Dualities in field theories and the role of K-theory

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Quantization and NCG, RIMS, Kyoto, Feb. 21–23, 2011
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Plan of the Lectures

Abstract: It is now known (or in some cases just believed) that many quantum field theories exhibit dualities, equivalences with the same or a different theory in which things appear very different, but the overall physical implications are the same. We will discuss some of these dualities from the point of view of a mathematician, focusing on "charge conservation" and the role played by K-theory and noncommutative geometry.

1. Overview with some classical examples.
2. Topological T-duality.
3. Problems presented by S-duality and other dualities.

Part I

Overview with Some Classical Examples

Fields

Most physical theories describe fields, e.g., the *gravitational field*, *electric field*, *magnetic field*, etc. Fields can be

- scalar-valued functions (scalars),
- sections of vector bundles (vectors),
- connections on principal bundles (special cases of gauge fields),
- sections of spinor bundles (spinors).

Lagrangians and Least Action

In classical physics, the fields satisfy a variational principle — they are critical points of the action S, which in turn is the integral of a *local* functional $\mathcal{L}$ called the Lagrangian. This is called the *principle of least action*, and can be traced back to Fermat's theory of optics (1662). The Euler-Lagrange equations for critical points of the action are the equations of motion.

Examples

Let M be a 4-manifold, say compact.

1. Yang-Mills Theory. Field is a connection A on a principal G-bundle. "Field strength" F is the curvature, a $\mathfrak{g}$-valued 2-form. Action is $S = \int_M \operatorname{Tr} F \wedge *F$.

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2. General Relativity (in Euclidean signature). Field is a Riemannian metric g on M. Action is $S = \int_M R \, d\mathrm{vol}$, $R =$ scalar curvature. Field equation is Einstein's equation.

Quantum Mechanics

Unlike classical mechanics, quantum mechanics is not deterministic, only probabilistic. The key property of quantum mechanics is the Heisenberg uncertainty principle, that observable quantities are represented by *noncommuting operators* A represented on a Hilbert space $\mathcal{H}$. In the quantum world, every particle has a wave-like aspect to it, and is represented by a wave function ψ, a unit vector in $\mathcal{H}$. The phase of ψ is not directly observable, only its amplitude, or more precisely, the state φ_ψ defined by ψ:

$$\varphi_\psi(A) = \langle A\psi, \psi\rangle .$$

But the phase is still important since *interference* depends on it.

Quantum Fields

The quantization of classical field theories is based on *path integrals*. The idea (not 100% rigorous in this formulation) is that *all fields contribute*, not just those that are critical points of the action (i.e., solutions of the classical field equations). Instead, one looks at the partition function

$$Z = \int e^{iS(\varphi)/\hbar}\, d\varphi \text{ or } \int e^{-S(\varphi)/\hbar}\, d\varphi\,,$$

depending on whether one is working in Lorentz or Euclidean signature. By the principle of stationary phase, only fields close to the classical solutions should contribute very much. Expectation values of physical quantities are given by

$$\langle A \rangle = \left(\int A(\varphi)\, e^{iS(\varphi)/\hbar}\, d\varphi \right) / Z\,.$$

Duality

A duality is a transformation between different-looking physical theories that, rather magically, have the same observable physics. Often, such dualities are part of a discrete group, such as $\mathbb{Z}/2$ or $\mathbb{Z}/4$ or $SL(2,\mathbb{Z})$.

Example (Electric-magnetic duality)

There is a symmetry of Maxwell's equations in free space

$$\begin{aligned} \nabla \cdot E &= 0, & \nabla \cdot B &= 0, \\ \frac{\partial E}{\partial t} &= c\,\nabla \times B, & \frac{\partial B}{\partial t} &= -c\,\nabla \times E, \end{aligned} \tag{1}$$

given by $E \mapsto -B$, $B \mapsto E$. This is a duality of order 4.

Fourier Duality

Example (Configuration space-momentum space duality)

Another example from standard quantum mechanics concerns the quantum harmonic oscillator (say in one dimension). For an object with mass m and a restoring force with "spring constant" k, the Hamiltonian is

$$H = \frac{k}{2}x^2 + \frac{1}{2m}p^2\,, \tag{2}$$

where p is the momentum. In classical mechanics, $p = m\dot{x}$. But in quantum mechanics (with $\hbar$ set to 1),

$$[x, p] = i\,. \tag{3}$$

We obtain a duality of (2) and (3) via $m \mapsto \frac{1}{k}$, $k \mapsto \frac{1}{m}$, $x \mapsto p$, $p \mapsto -x$. This is again a duality of order 4, and is closely related to the Fourier transform.

The Dirac Monopole

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Dirac (1931) proposed to solve both problems at once with a quantum theory of E&M that in modern terms we would call a $U(1)$ gauge theory.

The Dirac Monopole (cont'd)

In Dirac's theory, we assume spacetime is a 4-manifold M, say $\mathbb{R}^4 \setminus \mathbb{R} \cong \mathbb{R}^2 \times S^2$ (Minkowski space with the time trajectory of one particle taken out). The (magnetic) vector potential (A^1, A^2, A^3) and electric potential $A^0 = \phi$ of classical E&M are combined into a single entity A, a (unitary) connection on a complex line bundle L over M. Thus iA is *locally* a real-valued 1-form, and $F = i\mu dA$, μ a constant, is a 2-form encoding both of the fields E (via the $(0, j)$ components) and B (via the (j, k) components, $0 < j < k$). The Chern class $c_1(L) \in H^2(M, \mathbb{Z}) \cong \mathbb{Z}$ is an invariant of the topology of the situation. Of course, F should really be $i\mu$ times the curvature of A, and Chern-Weil theory says that the de Rham class $[F]$ is $2\pi\mu$ times the image of $c_1(L)$ in $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$. L is associated to a principal $U(1)$-bundle $P \to M$, and Dirac identifies a section of this bundle with the phase of a wave function of a charged particle in M.

Charge Quantization

In the above setup, if we integrate F over the S^2 that links the worldline we removed, we get $2\pi\mu c_1(L)$, and this is the flux of the magnetic field through S^2. So the deleted worldline can be identified with that of a magnetic monopole of charge $g = \mu c_1(L)$ in suitable units. Suppose we consider the motion of a test charge of electric charge q around a closed loop γ in M. In quantum E&M, by the Aharonov-Bohm effect, the exterior derivative is replaced by the covariant derivative (involving the vector potential A). So the phase change in the wave function is basically the holonomy of $(P \to M, A)$ around γ, or (taking $\hbar = 1$) $\exp\left(q\mu \oint_\gamma A\right)$. Since M is simply connected, γ bounds a disk D and this is $\exp\left(-iq \int_D F\right)$. Taking D in turn to be the two hemispheres in S^2, we get two answers which differ by a factor of

$$\exp\left(i\,q\int_{S^2} F\right) = e^{2\pi i\,q\,\mu\,c_1(L)}.$$

Since this must be 1, we get Dirac's quantization condition $qg \in \mathbb{Z}$.

Charge Groups

The upshot of this analysis is that we expect both electrical and magnetic charges to be quantized, but that *the basic quanta of electrical and magnetic charge should be inversely proportional in size*. In other words, the smallness of the fundamental electrical charge means that the charge of any magnetic monopole has to be large. In any event, we expect the electrical and magnetic charges (q, g) to take values in an abelian charge group C, in this case $\mathbb{Z}^2$. It is also reasonable to expect there to be particles, usually called dyons, with both charges q and g non-zero.

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Now think about the classical electric-magnetic duality that switches E and B. The Montonen-Olive conjecture, for which there is now some tantalizing evidence, is that in a wide variety of cases this should extend to a duality of quantum theories, which would necessarily give an isomorphism of charge groups between a theory and its dual.

The Role of Topology

In Dirac's theory, the quantization of magnetic charge and of electrical charge arise from different origins. The former is a purely topological phenomenon; it comes from the fact that the Chern classes live in *integral* cohomology. Quantization of electrical charge comes from the requirement that the *action* (for the field associated to a charged particle moving in the background electromagnetic field of a monopole) be well-defined and not multi-valued, so this can be viewed as a version of anomaly cancellation. However, since Maxwell's equations are invariant under electro-magnetic duality, we can imagine an equivalent dual theory in which electric charge is topological and magnetic charge is quantized to achieve anomaly cancellation.

A General Setup

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2. Each theory in $\mathcal{C}$ has an associated charge group C. If $g \in G$ gives an equivalence between two theories in $\mathcal{C}$, it must give an isomorphism between the associated charge groups. In particular, the stabilizer of a fixed theory operates by automorphisms on C.
3. In many cases, the charge groups arise as topological invariants. We have already seen how $\operatorname{Pic} X = H^2(X, \mathbb{Z})$ arises. We will see how K-theory arises in some cases.

String Theories

Many of the most interesting examples of duality (and of topological charge groups) arise in (supersymmetric) string theories. These are quantum field theories based on the idea of replacing point particles by strings or 1-manifolds (always compact, but maybe with boundary — contrary to mathematical usage, physicists call these "open strings"). For anomaly cancellation reasons, the spacetime manifold has to be 10-dimensional. The worldsheet traced out by a string in the spacetime X is a compact 2-manifold Σ (again, possibly with boundary), so we obtain fields that are maps $f\colon \Sigma \to X$, with the sigma-model action of the form

$$\int_\Sigma \|\nabla f\|^2 + \int_\Sigma f^*(B) + (\text{terms involving other fields}). \tag{4}$$

Here B is a 2-form on X called the B-field (*not* the magnetic field). The term $\int_\Sigma f^*(B)$ is called the Wess-Zumino term.

D-Brane Charges

In string theories, boundary conditions (of Dirichlet or Neumann type) must be imposed on the open string states. These are given by D-branes (D for Dirichlet), submanifolds of the spacetime X on which strings are allowed to "end." If we forget certain complications and look at type II string theory, then X is a 10-dimensional spin manifold and the D-branes are spin^c submanifolds, of even dimension for type IIB and of odd dimensional for type IIA. There is another piece of structure; each D-brane carries a Chan-Paton vector bundle that reflects a $U(N)$ gauge symmetry allowing for local exchanges between coincident D-branes.

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The D-branes carry charges which are not just numbers but elements of the K-group $K(X)$ (in the type IIB theory), $K^{-1}(X)$ (in the type IIA theory), or $KO(X)$ (in the type I theory).

Why K-Theory?

The idea that the D-brane charges should take values in K-theory comes from Minasian-Moore and Witten, around 1997–1998, with further elaboration by other authors later. Motivation comes from several sources:

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- compatibility with analysis of decay of unstable branes;
- compatibility with what is known about string duality.

We will not attempt to go through these arguments but will discuss some consequences.

D-Brane Charges and Duality

For a D-brane $W \overset{\iota}{\hookrightarrow} X$ with Chan-Paton bundle $E \to W$, the K-theory charge is $\iota_!([E])$, where $[E]$ is the class of E in $K(W)$, and $\iota_!$ is the Gysin map in K-theory (defined using the spin^c structures). While string dualities do not have to preserve the diffeomorphism type, or even the dimension, of D-branes, they do have to give rise to an isomorphism of the K-groups in which the D-brane charges lie.

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The most important kinds of string theory dualities are T-duality, an outgrowth of classical Fourier duality ("T" originally standing for "target space"), and S-duality, an outgrowth of classical electro-magnetic duality. The big difference between them is that T-duality preserves coupling strength and changes geometry, whereas S-duality ("S" standing for "strong-weak") interchanges strong and weak coupling and preserves the geometry of spacetime, just as electro-magnetic duality inverts the magnitude of charges.

Plan for the Other Lectures

T-duality replaces tori (of a fixed dimension k) in the spacetime manifold X by the dual tori (quotients of the *dual space* by the *dual lattice*), inverting the radii. If k is odd, T-duality interchanges the theories of types IIA and IIB, so one gets an isomorphism $K(X) \cong K^{-1}(X^\sharp)$ or $K^{-1}(X) \cong K(X^\sharp)$. S-duality interchanges type I string theory with the $SO(32)$ heterotic string theory, and also maps type IIB string theory to itself.

In my other two talks I plan to discuss T-duality and S-duality in more detail, and the way charge conservation in K-theory sheds more light on them.

Part II

Topological T-duality

(partially joint work with Mathai Varghese)

Gerbes and the H-Flux

It's now time to correct a slight oversimplification in Lecture 1: the "B-field" in the sigma-model action is not necessarily globally well-defined, though its *field strength* $H = dB$ does make sense globally. Properly normalized, one can show that H defines an integral de Rham class in H^3. This can be refined to an actual class in $[H] \in H^3(X, \mathbb{Z})$. Thus the Wess-Zumino term in the path integral should really be defined using a gerbe, for example a bundle gerbe in the sense of Murray with *curving* B and *Dixmier-Douady class* $[H]$. We usually refer to H (or to the associated class $[H] \in H^3(X, \mathbb{Z})$) as the H-flux.

Continuous-Trace Algebras and Twisted K-Theory

The association of H with a Dixmier-Douady class is not an accident, and indeed indicates a deeper connection with noncommutative geometry. To set this up in the simplest way, choose a stable continuous-trace algebra $A = CT(X, [H])$ with $\widehat{A} = X$ and with Dixmier-Douady class $[H]$. Thus A is the algebra of continuous sections vanishing at ∞ of a bundle over X with fibers $\mathcal{K}$ (the compact operators on a separable ∞-dimensional Hilbert space $\mathcal{H}$) and structure group $\operatorname{Aut}\mathcal{K} = PU(\mathcal{H}) \simeq K(\mathbb{Z}, 2)$.

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There are several possible definitions of twisted K-theory, but for our purposes we can define it as $K^{-i}(M, [H]) = K_i(A)$ with $A = CT(M, [H])$ as above. Up to isomorphism, this only depends on X and the cohomology class $[H] \in H^3(X, \mathbb{Z})$.

Twisted D-Brane Charges

In the presence of a topologically nontrivial H-flux, the K-theoretic classification of D-brane charges has to be modified. A D-brane $W \overset{\iota}{\hookrightarrow} X$ in type II string theory is no long a Spin^c manifold; instead it is Spin^c "up to a twist," according to the Freed-Witten anomaly cancellation condition $W_3(W) = \iota^*([H])$. Accordingly, the D-brane charge will live in the twisted K-group $K(X, [H])$ (in type IIB) or in $K^{-1}(X, [H])$ (in type IIA). Accordingly, if we have a T-duality between string theories on (X, H) and $(X^\sharp, H^\sharp)$, conservation of charge (for D-branes) requires an isomorphism of twisted K-groups of $(X, [H])$ and $(X^\sharp, [H^\sharp])$, with no degree shift if we dualize with respect to even-degree tori, and with a degree shift if we dualize with respect to odd-degree tori.

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One might wonder what happened to the K-groups of opposite parity, viz., $K^{-1}(X, [H])$ (in type IIB) and $K(X, [H])$ (in type IIA). These still have a physical significance in terms of Ramond-Ramond fields, so want these to match up under T-duality also.

Topology Change in T-Duality

Topological T-duality focuses on the topological aspects of T-duality. The first example of this phenomenon was studied by Alvarez, Alvarez-Gaumé, Barbón, and Lozano in 1993, and generalized 10 years later by Bouwknegt, Evslin, and Mathai. Let's start with the simplest nontrivial example of a circle fibration, where $X = S^3$, identified with $SU(2)$, T is a maximal torus. Then T acts freely on X (say by right translation) and the quotient X/T is $\mathbb{CP}^1 \cong S^2$, with quotient map $p\colon X \to S^2$ the Hopf fibration. Assume for simplicity that the B-field vanishes. We have $X = S^3$ fibering over $Z = X/T = S^2$. Think of Z as the union of the two hemispheres $Z^{\pm} \cong D^2$ intersecting in the equator $Z^0 \cong S^1$. The fibration is trivial over each hemisphere, so we have $p^{-1}(Z^{\pm}) \cong D^2 \times S^1$, with $p^{-1}(Z^0) \cong S^1 \times S^1$. So the T-dual also looks like the union of two copies of $D^2 \times S^1$, joined along $S^1 \times S^1$.

The Hopf fibration example (cont'd)

However, we have to be careful about the clutching that identifies the two copies of $S^1 \times S^1$. In the original Hopf fibration, the clutching function $S^1 \to S^1$ winds once around, with the result that the fundamental group $\mathbb{Z}$ of the fiber T dies in the total space X. But T-duality is supposed to interchange "winding" and "momentum" quantum numbers. So the T-dual $X^\sharp$ has no winding and is just $S^2 \times S^1$, while the winding of the original clutching function shows up in the H-flux of the dual.

In fact, following Buscher's method for dualizing a sigma-model, we find that the B-field on the dual side is different on the two copies of $D^2 \times S^1$; they differ by a closed 2-form, and so $H^\sharp$, the H-flux of the dual (for simplicity of notation we delete the brackets from now on), is nontrivial but well defined.

The Case of $S^2 \times S^1$ and S^3

Let's check the principle of K-theory matching in the case we've been considering, $X = S^3$ fibered by the Hopf fibration over $Z = S^2$. The H-flux on X is trivial, so D-brane charges lie in $K^*(S^3)$, with no twisting. And $K^0(S^3) \cong K^1(S^3) \cong \mathbb{Z}$.
On the T-dual side, we expect to find $X^\sharp = S^2 \times S^1$, also fibered over S^2, but simply by projection onto the first factor. If the H-flux on X were trivial, D-brane changes would lie in $K^0(S^2 \times S^1)$ and $K^1(S^2 \times S^1)$, both of which are isomorphic to $\mathbb{Z}^2$, which is too big.

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On the other hand, we can compute $K^*(S^2 \times S^1, H^\sharp)$ for the class $H^\sharp$ which is k times a generator of $H^3 \cong \mathbb{Z}$, using the Atiyah-Hirzebruch Spectral Sequence. The differential is

$$H^0(S^2 \times S^1) \xrightarrow{k} H^3(S^2 \times S^1),$$

so when $k = 1$, $K^*(S^2 \times S^1, H^\sharp) \cong K^*(S^3) \cong \mathbb{Z}$ for both $* = 0$ and $* = 1$.

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  - There is an involution (map of period 2) $(X, H) \mapsto (X^\sharp, H^\sharp)$ keeping the base Z fixed.
  - $K^*(X, H) \cong K^{*+1}(X^\sharp, H^\sharp)$.

The Bunke-Schick Construction

Bunke and Schick suggested constructing a theory satisfying these axioms by means of a universal example. It is known that (for reasonable spaces X, say CW complexes) all principal S^1-bundles $X \to Z$ come by pull-back from a diagram

$$\begin{tikzcd} X \arrow[r, dotted] \arrow[d] & ES^1 \simeq * \arrow[d] \\ Z \arrow[r, dotted] & BS^1 \simeq K(\mathbb{Z}, 2) \end{tikzcd}$$

Here the map $Z \dashrightarrow K(\mathbb{Z}, 2)$ is unique up to homotopy, and pulls the canonical class in $H^2(K(\mathbb{Z}, 2), \mathbb{Z})$ back to c_1 of the bundle.

Similarly, every class $H \in H^3(X, \mathbb{Z})$ comes by pull-back from a canonical class via a map $X \dashrightarrow K(\mathbb{Z}, 3)$ unique up to homotopy.

The Bunke-Schick Theorem

Theorem (Bunke-Schick)

There is a classifying space R, unique up to homotopy equivalence, with a fibration

$$\begin{tikzcd} K(\mathbb{Z},3) \arrow[r] & R \arrow[d] \\ & K(\mathbb{Z},2) \times K(\mathbb{Z},2), \end{tikzcd} \tag{5}$$

and any $(X, H) \to Z$ as in the axioms comes by a pull-back

$$\begin{tikzcd} X \arrow[r, dotted] \arrow[d] & E \arrow[d, "p"] \\ Z \arrow[r, dotted] & R, \end{tikzcd}$$

with the horizontal maps unique up to homotopy and H pulled back from a canonical class $h \in H^3(E, \mathbb{Z})$.

The Bunke-Schick Theorem (cont'd)

Theorem (Bunke-Schick)

Furthermore, the k-invariant of the Postnikov tower (5) characterizing R is the cup-product in

$$H^4(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2), \mathbb{Z})$$

of the two canonical classes in H^2. The space E in the fibration

$$\begin{tikzcd} S^1 \arrow[r] & E \arrow[d, "p"] \\ & R \end{tikzcd}$$

has the homotopy type of $K(\mathbb{Z}, 3) \times K(\mathbb{Z}, 2)$.

The Characteristic Class Formula

Corollary

If $(X \xrightarrow{p} Z, H)$ and $(X^\sharp \xrightarrow{p^\sharp} Z, H^\sharp)$ are a T-dual pair of circle bundles over a base space Z, then the bundles and fluxes are related by the formula

$$p_!(H) = [p^\sharp], \quad (p^\sharp)_!(H^\sharp) = [p].$$

Here $[p]$, $[p^\sharp]$ are the Euler classes of the bundles, and $p_!$, $(p^\sharp)_!$ are the "integration over the fiber" maps in the Gysin sequences. Furthermore, there is a pullback diagram of circle bundles

$$\begin{tikzcd} Y \arrow[r, "(p^\sharp)^*(p)"] \arrow[d, "p^*(p^\sharp)"'] & X \arrow[d, "p"] \\ X^\sharp \arrow[r, "p^\sharp"] & Z. \end{tikzcd}$$

in which H and $H^\sharp$ pull back to the same class on Y.

What is Higher-Dimensional T-Duality?

We now want to generalize T-duality to the case of spacetimes X "compactified on a higher-dimensional torus," or in other words, equipped with a principal $\mathbb{T}^n$-bundle $p\colon X \to Z$. In the simplest case, $X = Z \times \mathbb{T}^n = Z \times \overbrace{S^1 \times \cdots S^1}^{n}$. We can then perform a string of n T-dualities, one circle factor at a time. A single T-duality interchanges type IIA and type IIB string theories, so this n-dimensional T-duality "preserves type" when n is even and switches it when n is odd. In terms of our set of axioms for topological T-duality, we would therefore expect an isomorphism $K^*(X, H) \cong K^*(X^\sharp, H^\sharp)$ when n is even and $K^*(X, H) \cong K^{*+1}(X^\sharp, H^\sharp)$ when n is odd.

The Uniqueness Problem, Missing T-Duals, and the T-Duality Group

In the higher-dimensional case, a new problem presents itself: it is no longer clear that the T-dual should be unique. In fact, if we perform a string of n T-dualities, one circle factor at a time, it is not clear that the result should be independent of the order in which these operations are done. Furthermore, a higher-dimensional torus does not split as a product in only one way, so in principle there can be a lot of non-uniqueness.

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Another difficulty is that there are some spacetimes with H-flux that would appear to have no higher-dimensional T-duals at all, at least in the sense we have defined them so far, e.g., $X = T^3$, viewed as a principal $\mathbb{T}^3$-bundle over a point, with H the generator of $H^3(X, \mathbb{Z}) \cong \mathbb{Z}$.

Strategy of a NCG Approach (Mathai-Rosenberg)

Start with a principal $\mathbb{T}^n$-bundle $p\colon X \to Z$ and an "H-flux" $H \in H^3(X, \mathbb{Z})$. We assume that H is trivial when restricted to each $\mathbb{T}^n$-fiber of p. This of course is no restriction if $n = 2$, but it rules out cases with no T-dual in any sense.

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We want to lift the free action of $\mathbb{T}^n$ on X to an action on the continuous-trace algebra $A = CT(X, H)$. Usually there is no hope to get such a lifting for $\mathbb{T}^n$ itself, so we go to the universal covering group $\mathbb{R}^n$. If $\mathbb{R}^n$ acts on A so that the induced action on $\widehat{A}$ is trivial on $\mathbb{Z}^n$ and factors to the given action of $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, then we can take the crossed product $A \rtimes \mathbb{R}^n$ and use Connes' Thom Isomorphism Theorem to get an isomorphism between $K^{-*-n}(X, H) = K_{*+n}(A)$ and $K_*(A \rtimes \mathbb{R}^n)$.

Recovering Topological T-Duality

Under favorable circumstances, we can hope that the crossed product $A \rtimes \mathbb{R}^n$ will again be a continuous-trace algebra $CT(X^\sharp, H^\sharp)$, with $p^\sharp \colon X^\sharp \to Z$ a new principal $\mathbb{T}^n$-bundle and with $H^\sharp \in H^3(X^\sharp, \mathbb{Z})$. If we then act on $CT(X^\sharp, H^\sharp)$ with the dual action of $\widehat{\mathbb{R}}^n$, then by Takai Duality and stability, we come back to where we started. So we have a topological T-duality between (X, H) and $(X^\sharp, H^\sharp)$.

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$$K^{*+n}(X, H) \cong K^*(X^\sharp, H^\sharp),$$

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$$K^{*+n}(X, H) \cong K^*(X^\sharp, H^\sharp),$$

as required for matching of D-brane charges under T-duality. Now what about the problems we identified before, about potential non-uniqueness of the T-dual and "missing" T-duals? These can be explained either by non-uniqueness of the lift to an action of $\mathbb{R}^n$ on $A = CT(X, H)$, or by failure of the crossed product to be a continuous-trace algebra.

A Crucial Example

Let's now examine what happens when we try to carry out this program in one of our "problem cases," $n = 2$, $Z = S^1$, $X = T^3$ (a trivial $\mathbb{T}^2$-bundle over S^1), and H the usual generator of $H^3(T^3)$. First we show that there is an action of $\mathbb{R}^2$ on $CT(X, H)$ compatible with the free action of $\mathbb{T}^2$ on X with quotient S^1. We will need the notion of an *induced action*. We start with an action α of $\mathbb{Z}^2$ on $C(S^1, \mathcal{K})$ which is trivial on the spectrum. This is given by a map $\mathbb{Z}^2 \to C(S^1, \operatorname{Aut}\mathcal{K}) = C(S^1, PU(L^2(\mathbb{T})))$ sending the two generators of $\mathbb{Z}^2$ to the maps

$$w \mapsto \text{multiplication by } z,$$
$$w \mapsto \text{translation by } w.$$

(These unitaries commute in PU, not in U.)

A Calculation

Now form $A = \operatorname{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} C(S^1, \mathcal{K})$. This is a C^*-algebra with $\mathbb{R}^2$-action $\operatorname{Ind}\alpha$ whose spectrum (as an $\mathbb{R}^2$-space) is $\operatorname{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2} S^1 = S^1 \times \mathbb{T}^2 = X$. We can see that $A \cong CT(X, H)$ via "inducing in stages". Let $B = \operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}} C(S^1, \mathcal{K}(L^2(\mathbb{T})))$ be the result of inducing over the first copy of $\mathbb{R}$. It's clear that $B \cong C(S^1 \times \mathbb{T}, \mathcal{K})$. We still have another action of $\mathbb{Z}$ on B coming from the second generator of $\mathbb{Z}^2$, and $A = \operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}} B$. The action of $\mathbb{Z}$ on B is by means of a map $\sigma \colon S^1 \times \mathbb{T} \to PU(L^2(\mathbb{T})) = K(\mathbb{Z}, 2)$, whose value at (w, z) is the product of multiplication by z with translation by w. Thus A is a CT-algebra with Dixmier-Douady invariant $[\sigma] \times c = H$, where $[\sigma] \in H^2(S^1 \times \mathbb{T}, \mathbb{Z})$ is the homotopy class of σ and c is the usual generator of $H^1(S^1, \mathbb{Z})$.

A Calculation (cont'd)

Now that we have an action of $\mathbb{R}^2$ on $A = CT(X, H)$ inducing the free $\mathbb{T}^2$-action on the spectrum X, we can compute the crossed product to see what the associated "T-dual" is.

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$$A \rtimes_{\mathrm{Ind}\,\alpha} \mathbb{R}^2 \cong \left(C(S^1, \mathcal{K}) \rtimes_\alpha \mathbb{Z}^2 \right) \otimes \mathcal{K}.$$

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Recall that A_θ is the universal C^*-algebra generated by unitaries U and V with $UV = e^{2\pi i\theta} VU$. So if we look at the definition of α, we see that $A \rtimes_{\mathrm{Ind}\,\alpha} \mathbb{R}^2$ is the algebra of sections of a bundle of algebras over S^1, whose fiber over $e^{2\pi i\theta}$ is $A_\theta \otimes \mathcal{K}$. Alternatively, it is Morita equivalent to $C^*(\Gamma)$, where Γ is the discrete Heisenberg group of strictly upper-triangular 3×3 integral matrices.

Noncommutative T-Duals

Put another way, we could argue that we've shown that $C^*(\Gamma)$ is a noncommutative T-dual to (T^3, H), both viewed as fibering over S^1.

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Noncommutative T-Duals

Put another way, we could argue that we've shown that $C^*(\Gamma)$ is a noncommutative T-dual to (T^3, H), both viewed as fibering over S^1. So we have an explanation for the missing T-dual: we couldn't find it just in the world of topology alone because it's noncommutative. We will want to see how widely this phenomenon occurs, and also will want to resolve the question of nonuniqueness of T-duals when $n > 1$.

The Classification Theorem

Further analysis of this example leads to the following classification theorem:

Theorem (Mathai-Rosenberg)

Let $\mathbb{T}^2$ act freely on $X = T^3$ with quotient $Z = S^1$. Consider the set of all actions of $\mathbb{R}^2$ on algebras $CT(X, H)$ inducing this action on X, with H allowed to vary over $H^3(X, \mathbb{Z}) \cong \mathbb{Z}$. Then the set of exterior equivalence classes of such actions is parametrized by Maps$(Z, \mathbb{T})$. *The winding number of a map $f\colon Z \cong \mathbb{T} \to \mathbb{T}$ can be identified with the Dixmier-Douady invariant H. All these actions are given by the construction above, with f as the "Mackey obstruction map."*

The Mathai-Rosenberg Theorem

Consider a general $\mathbb{T}^2$-bundle $X \xrightarrow{p} Z$. We have an edge homomorphism

$$p_! \colon H^3(X, \mathbb{Z}) \to E_\infty^{1,2} \subseteq H^1(Z, H^2(\mathbb{T}^2, \mathbb{Z})) = H^1(Z, \mathbb{Z})$$

which turns out to play a major role.

Theorem (Mathai-Rosenberg)

Let $p\colon X \to Z$ be a principal $\mathbb{T}^2$-bundle as above, $H \in H^3(X, \mathbb{Z})$. Then we can always find a "generalized T-dual" by lifting the action of $\mathbb{T}^2$ on X to an action of $\mathbb{R}^2$ on $CT(X, H)$ and forming the crossed product. When $p_! H = 0$, we can always do this in such a way as to get a crossed product of the form $CT(X^\sharp, H^\sharp)$, where $(X^\sharp, H^\sharp)$ is a classical T-dual (e.g., as found though the purely topological theory). When $p_! H \neq 0$, the crossed product $CT(X, H) \rtimes \mathbb{R}^2$ is never locally stably commutative and should be viewed as a noncommutative T-dual.

Current Directions in Topological T-Duality

Here we just summarize some of the current trends in topological T-duality:

- the above approach with actions of $\mathbb{R}^n$ on continuous-trace algebras: more detailed study of non-uniqueness, extension to actions with more complicated isotropy.

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- a fancier approach using duality of sheaves (Bunke-Schick-Spitzweck-Thom).
- a generalization of the NCG approach using groupoids (Daenzer).
- algebraic analogues using Mukai duality with gerbes (Ben-Basset, Block, Pantev).

Part III

Problems presented by S-duality and other dualities

(partially joint work with Stefan Mendez-Diez)

Conjectured Dualities

As we mentioned before, there is believed to be an S-duality relating type I string theory to one of the heterotic string theories. There are also various other dualities relating these two theories to type IIA theory. Putting these together, we expect a (non-perturbative) duality between type I string theory on $T^4 \times \mathbb{R}^6$ and type IIA theory on $K3 \times \mathbb{R}^6$, at least at certain points in the moduli space.

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How can we reconcile this with the principle that brane charges in type I should take their values in KO, while brane charges in type IIA should take their values in K^{-1}?

On the face of it, this appears ridiculous: $KO(T^4 \times \mathbb{R}^6) = KO^{-6}(T^4)$ has lots of 2-torsion, while $K^*(K3)$ is all torsion-free and concentrated in even degree.

KO-Theory of T^4

One side is easy compute. Recall that for any space X,

$$KO^{-j}(X \times S^1) \cong KO^{-j}(X) \oplus KO^{-j-1}(X).$$

Iterating, we get

$$\begin{aligned} KO^{-6}(T^4) &\cong KO^{-6} \oplus 4KO^{-7} \oplus 6KO^{-8} \oplus 4KO^{-9} \oplus KO^{-10} \\ &\cong \mathbb{Z}^6 \oplus (\mathbb{Z}/2)^4 \oplus (\mathbb{Z}/2) \cong \mathbb{Z}^6 \oplus (\mathbb{Z}/2)^5. \end{aligned}$$

K-Theory of the Orbifold Limit of K3

The way we deal with the opposite side of the duality is to recall that a K3 surface can be obtained by blowing up the point singularities in T^4/G, where $G = \mathbb{Z}/2$ acting by multiplication by -1 on $\mathbb{R}^4/\mathbb{Z}^4$. This action is semi-free with 16 fixed points, the points with all four coordinates equal to 0 or $\frac{1}{2}$ mod $\mathbb{Z}$. If fact one way of deriving the (type I on T^4) $\leftrightarrow$ (type IIA on $K3$) duality explicitly uses the orbifold T^4/G.

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But what group should orbifold brane charges live in? Not just $K^*(T^4/G)$, as this ignores the orbifold structure. One solution that has been proposed is $K^*_G(T^4)$, which we computed. However, as we'll see, there appears to be a better candidate.

Cohomology Calculations

Let M be the result of removing an open ball around each G-fixed point in T^4. This is a compact manifold with boundary on which G acts freely; let $N = M/G$. We get a K3 surface back from N by gluing in 16 copies of the unit disk bundle of the tangent bundle of S^2 (known to physicists as the Eguchi-Hanson space), one along each $\mathbb{RP}^3$ boundary component in ∂N.

Theorem (with S. Mendez-Diez)

$$H^i(N, \partial N) \cong H_{4-i}(N) \cong \begin{cases} 0, & i = 0 \\ \mathbb{Z}^{15}, & i = 1 \\ \mathbb{Z}^6, & i = 2 \\ (\mathbb{Z}/2)^5, & i = 3 \\ \mathbb{Z}, & i = 4 \\ 0, & \textit{otherwise.} \end{cases}$$

K-Theory Calculations

Recall N is the manifold with boundary obtained from T^4/G by removing an open cone neighborhood of each singular point.

Theorem (with S. Mendez-Diez)

$K^0(N, \partial N) \cong K_0(N) \cong \mathbb{Z}^7$ *and*
$K^{-1}(N, \partial N) \cong K_1(N) \cong \mathbb{Z}^{15} \oplus (\mathbb{Z}/2)^5$.

Note that the reduced K-theory of (T^4/G) mod (singular points) is the same as $K^*(N, \partial N)$. Note the resemblance of $K^{-1}(N, \partial N)$ to $KO^{-6}(T^4) \cong \mathbb{Z}^6 \oplus (\mathbb{Z}/2)^5$. While they are not the same, the calculation suggests that the brane charges in type I string theory on $T^4 \times \mathbb{R}^6$ do indeed show up some way in type IIA string theory on the orbifold limit of $K3$.

Equivariant K-Theory Calculations

Again let $G = \mathbb{Z}/2$. Equivariant K-theory K_G^* is a module over the representation ring $R = R(G) = \mathbb{Z}[t]/(t^2 - 1)$. This ring has two important prime ideals, $I = (t - 1)$ and $J = (t + 1)$. We have $R/I \cong R/J \cong \mathbb{Z}$, $I \cdot J = 0$, $I + J = (I, 2) = (J, 2)$, $R/(I + J) = \mathbb{Z}/2$.

Theorem (with S. Mendez-Diez)

$K_G^0(\mathbb{T}^4) \cong R^8 \oplus (R/J)^8$, and $K_G^{-1}(\mathbb{T}^4) = 0$. Also, $K_G^0(M, \partial M) \cong (R/I)^7$, $K_G^{-1}(M, \partial M) \cong (R/I)^{10} \oplus (R/2I)^5$.

Discussion

Note that the equivariant K-theory calculation is a refinement of the ordinary K-theory calculation (since G acts freely on M and ∂M with quotients N and ∂N, so that $K^*_G(M)$ and $K^*_G(\partial M)$ are the same as $K^*(N)$ and $K^*(\partial N)$ *as abelian groups*, though with the addition of more structure). While we don't immediately need the extra structure, it may prove useful later in matching brane charges from $KO(T^4 \times \mathbb{R}^6)$ on specific classes of branes.

Other Cases of Type I/Type II Charge Matching

More generally, one could ask if there are circumstances where understanding of K-theory leads us to expect the possibility of a string duality between type I string theory on a spacetime Y and type II string theory on a spacetime Y'. For definiteness, we will assume we are dealing with type IIB on Y'. (This is no great loss of generality since as seen in the last lecture, types IIA and IIB are related via T-duality.) Matching of stable brane D-charges then leads us to look for an isomorphism of the form

$$KO^*(Y) \cong K^*(Y').$$

In general, such isomorphisms are quite rare, in part because of 2-torsion in KO^{-1} and KO^{-2}, and in part because KO-theory is usually 8-periodic rather than 2-periodic.

A Conjectural Duality

But there is one notable exception: one knows that

$$KO \wedge (S^0 \cup_\eta e^2) \simeq K,$$

where $S^0 \cup_\eta e^2$ is the stable cell complex obtained by attaching a stable 2-cell via the stable 1-stem η. This is stably the same (up to a degree shift) as $\mathbb{CP}^2$, since the attaching map $S^3 \to S^2 \cong \mathbb{CP}^1$ for the top cell of $\mathbb{CP}^2$ is the Hopf map, whose stable homotopy class is η. Thus one might expect a duality between type I string theory on $X^6 \times \left(\mathbb{CP}^2 \smallsetminus \{\mathrm{pt}\}\right)$ and type IIB string theory on $X^6 \times \mathbb{R}^4$. We plan to look for evidence for this.

Maldacena's Idea

The AdS/CFT correspondence or holographic duality is a conjectured physical duality, proposed by Juan Maldacena, of a different sort, relating IIB string theory on a 10-dimensional spacetime manifold to a gauge theory on another space. In the original version of this duality, the string theory lives on $AdS^5 \times S^5$, and the gauge theory is $\mathcal{N} = 4$ super-Yang-Mills theory on Minkowski space $\mathbb{R}^{1,3}$. Other versions involve slightly different spaces and gauge theories. Notation:

- $\mathcal{N}$ is the standard notation for the supersymmetry multiplicity. In other words, $\mathcal{N} = 4$ means there are 4 sets of supercharges, and there is a $U(4)$ R-symmetry group acting on them.
- AdS^5 is (up to coverings) $SO(4,2)/SO(4,1)$. Topologically, it's $\mathbb{R}^4 \times S^1$. It's better to pass to the universal cover, so that time isn't periodic.

Nature of the Correspondence

We have already explained that D-branes carry Chan-Paton bundles. In type IIB string theory, a collection of N coincident D3 branes have $3 + 1 = 4$ dimensions and carry a $U(N)$ gauge theory living on the Chan-Paton bundle. This gauge theory is the holographic dual of the string theory, and the number N can be recovered as the flux of the Ramond-Ramond (RR) 5-field G_5 through S^5. The rotation group $SO(6)$ of $\mathbb{R}^5$ is identified with the $SU(4)_R$ symmetry group of the $\mathcal{N} = 4$ gauge theory.

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The AdS/CFT correspondence looks like holography in that physics in the bulk of AdS space is described by a theory of one less dimension "on the boundary." This can be explained by the famous relation between the entropy of a black hole and the area of its boundary, which in turn forces quantum gravity theories to obey a holographic principle.

The Lagrangian for 4D SYM

Recall that the Montonen-Olive Conjecture asserts that classical electro-magnetic duality should extend to an exact symmetry of certain quantum field theories. 4-dimensional super-Yang-Mills (SYM) with $\mathcal{N} = 4$ supersymmetry is believed to be a case for which this conjecture applies. The Lagrangian involves the usual Yang-Mills term

$$\frac{-1}{4g_{YM}^2} \int \mathrm{Tr}(F \wedge *F)$$

and the theta angle term (related to the Pontrjagin number or instanton number)

$$\frac{\theta}{32\pi^2} \int \mathrm{Tr}(F \wedge F).$$

We combine these by introducing the tau parameter

$$\tau = \frac{4\pi i}{g_{YM}^2} + \frac{\theta}{2\pi}.$$

Charges in 4D SYM

The tau parameter measures the relative size of "magnetic" and "electric charges." Dyons in SYM have charges (m, n) living in the group $\mathbb{Z}^2$; the associated *complex charge* is $q + ig = q_0(m + n\tau)$. The electro-magnetic duality group $SL(2, \mathbb{Z})$ acts on τ by linear fractional transformations. More precisely, it is generated by two transformations: $T\colon\ \tau \mapsto \tau + 1$, which just increases the θ-angle by 2π, and has no effect on magnetic charges, and by $S\colon\ \tau \mapsto -\frac{1}{\tau}$, which effectively interchanges electric and magnetic charge. By the Montonen-Olive conjecture, the same group $SL(2, \mathbb{Z})$ should operate on type IIB string theory in a similar way, and θ should correspond in the string theory to the expectation value of the RR scalar field χ.

Puzzles About Charge Groups

An important constraint on variants of the AdS/CFT correspondence should come from the action of the $SL(2, \mathbb{Z})$ S-duality group on the various charges. For example, this group is expected to act on the pair (H, G_3) in $H^3(X, \mathbb{Z}) \times H^3(X, \mathbb{Z})$ by linear fractional transformations. Here G_3 denotes the RR 3-form field, or more precisely, its cohomology class. But now we have some puzzles:

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- The classes of RR fields are really supposed to live in K^{-1}, not cohomology. (Fortunately [Bouwknegt *et al.*], since the first differential in the Atiyah-Hirzebruch spectral sequence is Sq^3, there is no difference when it comes to classes in H^3, except when H^3 has 2-torsion.)

Puzzles About Charge Groups (cont'd)

- Since the S-duality group mixes the NS-NS and RR sectors, it is not clear how it should act on D-brane and RR field charges.

Puzzles About Charge Groups (cont'd)

- Since the S-duality group mixes the NS-NS and RR sectors, it is not clear how it should act on D-brane and RR field charges.
- It's also not so clear what conditions to impose at infinity when spacetime is not compact. For example, it would appear that the H-flux and RR fields do not have to have compact support, so perhaps K-theory with compact support is not the right home for the RR field charges. This point seems unclear in the literature.

An Example

Let's look again at the example of type IIB string theory on $AdS^5 \times S^5$, compared with $\mathcal{N} = 4$ SYM on 4-space. How do the K-theoretic charge groups match up? Our spacetime is topologically $X = \mathbb{R}^5 \times S^5$, where $\mathbb{R}^5$ is the universal cover of AdS^5. We think of $\mathbb{R}^5$ more exactly as $\mathbb{R}^4 \times \mathbb{R}_+$, so that $\mathbb{R}^4 \times \{0\}$, Minkowski space, is "at the boundary." The RR field charges should live in $K^{-1}(X)$, but we see this requires clarification: the RR field G_5 should represent the number N in $H^5(S^5)$, so we need to use homotopy theoretic K-theory K_h here instead of K-theory with compact support, which we've implicitly been using before. Indeed, note that $K^{-1}(X) \cong K^{-1}(\mathbb{R}^5) \otimes K^0(S^5) \cong H^0(S^5)$, while $K_h^{-1}(X) \cong K_h^0(\mathbb{R}^5) \otimes K^{-1}(S^5) \cong H^5(S^5)$, which is what we want.

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Now what about the D-brane charge group for the string theory? This should be $\mathbb{Z} \cong K^0(X) \cong K^0(\mathbb{R}^4 \times Y) \cong K^0(\mathbb{R}^4) \otimes K^0(Y)$, where Y is the D5-brane $\mathbb{R} \times S^5$, which has $K^0(Y) \cong \mathbb{Z}$. Note that this is naturally isomorphic to $K^0(\mathbb{R}^4) = \widehat{K}^0(S^4)$, which is where the instanton number lives in the dual gauge theory. But what charge group on X corresponds to the group of electric and magnetic charges in the gauge theory?

Conclusion

It is believed that the string/gauge correspondence should apply much more generally, to many type IIB string theories on spaces other than $AdS^5 \times S^5$, and to gauge theories with less supersymmetry than the $\mathcal{N} = 4$ theory that we've been considering. Analysis of the relevant charge groups on both the string and gauge sides of the correspondence should give us a guide as to what to expect. Study of these constraints is still in a very early stage.

Thank You

どうもありがとう

Thank you for listening, and a special thank you to the organizers!