Recent Progress on the Gromov-Lawson Conjecture

Jonathan Rosenberg
(University of Maryland)

in part joint work with
Boris Botvinnik
(University of Oregon)
Stephan Stolz
(University of Notre Dame)

June 7, 2002

The meaning of scalar curvature

If M^n a Riemannian manifold, *scalar curvature* $s(x)$ at a point $x \in M$ is (up to a normalizing factor) "average curvature" at x.

- When $n = 2$, scalar curvature$= 2\times$(Gauss curvature).

- In general, volume of a small geodesic ball of radius r about x is given by

  $$\mathrm{vol}(B_{\mathbb{R}^n}(r))\left(1 - \tfrac{1}{6(n+2)}s(x)r^2 + O(r^4)\right),$$

  so positive scalar curvature means small geodesic balls have smaller volume than balls of the same radius in $\mathbb{R}^n$.

- In a Riemannian product $M \times N$,

  $$s(x, y) = s(x) + s(y).$$

Trichotomy

If M^n a closed manifold (i.e., smooth, connected, compact, without boundary), exactly one of the following holds:

(A) M admits a Riemannian metric of positive scalar curvature. If $n = 2$, $M = S^2$ or $\mathbb{RP}^2$. If $n > 2$, any $s \in C^\infty(M)$ is the scalar curvature of some metric on M (Kazdan-Warner).

(B) M does not admit a Riemannian metric of positive scalar curvature, but admits a metric of non-negative scalar curvature. Any such metric is Ricci-flat (Kazdan-Warner). If $n = 2$, $M = T^2$ or a Klein bottle.

(C) M does not admit a Riemannian metric of non-negative scalar curvature.

Basic question: Which closed manifolds M^n belong to each of the various classes?

Review of the work of Gromov and Lawson

2 papers in *Ann. of Math.* **111** (1980), paper in *Publ. Math. IHES*, no. 58 (1983), discuss:

1. The surgery theorem and its applications.

2. Extensions of the Lichnerowicz-Hitchin Theorem.

3. Results on 3- and 4-manifolds.

4. Results on complete positive scalar curvature metrics on non-compact manifolds.

In this talk we will concentrate on items 1 and 2.

Notation: PSC = positive scalar curvature

Theorem 1 (Surgery) (Gromov-Lawson, Schoen-Yau) *If M is a compact manifold of PSC with connected components M_i, and if a closed connected manifold M' can be obtained from M by surgeries in codimension ≥ 3, then M' also admits a metric of PSC.*

Theorem 2 (Lichnerowicz) *If M is a spin manifold with Dirac operator D, then*

$$D^2 = \nabla^* \nabla + \frac{s}{4}.$$

Thus if M is closed with PSC, all index invariants of D vanish.

Fundamental idea of Gromov-Lawson: If M^n is a closed manifold with classifying map $M \to B\pi$, $\pi = \pi_1(M)$, then obstructions to PSC on M should depend on the class of M in $H_n(B\pi, \mathbb{Z})$ or (in the spin case) $KO_n(B\pi, \mathbb{Z})$.

Theorem 3 (Jung, Stolz) *Let M^n be a compact connected manifold with $n = \dim M \geq 5$, with fundamental group π, and with classifying map $M \to B\pi$.*

1. *If M is spin, then M admits PSC if and only the class of $M \to B\pi$ in $ko_n(B\pi)$ lies in the subgroup generated by classes of $M' \to B\pi$ with M' a spin manifold with PSC.*

2. *If M is oriented and the universal cover of M is* **not** *spin, then M admits PSC if and only the class of $M \to B\pi$ in $H_n(B\pi, \mathbb{Z})$ lies in the subgroup generated by classes of $M' \to B\pi$ with M' oriented with PSC.*

3. *If M is non-orientable and the universal cover of M is* **not** *spin, then M admits PSC if and only the class of $M \to B\pi$ in $H_n(B\pi, \mathbb{Z}/2)$ lies in the subgroup generated by classes of $M' \to B\pi$ with PSC.*

Dirac obstructions and assembly

Suppose M^n is a spin manifold with fundamental group π and classifying map $f\colon M \to B\pi$. The spin structure defines fundamental classes $[M] \in ko_n(M)$ and $\mathrm{per}_*([M]) \in KO_n(M)$.

Let $C_r^*(\pi)$ denote the completion of the group ring $\mathbb{R}\pi$ in the operator norm for its action on $L^2(\pi)$. There are maps (natural in π)

$$ko_n(B\pi) \xrightarrow{\mathrm{per}_*} KO_n(B\pi) \xrightarrow{\mathrm{ass}_*} KO_n(C_r^*(\pi)).$$

Theorem 4 (Rosenberg) *Suppose M^n is a spin manifold with fundamental group π and classifying map $f\colon M \to B\pi$. If M has a metric of positive scalar curvature, then*

$$\mathit{ass}_* \circ \mathit{per}_* \circ f_*([M]) = 0.$$

Corollary 5 (Easy case of the Gromov-Lawson Conjecture) *Suppose M^n is a spin manifold with torsion-free fundamental group π and classifying map $f\colon M \to B\pi$. If ass$_*$ is injective for π* (the Strong Novikov Conjecture) *and the periodicity map $ko_n(B\pi) \to KO_n(B\pi)$ is injective, then M admits a metric of PSC if and only it its "Dirac obstruction" vanishes in $KO_n(B\pi)$.*

This applies to free abelian groups, surface groups, many other "standard" cases.

Sadly, for finite groups, one has almost the opposite extreme. The assembly map is computable (for example, the reduced assembly map $\widetilde{KO}_n(B\pi) \to \widetilde{KO}_n(C_r^*(\pi)) = KO_n(\mathbb{R}\pi/\mathbb{R})$ vanishes identically if $|\pi|$ is odd) but never injective, and the periodicity map is rarely injective also.

Modifications of the Gromov-Lawson Conjecture

"Gromov-Lawson-Rosenberg Conjecture": ("Dirac tells all") If M^n is a spin manifold, $n \geq 5$ with fundamental group π and classifying map $f\colon M \to B\pi$, then M admits a metric of PSC if and only it its "Dirac obstruction" $\text{ass}_* \circ \text{per}_* \circ f_*([M])$ vanishes in $KO_n(C_r^*(\pi))$. If the universal cover of M^n is non-spin (so there are no Dirac obstructions), then M admits a metric of PSC.

This is known to fail in some cases, but there are no known counterexamples with π finite.

We stand a better chance if we build in periodicity in the geometry, not just in the algebraic topology.

Fix a Bott manifold J^8 which is simply connected, spin, with $\widehat{A}(J) = 1$.

Example: a Joyce manifold with holonomy Spin(7). This represents Bott periodicity in KO_*.

Say M stably admits a metric of PSC if

$$M \times \overbrace{J^8 \times \cdots \times J^8}^{k}$$

admits a metric of PSC for k sufficiently large.

"Stable Gromov-Lawson-Rosenberg Conjecture": If M^n is a spin manifold with fundamental group π and classifying map $f\colon\ M \to B\pi$, then M stably admits a metric of PSC if and only it its "Dirac obstruction" $\mathrm{ass}_* \circ \mathrm{per}_* \circ f_*([M])$ vanishes in $KO_n(C_r^*(\pi))$. If the universal cover of M^n is non-spin (so there are no Dirac obstructions), then M stably admits a metric of PSC.

A few known results

Theorem 6 (Botvinnik-Gilkey-Stolz) *The Gromov-Lawson-Rosenberg Conjecture holds for spin manifolds with finite* π_1 *with periodic cohomology (i.e., all Sylow subgroups cylic or quaternionic).*

Theorem 7 (Stolz) *The Stable Gromov-Lawson-Rosenberg Conjecture holds for spin manifolds with fundamental group* π *as long as* π *stisfies the Baum-Connes Conjecture.*

Thus for "reasonable groups" we expect the Stable GLR Conjecture to hold. The unstable conjecture is much tougher to approach.

A result of Kwasik-Schultz makes it possible to reduce the study of the conjecture for finite groups to the easier case of p-groups.

Products and Toda brackets

In what follows, we specialize further to the key test case of elementary abelian p-groups (i.e., $(\mathbb{Z}/p)^r$). To avoid some technicalities, we'll stick to the case p odd, M non-spin.

Then $H_*(B(\mathbb{Z}/p)^r)$ and $\Omega_*(B(\mathbb{Z}/p)^r)$ can be computed by the Künneth Theorem. Tensor terms correspond to products of manifolds, but Tor terms correspond to Toda brackets.

Example: If $M \to B\pi$ and $M' \to B\pi'$ represent bordism classes each of order p, the corresponding Tor term in $\Omega_*(B\pi \times B\pi')$ is given by the Toda bracket $\langle M, p, M' \rangle$, manufactured as follows:

Choose $W \to B\pi$ bounding $pM = \overbrace{M \amalg \cdots \amalg M}^{p}$ and W' bounding pM'. Then glue $W \times M'$ with boundary $pM \times M' \cong M \times pM'$ to $M \times W'$.

Theorem 8 *A Toda bracket $\langle M, P, M' \rangle$ admits PSC if M and M' do. In other words, if $M \times P = \partial W_0$ and $P \times M' = \partial W_1$ for some compact manifolds with boundary, W_0 and W_1, then*

$$N = \left(W_0 \times M'\right) \cup_{M \times P \times M'} (M \times W_1)$$

admits a metric of PSC.

Proof. Rescale M' to have small diameter so $W_0 \times M'$ has PSC. Also rescale M to have small diameter so $M \times W_1$ has PSC. Then one just needs to interpolate between two rescaled metrics on $M \times P \times M'$ using a Gromov-Lawson trick:

In $H_n(B(\mathbb{Z}/p)^r)$, $n \leq r$, we have toral classes generated by

$$T^n = B\mathbb{Z}^n \to B(\mathbb{Z}/p)^n \hookrightarrow B(\mathbb{Z}/p)^r.$$

We can define a complement to the atoral classes which we call the atoral part.

Theorem 9 *Let p be an odd prime, let π be an elementary abelian p-group, and let $n \geq 5$. Let M^n be a non-spin manifold with $f\colon M \to B\pi$ the classifying map for its universal covering. If the class $[M \xrightarrow{f} B\pi] \in H_n(B\pi)$ is atoral, then M has a metric with PSC. In particular, if $n > \operatorname{rk}\pi$, then every non-spin n-manifold with fundamental group π has a metric of PSC.*

Sketch of proof of Theorem 9

The key fact that makes this work is the following. For any space X, we denote by $RH_*(X)$ the image of the Thom map $\Omega_*(X) \to H_*(X, \mathbb{Z})$, and call it the representable homology.

Theorem 10 *Let π be an elementary abelian p-group, where p is an odd prime. Then*

$$RH_*(B\pi)$$

is generated (as an abelian group) by elements $x_1 \otimes \cdots \otimes x_j \in H_*(B\sigma_1) \otimes \cdots \otimes H_*(B\sigma_j)$, with $\sigma_1 \times \cdots \times \sigma_j$ a subgroup of π with each σ_i a cyclic p-group.

This is based on an η-invariant calculation together with the structure of $BP_*(B\pi)$. Product classes as in Theorem 10 are either toral or else have at least one factor which is a lens space with PSC. It's not clear whether the toral homology classes contain manifolds of PSC.