0'' \text{ and } \mathcal{H}_1'' \text{ finitely generated projective,}$$
$$D\colon \mathcal{H}_0' \xrightarrow{\cong} \mathcal{H}_1', \quad D\colon \mathcal{H}_0'' \to \mathcal{H}_1''.$$

This means that "up to A-compact perturbation" the kernel and cokernel of D are finitely generated projective A-modules. The *index* of D is

$$\operatorname{Ind} D = [\mathcal{H}_0''] - [\mathcal{H}_1''],$$

computed in the group of formal differences of isomorphism classes of such modules, $K_0(A)$.

III.2. Flat C^*-Algebra Bundles and the Assembly Map

If X is a compact space and A is a C^*-algebra with unit, the group of formal differences of isomorphism classes of A-vector bundles over X is denoted $K^0(X; A)$. The following is analogous to Swan's Theorem.

Proposition 16 *If X is a compact space and A is a C^*-algebra with unit, then $K^0(X; A)$ is naturally isomorphic to $K_0(C(X) \otimes A)$.*

Definition 17 Let X be a compact space, $\widetilde{X} \to X$ a normal covering with covering group π. Let $C_r^*(\pi)$ be the reduced group C^*-algebra of π (the completion of the group ring in the operator norm for its action on $L^2(\pi)$). The *universal $C_r^*(\pi)$-bundle* over X is

$$\mathcal{V}_X = \widetilde{X} \times_\pi C_r^*(\pi) \to X.$$

It has a class $[\mathcal{V}_X] \in K^0(X; C_r^*(\pi))$ which is pulled back (via the classifying map $X \to B\pi$) from the class of

$$\mathcal{V} = E\pi \times_\pi C_r^*(\pi) \to B\pi$$

in $K^0(B\pi; C_r^*(\pi))$. ($B\pi$ may not be compact, but it can always be approximated by finite CW complexes.) "Slant product" with $[\mathcal{V}]$ defines the *assembly map*

$$\mathcal{A}\colon K_*(B\pi) \to K_*(C_r^*(\pi)).$$

III.3. Kasparov Theory and the Index Theorem

The formalism of *Kasparov theory* attaches, to an elliptic operator D on a manifold M, a K-homology class $[D] \in K_*(M)$. If M is compact, the *collapse map* $c\colon M \to \mathrm{pt}$ is proper and $\operatorname{Ind} D = c_*([D]) \in K_*(\mathrm{pt})$.

Now if E is an A-vector bundle over M and D is an elliptic operator over M, we can form "D with coefficients in E," an A-elliptic operator. The Mishchenko-Fomenko index of this operator is computed by pairing

$$[D] \in K_*(M) \quad \text{with} \quad [E] \in K^0(M; A).$$

In particular, if $\widetilde{M} \to M$ is a normal covering of M with covering group π, then we can form D with coefficients in $\mathcal{V}_X$, and its index is $\mathcal{A} \circ u_*([D])$, where $u \colon M \to B\pi$ is the classifying map for the covering.

Conjecture 18 (Novikov Conjecture) *The assembly map $\mathcal{A} \colon K_*(B\pi) \to K_*(C_r^*(\pi))$ is rationally injective for all groups π, and is injective for all torsion-free groups π.*

There are no known counterexamples. Conjecture 18 is known for discrete subgroups of Lie groups [Kasparov], amenable groups [Higson-Kasparov], and many other classes.

III.4. Applications

1. The L^2-Index Theorem. The connection with Atiyah's Theorem from Lecture II is as follows. Suppose D is an elliptic operator on a compact manifold M, and $\widetilde{M} \to M$ is a normal covering of M with covering group π. The group C^*-algebra $C_r^*(\pi)$ embeds in the group von Neumann algebra, and the trace τ then induces a homomorphism $\tau_*\colon K_0(C_r^*(\pi)) \to \mathbb{R}$. The image under τ_* of the index of D with coefficients in $C_r^*(\pi)$ can be identified with the L^2-index of $\widetilde{D}$, the lift of D to $\widetilde{M}$. Atiyah's Theorem thus becomes the assertion that the following diagram commutes:

$$
\begin{array}{ccc}
K_0(M) & \xrightarrow{\;u_*\;} & K_0(B\pi) \\
\downarrow{\scriptstyle c_*} & & \downarrow{\scriptstyle \mathcal{A}} \\
K_0(\mathrm{pt}) = \mathbb{Z} & & K_0(C_r^*(\pi)) \\
& \searrow & \downarrow{\scriptstyle \tau_*} \\
& & \mathbb{R}.
\end{array}
$$

2. Original Version of the Novikov Conjecture. Consider the signature operator D on a closed oriented manifold M^{4k}. This is constructed so that $\operatorname{Ind} D$ is the *signature* of M, i.e., the signature of the form

$$\langle x, y \rangle = \langle x \cup y, [M] \rangle$$

on middle cohomology $H^{2k}(M, \mathbb{R})$. The signature is obviously an oriented homotopy invariant and Hirzebruch's formula says $\operatorname{sign} M = \langle \mathcal{L}(M), [M] \rangle$, where $\mathcal{L}(M)$ is a power series in the rational Pontryagin classes, the Poincaré dual of $\operatorname{Ch}[D]$. Here $\operatorname{Ch}\colon K_0(M) \to H_*(M, \mathbb{Q})$ is the Chern character, a natural transformation of homology theories.

If $u\colon M \to B\pi$, $u_*(\operatorname{Ch}[D]) \in H_*(B\pi, \mathbb{Q})$ is called a *higher signature* of M and Novikov conjectured that, like the ordinary signature (the case $\pi = 1$), it is an oriented homotopy invariant. The conjecture follows from injectivity of the assembly map, since Kasparov and Mishchenko showed that $\mathcal{A} \circ u_*([D])$ is an oriented homotopy invariant.

3. Positive Scalar Curvature. If M^n is a closed spin manifold, then M carries a special first-order elliptic operator, the *Dirac operator* D with a class $[D] \in KO_n(M)$. The operator D depends on a choice of Riemannian metric, though its K-homology class is independent of the choice. Lichnerowicz proved that

$$D^2 = \nabla^*\nabla + \frac{\kappa}{4},$$

where κ is the scalar curvature of the metric. Thus if $\kappa > 0$, the spectrum of D is bounded away from 0 and $\operatorname{Ind} D = 0$ in

$$KO_n(\text{pt}) = \begin{cases} \mathbb{Z}, & n \equiv 0 \bmod 4, \\ \mathbb{Z}/2, & n \equiv 1 \text{ or } 2 \bmod 8, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 19 (Stolz) *If M^n is a closed simply connected spin manifold with Dirac operator class $[D] \in KO_n(M)$, and if $n \geq 5$, then M admits a metric of positive scalar curvature if and only if $\operatorname{Ind} D = 0$ in $KO_n(pt)$.*

What if M is not simply connected? Then Gromov-Lawson, Schoen-Yau showed there are other obstructions coming from the fundamental group, and Gromov-Lawson suggested that the "higher index" of D is responsible.

Theorem 20 (Rosenberg) *Suppose M is a closed spin manifold and $u\colon M \to B\pi$ classifies the universal cover of M. If M admits a metric of positive scalar curvature and if the (strong) Novikov Conjecture holds for π, then $u_*([D]) = 0$ in $KO_n(B\pi)$.*

For some torsion-free groups, the converse is known to hold for $n \geq 5$, generalizing Theorem 19.

Conjecture 21 (Gromov-Lawson) *A closed aspherical manifold cannot admit a metric of positive scalar curvature.*

Theorem 20 shows that the Strong Novikov Conjecture implies Conjecture 21, at least for spin manifolds.

For groups with torsion, the assembly map is usually not an isomorphism, so the converse of Theorem 20 is quite unlikely.

Definition 22 Fix a simply connected spin manifold J^8 of dimension 8 with $\widehat{A}$-genus 1. (Such a manifold is known to exist, and Joyce constructed an explicit example with Spin(7) holonomy.) Taking a product with J does not change the KO-index of the Dirac operator. Say that a manifold M *stably admits a metric of positive scalar curvature* if there is a metric on $M \times J \times \cdots \times J$ with positive scalar curvature, for sufficiently many J factors.

Proposition 23 *A simply connected manifold of dimension* $\neq 3, 4$ *stably admits a metric of positive scalar curvature if and only if it actually admits a metric of positive scalar curvature.*

For finite fundamental group, the best general result is:

Theorem 24 (Rosenberg-Stolz) *Let M^n be a spin manifold with finite fundamental group π, with Dirac operator class $[D]$, and with classifying map $u\colon M \to B\pi$ for the universal cover. Then M stably admits a metric of positive scalar curvature if and only $\mathcal{A} \circ u_*([D]) = 0$ in $KO_n(C_r^*(\pi))$. (Of course, for π finite, $C_r^*(\pi) = \mathbb{R}[\pi]$.)*

This has been generalized by Stolz to those groups π for which the Baum-Connes assembly map in KO (a generalization of our $\mathcal{A}$, taking the torsion in π into account) is injective. This is a fairly large class including all discrete subgroups of Lie groups.

IV. Other C^*-Algebras and Applications in Topology: Group Actions, Foliations, $\mathbb{Z}/k$-Indices, and Coarse Geometry

IV.1. Crossed Products and Invariants of Group Actions

If a (locally compact) group G acts on a locally compact space X, one can form the *transformation group algebra* or *crossed product* $C^*(G, X)$ or $C_0(X) \rtimes G$. When G acts freely and properly on X, $C^*(G, X)$ is strongly Morita equivalent to $C_0(G/X)$. It thus plays the role of the algebra of functions on G/X, even when the latter is a "bad" space, and captures much of the equivariant topology, as we see from:

Theorem 25 (Green-Julg) *If G is compact, there is a natural isomorphism*

$$K_*(C^*(G, X)) \cong K_G^{-*}(X).$$

Definition 26 An n-dimensional *orbifold* X is a space covered by charts each homeomorphic to $\mathbb{R}^n/G$, where G is a finite group (which may vary from chart to chart) acting linearly on $\mathbb{R}^n$, and with compatible transition functions. Example: a quotient of a manifold by a locally linear action of a finite group. Each smooth orbifold X is of the form $\widetilde{X}/O(n)$, where $\widetilde{X}$ is its frame bundle, and the action of $O(n)$ is locally free. $C^*_{\text{orb}}(X) = C^*(O(n), \widetilde{X})$ is called the *orbifold algebra* of X [Farsi]. (It depends on the orbifold structure, not just the homeomorphism class of X as a space.) Note that $C^*_{\text{orb}}(X)$ is strongly Morita equivalent to $C_0(X)$ when X is a manifold, or to $C^*(G, M)$ when X is the quotient of a manifold M by an action of a finite group G.

An elliptic operator D on the orbifold (which in each local chart $\mathbb{R}^n/G$ is a G-invariant elliptic operator on $\mathbb{R}^n$) defines a class $[D] \in K^{-*}(C^*_{\text{orb}}(X))$ (which we think of as $K^{\text{orb}}_*(X)$. If X is compact, then as in the manifold case, $\operatorname{Ind} D = c_*([D]) \in K_*(\text{pt})$.

Applying the Kasparov formalism and working out all the terms, one can deduce various index theorems for orbifolds, originally obtained by Kawasaki by a different method.

IV.2. Foliation C^*-Algebras and Applications

Definition 27 Let M^n be a compact smooth manifold, $\mathcal{F}$ a foliation of M by leaves L^p of dimension p, codimension $q = n - p$. Then one can define a C^*-algebra $C^*(M, \mathcal{F})$ encoding the structure of the foliation. (This is the C^*-completion of the convolution algebra of functions on the holonomy groupoid.) When the foliation is a fibration $L \to M \to X$, where X is a compact q-manifold, then $C^*(M, \mathcal{F})$ is strongly Morita equivalent to $C(X)$. This justifies thinking of $K_*(C^*(M, \mathcal{F}))$ as $K^{-*}(M/\mathcal{F})$, the *K-theory of the space of leaves*. When the foliation comes from a locally free action of a Lie group G on M, then $C^*(M, \mathcal{F})$ is just the crossed product $C^*(G, M)$.

Introducing $C^*(M, \mathcal{F})$ makes it possible to extend the Connes index theorem for foliations. If D is an operator elliptic along the leaves, then in general $\operatorname{Ind} D$ is an element of the group $K_0(C^*(M, \mathcal{F}))$. If there is an invariant transverse measure μ, then one obtains a real-valued index by composing with the map

$$\int d\mu \colon K_0(C^*(M, \mathcal{F})) \to \mathbb{R}.$$

Theorem 28 (Connes-Skandalis) *Let* $(M, \mathcal{F})$ *be a compact (smooth) foliated manifold and let*

$$D \colon C^\infty(M, E_0) \to C^\infty(M, E_1)$$

be elliptic along the leaves. Then $\operatorname{Ind} D \in K_0(C^*(M, \mathcal{F}))$ *agrees with a "topological index"* $\operatorname{Ind}_{\text{top}}(D)$ *computed from the characteristic classes of* $\sigma(D)$ *just as in the usual Atiyah-Singer index theorem.*

Corollary 29 (Connes-Skandalis) *Let $(M,\mathcal{F})$ be a compact foliated manifold and let D be the Euler characteristic operator along the leaves. Then* $\operatorname{Ind} D$ *is the class of the zeros Z of a generic vector field along the fibers, counting signs appropriately.* (*Compare Poincaré-Hopf.*)

Example 30 M a compact Riemann surface of genus $g \geq 2$, $\widetilde{M}$ the hyperbolic plane, π a discrete torsion-free cocompact subgroup of $G = PSL(2,\mathbb{R})$. Foliate $V = \widetilde{M} \times_\pi S^2$, π acting on $S^2 = \mathbb{CP}^1$ by projective transformations, by the images of $\widetilde{M} \times \{x\}$. Note that V is an S^2-bundle over M. In this case there is no invariant transverse measure, but $\operatorname{Ind} D$ is non-zero in $K_0(C^*(V,\mathcal{F}))$. (It is $-2(g-1) \cdot [S^2]$.)

IV.3. C^*-Algebras and $\mathbb{Z}/k$-Index Theory

Definition 31 A *$\mathbb{Z}/k$-manifold* is a smooth compact manifold with boundary, M^m, along with an identification of ∂M with a disjoint union of k copies of a fixed manifold N^{m-1}. It is oriented if M is oriented, the boundary components have the induced orientation, and the identifications are orientation-preserving.

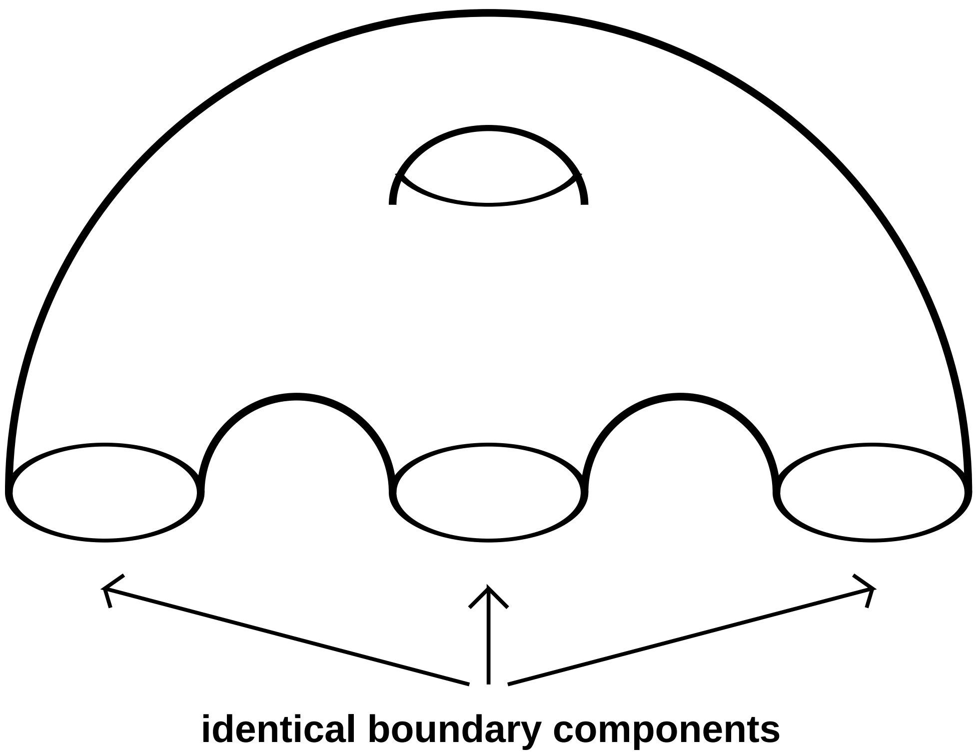

Figure: A $\mathbb{Z}/3$-manifold

One should really think of a $\mathbb{Z}/k$-manifold M as the singular space $M/\sim$ obtained by identifying all k of the boundary components with one another. This space is not a manifold (if $k > 2$), and so does not satisfy Poincaré duality. Thus, for example, an oriented $\mathbb{Z}/k$-manifold of dimension $4n$ does not have a signature in the usual sense. But it does have a signature mod k since it has a fundamental class in homology mod k, and there is a $\mathbb{Z}/k$-version of Hirzebruch's formula,

$$\operatorname{sign} M = \langle \mathcal{L}(M), [M] \rangle \in \mathbb{Z}/k.$$

Higson showed this formula may be obtained by showing that the signature operator gives an element in the C^*-algebra $\mathcal{D}_k$ of operators on $\mathcal{H} \otimes \mathbb{C}^{k+1}$ which modulo compacts have the form

$$\operatorname{diag}(\overbrace{A, A, \cdots, A}^{k}, 0).$$

We can recast $\mathbb{Z}/k$-index theory by attaching to each $\mathbb{Z}/k$-manifold M, $\partial M \cong N \times \mathbb{Z}/k$, a non-commutative C^*-algebra $C^*(M;\mathbb{Z}/k)$, so that objects like the signature operator D give a K-homology class in $K^0(C^*(M;\mathbb{Z}/k))$, whose $\mathbb{Z}/k$-index is obtained as

$$c_*([D]) \in K^0(C^*(\text{pt};\mathbb{Z}/k)) = \mathbb{Z}/k.$$

Let

$$\begin{aligned} C^*(M;\mathbb{Z}/k) = \{(f,g) : \; & f \in C(M), \\ & g \in C_0(N \times [0,\infty), M_k), \\ & g|_{N\times\{0\}} \text{ diagonal}, \\ & f|\partial M \text{ matching } g|_{N\times\{0\}}\}. \end{aligned}$$

$C^*(\text{pt};\mathbb{Z}/k)$ is defined similarly with M and N replaced by points. Note the use of the philosophy of non-commutative geometry: instead of collapsing the cylinders together, we let them "talk to each other."

IV.4. Roe C^*-Algebras and Coarse Geometry

Definition 32 [Roe] Let M be a complete Riemannian manifold (usually noncompact). Fix a suitable Hilbert space $\mathcal{H}$ on which $C_0(M)$ acts (for example, $L^2(M, d\,\text{vol})$). A bounded operator T on $\mathcal{H}$ is called *locally compact* if $\varphi T, T\varphi \in \mathcal{K}(\mathcal{H})$ for $\varphi \in C_c(M)$, *of finite propagation* if for some $R > 0$ (depending on T), $\varphi T \psi = 0$ for $\varphi, \psi \in C_c(M)$, $\text{dist}(\text{supp}\,\varphi, \text{supp}\,\psi) > R$. Let $C^*_{\text{Roe}}(M)$ be the C^*-algebra generated by the locally compact, finite propagation, operators.

Example 33 If M is compact, $C^*_{\text{Roe}}(M) = \mathcal{K}$, the compact operators. If $M = \mathbb{R}^n$ with the usual Euclidean metric, then $K_i(C^*_{\text{Roe}}(M)) \cong \mathbb{Z}$ for $i \equiv n \mod 2$, and $K_i(C^*_{\text{Roe}}(M)) = 0$ for $i \equiv n - 1 \mod 2$.

Theorem 34 (Roe) *If M is a complete Riemannian manifold, there is a functorial "assembly map" $\mathcal{A}\colon K_*(M) \to K_*(C^*_{Roe}(M))$. If D is a geometric elliptic operator on M (say the Dirac operator or the signature operator), it has a class in $K_0(M)$, and $\mathcal{A}([D])$ is its "coarse index." For noncompact spin manifolds, vanishing of $\mathcal{A}([D])$ (for the Dirac operator) is a necessary condition for there being a metric of uniformly positive scalar curvature in the quasi-isometry class of the original metric on M.*

Another application:

Theorem 35 (Principle of descent) *"Coarse Baum-Connes" for $C^*_{Roe}(\pi)$, π a group, but viewed as a discrete metric space, implies the Novikov Conjecture for π.*