The KO-Assembly Map and Positive Scalar Curvature

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Abstract. We state a geometrically appealing conjecture about when a closed manifold with finite fundamental group π admits a Riemannian metric with positive scalar curvature: this should happen exactly when there are no KO_*-valued obstructions coming from Dirac operators. When the universal cover does not have a spin structure, the conjecture says there should always be a metric of positive scalar curvature, and we prove this if the dimension is ≥ 5 and if all Sylow subgroups of π are cyclic. In the spin case, the conjecture is closely tied to the structure of the assembly map $KO_*(B\pi) \to KO_*(\mathbf{R}\pi)$, and we compute this map explicitly for all finite groups π. Finally, we give some evidence for the conjecture in the case of spin manifolds with $\pi = \mathbf{Z}/2$.

§0. INTRODUCTION

This paper is a continuation of my previous papers [**R1**], [**R2**], and [**R3**], but with an emphasis on manifolds with *finite* fundamental group. In other words, I shall try to answer the following question: given a smooth closed connected manifold M^n with finite fundamental group π, when does it admit a metric of positive scalar curvature? A few very partial results on this problem were given in [**R2**] and [**R3**], and some further cases were studied in [**KS1**] and [**KS2**]. Extrapolating from these and other cases, I would like to make here a somewhat audacious but intuitively appealing conjecture:

CONJECTURE 0.1. *A closed manifold M^n with finite fundamental group admits a metric of positive scalar curvature if and only if all (KO_*-valued) index obstructions associated to Dirac operators with coefficients in flat bundles (on M and it covers) vanish, at least if $n \geq 5$.*

The rest of this paper will be devoted to explaining exactly what are the obstructions described in the Conjecture, and to proving that the Conjecture is valid in many cases. As explained in [**GL2**] and in [**R2**], the problem naturally splits into two cases, depending on whether or not $w_2(\tilde{M})$, where $\tilde{M}$ is the universal cover of M, vanishes. If $w_2(\tilde{M}) \neq 0$, so that $\tilde{M}$ (and *a fortiori* M) doesn't admit a spin structure, then there are no Dirac operators with coefficients in flat bundles defined on M or on any of its covers. Thus the Conjecture reduces to:

CONJECTURE 0.2. *If M^n is a closed connected manifold with finite fundamental group π, and if $w_2(\tilde{M}) \neq 0$ and $n \geq 5$, then M admits a metric of positive scalar curvature.*

Section 1 will be devoted to the proof of an interesting case of Conjecture 0.2. I would like to thank the referee for some corrections to the proofs and improvements in the exposition. By the way, the condition in Conjecture 0.2 that π be finite cannot be

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omitted in general, as shown by the example in [**GL3**, p. 186] of $\mathbf{CP}^2 \# T^4$. (The reader concerned about the fact that this example has the exceptional dimension 4 can easly replace it by $(\mathbf{CP}^2 \times S^2) \# T^6$.)

The rest of the paper, §§2 and 3, will deal with the spin case, that is, the case where $w_2(M) = 0$. Section 2 actually involves no geometry, only pure algebraic topology and algebra, and may have some independent interest because of its parallels to known results about assembly maps in L-theory and algebraic K-theory. Theorem 2.5 was proved during a visit to Århus in 1985, and I would like to thank Ib Madsen and Gunnar Carlsson for helpful suggestions at that time.

The concluding section, §3, returns to the problem of positive scalar curvature. Here Conjecture 0.1 is restated in the spin case, using the language of §2, and we give some evidence for the Conjecture in the "hard case" of spin manifolds with fundamental groups of *even* order. We also briefly indicate how to interpret the Conjecture when $w_2(\tilde{M}) = 0$ but $w_2(M) \neq 0$, though there are substantial technical difficulties in getting any good results for this case.

§1. Positive scalar curvature when the universal cover is non-spin

The object of this section is to give some evidence for Conjecture 0.2 above. In fact, this conjecture was proved in [**R2**, Theorem 2.14] in the case where π is cyclic of odd order, and this result was strengthened in [**KS1**] to cover the case of any group of odd order with periodic cohomology (or equivalently, with all Sylow subgroups cyclic). One of the technical advances in [**KS1**] was Corollary 1.6 of that paper, which showed that the conjecture holds for a finite group π if and only if it holds for all its Sylow subgroups. However, as is clear from [**R2**], [**R3**], [**KS1**], and [**KS2**], it is much harder to prove results for even-order groups than for the odd order case. Thus the following theorem is in a way much more convincing evidence for Conjecture 0.2.

THEOREM 1.1. *If M^n is a closed orientable connected manifold with cyclic finite fundamental group π, and if $w_2(\tilde{M}) \neq 0$ and $n \geq 5$, then M admits a metric of positive scalar curvature.*

COROLLARY 1.2. *If M^n is a closed orientable connected manifold with a finite fundamental group π, all of whose Sylow subgroups are cyclic, and if $w_2(\tilde{M}) \neq 0$ and $n \geq 5$, then M admits a metric of positive scalar curvature.*

PROOF OF COROLLARY: This follows immediately from the Theorem and from [**KS1**, Proposition 1.5]. ∎

PROOF OF THEOREM: Because of the results of [**R2**] and [**KS1**] just quoted, it's enough to consider the case where our cyclic group has order a power of two. We begin with the key case where π is of order 2. By [**R2**, Theorem 2.13], it is enough to exhibit an oriented Riemannian manifold X^n of positive scalar curvature, together with a map $X^n \to \mathbf{RP}^\infty$, in every class in $\Omega_n(\mathbf{RP}^\infty)$, for all $n \geq 5$. For this we use the well-known isomorphism of [**S**, pp. 216–217]:

$$\Omega_n(\mathbf{RP}^\infty) \cong \Omega_n \oplus \mathfrak{N}_{n-1}.$$

The summand of Ω_n corresponds to the case where X is simply connected (or at least the map $X^n \to \mathbf{RP}^\infty$ is null-homotopic), so this case is handled by [**GL2**, Proof of Theorem C]. So it remains to deal with the summand $\mathfrak{N}_{n-1}$. Suppose Y^{n-1} represents a class in $\mathfrak{N}_{n-1}$. By the analysis in [S, pp. 216–217], the corresponding element of $\Omega_n(\mathbf{RP}^\infty)$ is represented by $f : X^n \to \mathbf{RP}^\infty$, where Y is the submanifold of X of codimension 1 which is dual to the line bundle defined by f. Note that Y doesn't determine (X, f) uniquely; however, the class of (X, f) in $\Omega_n(\mathbf{RP}^\infty)$ is determined up to an element of Ω_n (which we can "subtract off" by what we already know). Now given the manifold Y^{n-1}, if Y is orientable, we can simply orient Y and take $X = Y \times S^1$, with f factoring through S^1 and inducing a surjection on π_1. If Y has a metric of positive scalar curvature, we can give X a product metric, and then X will have positive scalar curvature as well. So suppose Y is not orientable, and let $\tilde{Y}$ be its orientable double cover, which carries a canonical orientation-reversing involution τ. Let σ be the orientation-reversing involution on S^1 defined by complex conjugation on the unit circle in $\mathbf{C}$. Then $\tau \times \sigma$ is an orientation-**preserving** involution on $\tilde{Y} \times S^1$, so $X = (\tilde{Y} \times S^1)/(\tau \times \sigma)$ can be oriented. Furthermore, there is a map $\pi_1(X) \to \mathbf{Z}/2$, and thus a map $f : X \to \mathbf{RP}^\infty$, associated to this construction of X, for which Y is the dual submanifold. Finally, if Y has a metric of positive scalar curvature, we lift the metric to $\tilde{Y}$ and give $\tilde{Y} \times S^1$ the product metric, and this descends to a metric of positive scalar curvature on X.

Hence to complete the proof for the case where π has order 2, it will suffice to construct additive generators with positive scalar curvature for $\mathfrak{N}_n$, for all $n \geq 4$. In fact, since the property of positive scalar curvature is preserved under taking products, it's in fact enough to find **multiplicative** generators for $\mathfrak{N}_*$ with positive scalar curvature. But by the structure theory for unoriented bordism (see for instance [S, pp. 96–98]), $\mathfrak{N}_*$ is a polynomial algebra over the field $\mathbf{F}_2$ of two elements, with generators represented by even-dimensional real projective spaces and by hypersurfaces of degree $(1, 1)$ in products of pairs of real projective spaces. These manifolds all have natural metrics of positive scalar curvature (cf. [**GL2**, p. 43]), so this completes the first part of the proof.

Now we have to go on to the case where the order of π is any positive power of 2. The key fact we need, which is proved in [S, pp. 209–212 and 233–236], is that the oriented bordism spectrum is Eilenberg-MacLane at 2, and thus that for π a 2-group, the Atiyah-Hirzebruch spectral sequence

$$H_*(\pi, \Omega_*) \Longrightarrow \Omega_*(B\pi)$$

collapses, and

$$\Omega_n(B\pi) \cong \oplus_{p+q=n} H_p(B\pi, \Omega_q). \tag{1.3}$$

Note that the natural map $\Omega_n(B\pi) \to H_n(B\pi, \mathbf{Z})$ corresponds to projection onto the $(p = n,\ q = 0)$ summand.

In order to facilitate future improvements of Theorem 1.1, we first prove the following:

LEMMA 1.4. *Let π be a finite 2-group and let M be a closed connected oriented n-manifold with fundamental group π such that $w_2(\tilde{M}) \neq 0$, $n \geq 5$, and the bordism class of M maps to zero in $H_n(B\pi, \mathbf{Z})$. Then M admits a metric of positive scalar curvature.*

PROOF OF LEMMA: We need to produce enough manifolds of positive scalar curvature to generate the summands in (1.3) other than the ($p = n$, $q = 0$) summand. These are of two types, copies of $H_p(B\pi, \mathbf{Z})$ in bidegrees (p, q) with $q \geq 4$ divisible by 4, and copies of $H_p(B\pi, \mathbf{Z}/2)$ in bidegrees (p, q) for which Ω_q contains a $\mathbf{Z}/2$ summand.

The summands of the first type are no problem, since they correspond to oriented bordism classes (over $B\pi$) of the form

$$N^p \times Y^{4t} \xrightarrow{\phi} B\pi,$$

where ϕ only depends on the first coordinate, where Y^{4t} is a generator for a torsion-free summand in Ω_{4t}, where $N^p \xrightarrow{\phi} B\pi$ generates a cyclic summand in $H_p(B\pi, \mathbf{Z})$, and where $p + 4t = n \geq 5$. Since $t \geq 1$, then by [**GL2**, Theorem C], we may choose Y^{4t} to have positive scalar curvature, and then so does $N^p \times Y^{4t}$ for suitable product metric.

Consider now the summands of $\Omega_n(B\pi)$ coming from $H_*(\pi, \mathbf{Z}/2)$. If a class in $H_*(\pi, \mathbf{Z}/2)$ is the reduction of an integral class, it can be realized by some $N^p \xrightarrow{\phi} B\pi$ with N^p a closed oriented p-manifold, and as before, the corresponding classes in $\Omega_*(B\pi)$ are represented by $N^p \times Y \xrightarrow{\phi} B\pi$, where ϕ only depends on the first coordinate, where Y is a closed oriented manifold giving a 2-torsion summand in Ω_*. Since all such Y's can be chosen to admit metrics of positive scalar curvature [**GL2**, Theorem C], so can $N \times Y$. So it remains to deal with classes in $H_*(\pi, \mathbf{Z}/2)$ which are **not** reductions of integral classes. Such classes only occur in even degree and **cannot** be represented by oriented manifolds mapping into $B\pi$. They can, however, be represented by **non-orientable** manifolds, since $\mathfrak{N}_*(B\pi)$ surjects onto $H_*(\pi, \mathbf{Z}/2)$. Thus consider a class in $\Omega_*(B\pi)$ corresponding to $\phi_*([N]) \times Y$, where $N^p \xrightarrow{\phi} B\pi$, N is non-orientable, $[N]$ is its $\mathbf{Z}/2$-fundamental class, and Y is an orientable manifold giving a $\mathbf{Z}/2$-torsion class in Ω_*. Fortunately, we can construct an oriented manifold mapping into $B\pi$ and defining the same bordism class.

Namely, observe that the metrics of positive scalar curvature on the standard generators of the torsion classes in Ω_*, the Dold manifolds appearing in the proof of [**GL2**, Theorem C], admit orientation-reversing (not necessarily free) involutions. If we choose such an involution σ' on Y and let σ be the orientation-reversing free involution on the oriented double cover $\tilde{N}$ of N, then $\sigma \times \sigma'$ is free and orientation-preserving, and we have a fibration

$$Y \to (\tilde{N} \times Y)/(\sigma \times \sigma') \to N.$$

The composite $(\tilde{N} \times Y)/(\sigma \times \sigma') \to N \xrightarrow{\phi} B\pi$ now represents our class in $\Omega_*(B\pi)$ by an oriented manifold of positive scalar curvature. This completes the proof. ∎

PROOF OF THEOREM 1.1, CONTINUED: Suppose now that π is a cyclic 2-group. By the lemma, it's enough to exhibit an oriented manifold of positive scalar curvature corresponding to each cyclic summand in $H_*(\pi, \mathbf{Z})$. But lens spaces obviously do the trick. ∎

In fact we can improve Corollary 1.2 considerably by allowing a much greater variety of Sylow 2-subgroups. The following two theorems give sample results along these lines.

THEOREM 1.5. *If M^n is a closed orientable connected manifold with fundamental group $\pi = Q$, the quaternion group of order 8, and if $w_2(\tilde{M}) \neq 0$ and $n \geq 5$, then M admits a metric of positive scalar curvature.*

PROOF: By Lemma 1.4, it is enough to exhibit an oriented Riemannian manifold X^n of positive scalar curvature, together with a map $X^n \to BQ$, in every class in $H_n(Q, \mathbf{Z})$, for all $n \geq 5$. So we only have to worry about the case of manifolds of the form $N^n \xrightarrow{\phi} BQ$ generating a cyclic summand in $H_n(Q, \mathbf{Z})$. By [CE, pp. 253–254], such summands occur only for n odd. If $n \equiv 3 \pmod 4$, there is only one such summand, generated by a quaternionic lens space, which can be given a metric of constant positive sectional curvature. If $n \equiv 1 \pmod 4$, there are two such summands, each of order 2, and since one can be taken to the other by an automorphism of Q, we only have to worry about one of them. Such a summand is represented by a submanifold of codimension 2 in a quaternionic lens space S^{4n-1}/Q, dual to a flat complex line bundle. Note that $Q \lhd H$, where H is the normalizer of a maximal torus in $SU(2)$, which also acts freely on S^{4n-1}, and that $H/Q \cong S^1$. Thus we have a fibration

$$S^1 \to S^{4n-1}/Q \to S^{4n-1}/H = \mathbf{C}\mathrm{P}^{2n-1}/(\mathbf{Z}/2),$$

and it's easy to see that the appropriate flat line bundle on S^{4n-1}/Q is pulled back from the quotient. Thus N^{4n-3} projects onto the submanifold R of $\mathbf{C}\mathrm{P}^{2n-1}/(\mathbf{Z}/2)$ dual to the non-trivial flat line bundle on this manifold. By [BB, Theorem C], it's enough to show that R admits a metric of positive scalar curvature. But a little calculation shows that R is a homogeneous manifold for a "large" compact Lie group (its double cover is a homogeneous complex quadric hypersurface in $\mathbf{C}\mathrm{P}^{2n-1}$) and so this is easy to check (it even has a metric with non-negative sectional curvature). This completes the proof. ∎

THEOREM 1.6. *If M^n is a closed orientable connected manifold with fundamental group π a product of $k \leq 4$ cyclic groups of order 2, and if $w_2(\tilde{M}) \neq 0$ and $n \geq 5$, then M admits a metric of positive scalar curvature.*

PROOF: The proof proceeds like that of Theorem 1.5. The same reasoning will work, provided we can realize every additive generator in $H_n((\mathbf{Z}/2)^k, \mathbf{Z})$, $n \geq 5$, by an orientable manifold of positive scalar curvature mapping into $B\pi = (\mathbf{R}\mathrm{P}^\infty)^k$. However, these classes are all represented by products of either odd-dimensional real projective spaces (which are orientable) or else manifolds of the form $(S^{2m} \times S^{2j})/(\sigma \times \sigma')$, where σ and σ' are the antipodal involutions on even-dimensional spheres. (The latter represent the Tor terms in the Künneth formula.) In any event, these manifolds all have non-negative sectional curvature, and zero curvature only occurs in the case of a torus $(S^1)^k$, which can't have the requisite dimension if $k \leq 4$. (Compare [**R3**, Theorem 3.6].) ∎

§2. The KO-Assembly Map

For applications to the positive scalar curvature problem in §3, we need now to examine in some detail the "assembly map" $\beta : KO_*(B\pi) \to KO_*(C^*_{\mathbf{R}}(\pi))$ introduced in [**K**] (in the complex case) and in [**R3**, §2]. Here π is any group with the discrete topology (for the moment not necessarily finite), $B\pi$ is a $K(\pi, 1)$-space, and $C^*_{\mathbf{R}}(\pi)$ is some C*-completion of the real group ring $\mathbf{R}\pi$. We do not need to worry about which C*-completion is to be used, though in practice the usual choice would be the reduced C*-algebra, that is, the completion of the group ring in the operator norm for its action on $\ell^2(\pi)$ by convolution. Since the space $B\pi$ will rarely be a finite complex, the KO-homology groups $KO_*(B\pi)$ are to be interpreted as what Kasparov called $RKO_*(B\pi)$, that is, as the inductive limit

$$\varinjlim_{X \hookrightarrow B\pi} KO_*(X),$$

where X runs over the finite subcomplexes of $B\pi$.

The first result of this section, for which we don't claim any originality (in fact the theorem is known to most workers in the subject, though it seems never to have been stated anywhere in print) identifies Kasparov's β with a map with a homotopy-theoretic construction similar to that used in [**L**]. Recall that from the point of view of a homotopy theorist, we may also identify $KO_*(B\pi)$ with $\pi_*(B\pi^+ \wedge (BO \times \mathbf{Z}))$, since $BO \times \mathbf{Z}$ is the classifying space for real K-theory. Here the $^+$ means that a disjoint basepoint is to be added—this is to avoid getting the reduced homology groups. (More accurately, the homotopy groups here are those of spectra, and we use the usual periodic spectrum whose zeroth space is the infinite loop space $BO \times \mathbf{Z}$.) Similarly, $KO_*(C^*_{\mathbf{R}}(\pi)) \cong \pi_*\left(BO(C^*_{\mathbf{R}}(\pi)) \times KO_0(C^*_{\mathbf{R}}(\pi))\right)$. Thus to construct an assembly map between these homotopy groups, it suffices to construct a map of spaces (or of spectra)

$$\gamma : B\pi^+ \wedge (BO \times \mathbf{Z}) \to BO(C^*_{\mathbf{R}}(\pi)) \times KO_0(C^*_{\mathbf{R}}(\pi)). \tag{2.1}$$

The actual assembly map itself will then be the induced map γ_* on homotopy groups.

Theorem 2.2 (Folklore). *The map $\beta : KO_*(B\pi) \to KO_*(C^*_{\mathbf{R}}(\pi))$ introduced by Kasparov coincides with the assembly map γ_*, where γ as in (2.1) is constructed as the composite*

$$\mu \circ (B\iota \wedge id_{BO\times\mathbf{Z}}).$$

Here ι is the inclusion $\pi \hookrightarrow O(C^*_{\mathbf{R}}(\pi))$ and

$$\mu : (BO(C^*_{\mathbf{R}}(\pi)) \times KO_0(C^*_{\mathbf{R}}(\pi))) \wedge (BO \times \mathbf{Z}) \to BO(C^*_{\mathbf{R}}(\pi)) \times KO_0(C^*_{\mathbf{R}}(\pi))$$

is the multiplication map corresponding to the action of $KO_*(\mathbf{R})$ on $KO_*(C^*_{\mathbf{R}}(\pi))$.

Proof: Let's go back to the original definition of Kasparov's map. For convenience set $A = C^*_{\mathbf{R}}(\pi)$. There is a canonical flat A-line bundle $\mathcal{V}_{B\pi}$ on $B\pi$ defined as $E\pi \times_\pi A$, and given $X \hookrightarrow B\pi$, this pulls back to an A-line bundle $\mathcal{V}_X$ on X, which has a class $[\mathcal{V}_X]$ in $KO^0(X; A)$. The map β is obtained upon passage to the limit over X from the slant (or Kasparov) product

$$KO_*(X) \xrightarrow{\otimes_X [\mathcal{V}_X]} KO_*(A).$$

On the other hand, products in homology and cohomology theories, such as this slant product pairing, come homotopy-theoretically from pairings of spectra. Thus given a class $x \in KO^0(X; A)$, it corresponds to the homotopy class of the classifying map $f_x : X \to (BO(A) \times KO_0(A))$ (or of the corresponding pointed map on X^+). The associated pairing

$$(2.3) \qquad KO_*(X) \xrightarrow{\otimes x\, f_x} KO_*(A)$$

is then given by δ_*, where δ is the composite

$$X^+ \wedge (BO \times \mathbf{Z}) \xrightarrow{f_x \wedge \mathrm{id}} (BO(A) \times KO_0(A)) \wedge (BO \times \mathbf{Z}) \xrightarrow{\mu} (BO(A) \times KO_0(A)).$$

We apply this with $x = [\mathcal{V}_X]$, for which the classifying map f_x is clearly just the composite

$$X \hookrightarrow B\pi \xrightarrow{B\iota} (BO(A) \times KO_0(A)).$$

The result now follows on taking the limit over X. ∎

We proceed now to compute β explicitly in the case where π is a finite group. In this case, the real group ring $\mathbf{R}\pi$ is finite-dimensional, and although there are many different Banach algebra norms on this algebra, they are all equivalent and give the same K-theory. Furthermore, by Maschke's Theorem, $\mathbf{R}\pi$ is semisimple; hence by Wedderburn theory, this algebra is a finite direct sum of matrix algebras over $\mathbf{R}$, $\mathbf{C}$, and $\mathbf{H}$ (the reals, complexes, and quaternions). There is one summand of given type for each irreducible representation of π of the same type.

Since K-theory is invariant under Morita equivalence, we instantly deduce:

LEMMA 2.4. *For π a finite group,*

$$KO_*(C^*_{\mathbf{R}}(\pi)) = KO_*(\mathbf{R}\pi) \cong (\oplus_r KO_*(pt)) \oplus (\oplus_c K_*(pt)) \oplus (\oplus_h KSp_*(pt)).$$

Here r, c, and h are the numbers of irreducible representations of π of real, complex, and quaternionic type (respectively). ∎

Note that it follows that all torsion-free summands in $KO_*(\mathbf{R}\pi)$ occur in even degree (in fact divisible by 4, except for summands associated to representations of complex type), and that all torsion is of order 2 and occurs in degrees 1 and 2 (mod 8) (if coming from representations of real type) and in degrees 5 and 6 (mod 8) (if coming from representations of quaternionic type). On the other hand, for π finite, $\widetilde{KO}_*(B\pi)$ consists **entirely** of torsion, and thus its image under β can hit **only** the $\mathbf{Z}/2$ summands in degrees 1, 2, 5, or 6 (mod 8). The following theorem now completely describes the map.

THEOREM 2.5. *If π is a finite 2-group, the map $\beta : KO_*(B\pi) \to KO_*(\mathbf{R}\pi)$ gives a split surjection onto each $\mathbf{Z}/2$ summand on the right. If π is a general finite group, the image of β consists exactly of the $KO_*(pt)$ summand corresponding to the trivial representation, plus the image under the map induced by the inclusion of the Sylow 2-subgroup π_2 of all torsion in $KO_*(\pi_2)$.*

PROOF: First note that we can reduce immediately to the case of a 2-group, because of the fact that all torsion in $KO_*(\mathbf{R}\pi)$ is of order 2 and because of commutativity of the diagram

$$\begin{array}{ccc} KO_*(B\pi_2) & \xrightarrow{\beta_{\pi_2}} & KO_*(\mathbf{R}\pi_2) \\ {\scriptstyle i_*}\downarrow & & {\scriptstyle i_*}\downarrow \\ KO_*(B\pi) & \xrightarrow{\beta_\pi} & KO_*(\mathbf{R}\pi) \end{array}$$

(cf. [**R1**, proof of Proposition 2.7]) and the fact that the map on the left is a split epimorphism when localized at the prime 2 (with the transfer as a splitting).

Thus suppose π is a 2-group. The idea is to use the results of [**AS**], which describe $KO^*(B\pi)$ (and in fact the pro-ring $\{KO^*(X) : X \hookrightarrow B\pi\}$), together with the universal coefficient theorem for KO, due to Yosimura [**Y**]. (The latter also works for real C*-algebras such as $\mathbf{R}\pi$—see [**MR**].) Since the Atiyah-Segal results refer to I-adic completions, where I is the augmentation ideal of the representation ring $R(\pi)$, we will also have to use the following well-known fact:

LEMMA 2.6. *If G is a p-group and I is the augmentation ideal in the representation ring $R(G)$, then $I \otimes_{\mathbf{Z}} \mathbf{Z}/p$ is nilpotent.*

PROOF OF LEMMA: This is a special case of [**Se**, Lemma 3.6], which asserts that if one has a split extension $S \rightarrowtail G \twoheadrightarrow P$ of a p-group P by a cyclic group S of order prime to p, then the restriction map $r : R(G) \otimes_{\mathbf{Z}} \mathbf{Z}/p \to (R(S) \otimes_{\mathbf{Z}} \mathbf{Z}/p)^P$ is surjective with kernel the nilradical of $R(G) \otimes_{\mathbf{Z}} \mathbf{Z}/p$. Take $S = \{1\}$, $P = G$; then $I(G)$ is just the kernel of $r : R(G) \to R(S)$, and so $I(G) \otimes_{\mathbf{Z}} \mathbf{Z}/p \subseteq \operatorname{nilrad} R(G) \otimes_{\mathbf{Z}} \mathbf{Z}/p$. This of course means $I(G)$ is nilpotent mod p, as claimed. ∎

PROOF OF THEOREM (CONTINUED): We return to the case of π a 2-group. By Lemma 2.6, $I(\pi)$ acts nilpotently on $R(\pi) \otimes_{\mathbf{Z}} \mathbf{Z}/2$. Thus some power of $I(\pi)$ acts trivially on each $\mathbf{Z}/2$ summand in $KO_\pi^{-j}(pt)$ (such summands can occur for $j \equiv 1$, 2, 5, or 6 (mod 8)), and so nothing happens to these summands upon I-adic completion.

We refer to the description of $KO^*(B\pi)$ in [**AS**, p. 17] (though there is one misprint to correct—$KO_\pi^{-6}(pt) = R(\pi)/R_{\mathbf{R}}(\pi)$, not $R(\pi)/R_{\mathbf{H}}(\pi)$). Thus for $j = 1$, 2, 5, or 6 (mod 8), respectively, $KO^{-j}(B\pi)$ is gotten from I-adic completion of $KO_\pi^{-1}(pt) = R_{\mathbf{R}}(\pi)/\rho R(\pi)$, $KO_\pi^{-2}(pt) = R(\pi)/R_{\mathbf{H}}(\pi)$, $KO_\pi^{-5}(pt) = R_{\mathbf{H}}(\pi)/\eta R(\pi)$, and $KO_\pi^{-6}(pt) = R(\pi)/R_{\mathbf{R}}(\pi)$. Matching these up with the description of $KO_*(\mathbf{R}\pi)$, we see that the "dual Kasparov map"

$$\alpha : KO^{-j}(\mathbf{R}\pi) \to KO^{-j}(B\pi)$$

as described in [**K**, §6.2] is an injection on the torsion for $j \equiv 1$, 2, 5, and 6 (mod 8), and even an isomorphism of $\mathbf{F}_2$-vector spaces when $j \equiv 1$ or 5 (mod 8). (Note: As pointed out in the Remarks following [**K**, Corollary 2.15], there is a natural identification of

$KO^{-j}(\mathbf{R}\pi)$ with $KO_\pi^{-j}(pt)$, whereby one can identify α with a similar map studied by Atiyah and Segal.)

Now applying the universal coefficient theorem of [Y], together with the fact that $\mathbf{Q}/\mathbf{Z}$ is an injective $\mathbf{Z}$-module, gives a commutative diagram

$$\begin{array}{ccc} KSp^*(\mathbf{R}\pi;\,\mathbf{Q}/\mathbf{Z}) & \xrightarrow{\cong} & \mathrm{Hom}(KO_*(\mathbf{R}\pi),\,\mathbf{Q}/\mathbf{Z}) \\ {\scriptstyle\alpha}\big\downarrow & & \big\downarrow{\scriptstyle\beta^*} \\ KSp^*(B\pi;\,\mathbf{Q}/\mathbf{Z}) & \xrightarrow{\cong} & \mathrm{Hom}(KO_*(B\pi),\,\mathbf{Q}/\mathbf{Z}). \end{array}$$

Let $x \in KO_j(\mathbf{R}\pi)$ with $j \equiv 1$ or $5 \pmod 8$ and let $z_1, \dots, z_n$ be an $\mathbf{F}_2$-basis for the torsion subgroup of $KO^{j-3}(\mathbf{R}\pi) \cong KSp^{j+1}(\mathbf{R}\pi)$. The exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & KSp^j(\mathbf{R}\pi)\otimes_{\mathbf{Z}}\mathbf{Q}/\mathbf{Z} & \longrightarrow & KSp^j(\mathbf{R}\pi;\,\mathbf{Q}/\mathbf{Z}) & \xrightarrow{\cong} & \mathrm{Tor}(KSp^{j+1}(\mathbf{R}\pi),\,\mathbf{Q}/\mathbf{Z}) & \longrightarrow & 0 \\ & & \big\| & & & & & & \\ & & 0 & & & & & & \end{array}$$

shows these lift to unique elements $\bar{z}_1, \dots, \bar{z}_n$ in $KSp^j(\mathbf{R}\pi;\,\mathbf{Q}/\mathbf{Z})$, each of order 2. Then $\alpha(\bar{z}_1), \dots, \alpha(\bar{z}_n)$ must be linearly independent elements in $KSp^j(B\pi;\,\mathbf{Q}/\mathbf{Z})$ (since $\alpha(z_1)$, $\dots$, $\alpha(z_n)$ are linearly independent in $KSp^{j+1}(B\pi)$ by the application of Atiyah-Segal). However,

$$KSp^j(B\pi;\,\mathbf{Q}/\mathbf{Z}) \cong \mathrm{Hom}(KO_j(B\pi),\,\mathbf{Q}/\mathbf{Z}),$$

so $\alpha(\bar{z}_1), \dots, \alpha(\bar{z}_n)$ may be viewed as homomorphisms $KO_j(B\pi) \to \mathbf{F}_2$. Since these are linearly independent, we may choose an element $y \in KO_j(B\pi)$ with

$$\langle y,\, \alpha(\bar{z}_j)\rangle = \langle x,\, \bar{z}_j\rangle \in \mathbf{Z}/2 \hookrightarrow \mathbf{Q}/\mathbf{Z},$$

and then

$$\langle \beta(y),\, \bar{z}_j\rangle = \langle x,\, \bar{z}_j\rangle \quad \text{for all } j,$$

hence $x = \beta(y)$.

This proves that for π a 2-group, β is surjective in degrees $\equiv 1$ or $5 \pmod 8$. But multiplication by the generator θ of $KO_1(pt)$ gives isomorphisms

$$KO_1(\mathbf{R}\pi) \xrightarrow{\cong} \mathrm{Tors}(KO_2(\mathbf{R}\pi)), \quad KO_5(\mathbf{R}\pi) \xrightarrow{\cong} \mathrm{Tors}(KO_6(\mathbf{R}\pi)),$$

hence since β commutes with multiplication by θ, β is also surjective onto the torsion in degrees $\equiv 2$ or $6 \pmod 8$. Finally, for the statement about splitting, recall that $KO_1(B\pi)$ and $KO_5(B\pi)$ are countable abelian torsion groups to which we can apply Pontryagin duality. For such a group H (which we give the discrete topology), the dual group $\hat{H} = \mathrm{Hom}(H,\,\mathbf{T})$ is compact and coincides with $\mathrm{Hom}(H,\,\mathbf{Q}/\mathbf{Z})$; furthermore, we can recover H as $\mathrm{Hom}_{\mathrm{cont}}(\hat{H},\,\mathbf{T})$. From the analysis above, $\mathrm{Hom}(KO_1(B\pi),\,\mathbf{Q}/\mathbf{Z})$ and $\mathrm{Hom}(KO_5(B\pi),\,\mathbf{Q}/\mathbf{Z})$ contain $\mathbf{Z}/2$ summands which map isomorphically under β^*. But direct sum decompositions of these dual groups dualize to give direct sum decompositions of $KO_1(B\pi)$ and $KO_5(B\pi)$ that give a splitting of β. ∎

REMARK 2.7.

Unfortunately, we do not know of any way to avoid mention of the map $KO_*(\mathbf{R}\pi_2) \to KO_*(\mathbf{R}\pi)$ in Theorem 2.6, since this map fail to be injective or may fail to be surjective (or both) on the torsion. For instance, if $\pi = S_3$, $\pi_2 = \mathbf{Z}/2$, the map is injective but not surjective on torsion since π has 3 irreducible representations of real type and π_2 has only 2. On the other hand, if $\pi = A_4$, then $\pi_2 \cong \mathbf{Z}/2 \times \mathbf{Z}/2$ is normal, π_2 has 4 irreducible representations of real type, but π has only 2; the map induced by the inclusion in this case is surjective on torsion but not injective.

§3. POSITIVE SCALAR CURVATURE WHEN THE UNIVERSAL COVER IS SPIN

In this final section, we use the results of §2 to return to the positive scalar curvature problem for manifolds with finite fundamental group. In a while we shall mention how to understand Conjecture 0.1 when $\tilde{M}$ has a spin structure but M does not, but first we discuss the case where it is easiest to give specific results, namely the case where M itself is a spin manifold. First we recall one of the main results of [**R3**], which explains why Theorem 2.5 above is relevant to our problem.

THEOREM 3.1 [**R3**, THEOREM 3.4]. *Let M^n be a closed Riemannian manifold with positive scalar curvature and with a spin structure s. Let $f : M \to B\pi$ be the classifying map for the universal cover of M, and let $[M, s] \in KO_n(M)$ be the KO-fundamental class defined by the spin structure. Then $\beta \circ f_*([M, s]) = 0$ in $KO_n(C^*_{\mathbf{R}}(\pi))$.*

The precise meaning of Conjecture 0.1 in the spin case is that we conjecture that the necessary condition for positive scalar curvature in Theorem 3.1 is actually sufficient. Note that there are two ways of viewing the obstructions given by Theorem 3.1. For a spin manifold with finite fundamental group π, one may think of there being a single obstruction to positive scalar curvature in $KO_n(\mathbf{R}\pi)$, or (by Lemma 2.4) of there being a whole family of obstructions, corresponding to indices of Dirac operators with coefficients in the various flat bundles parameterized by the irreducible representations of π. The content of Theorem 2.5 is that when π is a 2-group, **every** irreducible representation of real or quaternionic type gives rise to genuine obstructions to positive scalar curvature (in dimensions 1 and 2 (mod 8) in the real case, 5 and 6 (mod 8) in the quaternionic case) for manifolds with the given fundamental group. This is because every class in $KO_*(B\pi)$ can be realized by a spin manifold with the correct fundamental group, provided we jack up the dimension sufficiently using Bott periodicity.

Now we consider the positive evidence for Conjecture 0.1 in the spin case. When the manifold is simply connected, the conjecture just becomes the original conjecture of Gromov and Lawson [**GL2**] about simply connected manifolds of positive scalar curvature, which it seems has now been verified by Stolz [**Sz**]. And by [**R3**] and [**KS1**], the conjecture in the simply connected case implies the conjecture for manifolds with fundamental group of odd order, provided the Sylow subgroups of the fundamental group are all cyclic. Even more than in the non-spin case treated in §1 above, the crucial case to consider is that where the fundamental group π is a 2-group. Even the simplest case of a cyclic group of order 2 is quite hard; however, we do have the following positive result. Note that when $\pi = \mathbf{Z}/2$, M admits exactly one non-trivial flat line bundle, giving us a "twisted Dirac operator" having (by Lemma 2.4 and Theorem 2.5) a $(\mathbb{Z}/2)$-valued index which is an obstruction to positive scalar curvature in dimensions $\equiv 1$ and $2 \pmod 8$.

THEOREM 3.2. *Let M^n be a closed spin manifold with fundamental group $\mathbb{Z}/2$. If $5 \leq n \leq 15$, then Conjecture 0.1 holds for M; that is, M admits a metric of positive scalar curvature if and only if the KO_*-valued index obstructions associated to the Dirac operator on M and the twisted Dirac operator on M vanish.*

Furthermore, for arbitrary $n \geq 5$, if M^n is a spin boundary (forgetting the fundamental group) and if $[M^n] \in \tilde{\Omega}_n^{\mathrm{Spin}}(\mathbb{RP}^\infty)$ has order greater than 2, then M admits a metric of positive scalar curvature.

PROOF: We use the isomorphism

$$\Omega_n^{\mathrm{Spin}}(\mathbb{RP}^\infty) \cong \Omega_n^{\mathrm{Spin}} \oplus \tilde{\Omega}_n^{\mathrm{Spin}}(\mathbb{RP}^\infty) \cong \Omega_n^{\mathrm{Spin}} \oplus \Omega_{n-1}^{\mathrm{Pin}} \tag{3.3}$$

(the analogue of the decomposition of $\Omega_n(\mathbb{RP}^\infty)$ used in the proof of Theorem 1.1) and the results of [**ABP**] and [**G**]. We may restrict attention to the second summand, since the first summand corresponds to the simply connected case of the positive scalar curvature problem.

Let's handle the second statement first, since it will take care of much of the first statement (the "low-dimensional" case) as well. By [**G**, Corollary 3.5], the subgroup of Ω_*^{Pin} generated by elements of order greater than two is generated by products of certain spin manifolds M_J with $\mathbb{RP}^{4k+2}$'s. Under the isomorphism of (3.3), such products correspond in $\Omega_n^{\mathrm{Spin}}(\mathbb{RP}^\infty)$ with $M_J \times \mathbb{RP}^{4k+3}$ (note $\mathbb{RP}^{4k+3}$ is a spin manifold with fundamental group $\mathbb{Z}/2$). Since $\mathbb{RP}^{4k+3}$ has a metric of positive curvature, $M_J \times \mathbb{RP}^{4k+3}$ has a metric of positive scalar curvature.

Now let's go back to the case where $5 \leq n \leq 15$. [**ABP**, Theorem 5.1] gives us a precise calculation of $\tilde{\Omega}_n^{\mathrm{Spin}}(\mathbb{RP}^\infty) \cong \Omega_{n-1}^{\mathrm{Pin}}$. Consider first the summands coming from $\mathbf{BO}\langle 0\rangle$. Aside from cyclic summands of large order, which we've already handled, we have, in our range of values of n, $\mathbb{Z}/2$-summands in dimensions 9 and 10. These correspond to manifolds for which the KO_*-valued index of the twisted Dirac operator is non-zero (e.g., $M_0^8 \times S^1$ with spin surgery to reduce the fundamental group to $\mathbb{Z}/2$; here M_0^8 is a spin 8-manifold with $\hat{A}$-genus $= 1$). We've seen these manifolds do **not** have metrics of positive scalar curvature.

Next consider the remaining summands, which come from $\mathbf{BO}\langle 8\rangle$ and from $\mathbf{BO}\langle 10\rangle$. We know the cyclic summands of large order correspond to manifolds of positive scalar curvature, and the remaining $\mathbb{Z}/2$-summands occur in dimensions 9 and 10 (coming from $\mathbf{BO}\langle 8\rangle$) and in dimensions 12, 13, and 14 (coming from $\mathbf{BO}\langle 10\rangle$). We can find representatives for all of these with positive scalar curvature. For the generators in dimensions 9 and 10, one can take $\mathbb{HP}^2 \times S^1$ and $\mathbb{HP}^2 \times \bar{S}^1 \times S^1$, with suitable spin surgeries to reduce the fundamental group to $\mathbb{Z}/2$. The generators in dimensions 12, 13, and 14 can be built from a spin manifold M^{10} of positive scalar curvature (representing a $\mathbb{Z}/2$ summand in $\Omega_{10}^{\mathrm{Spin}}$) as $M^{10} \times S^1$ with spin surgery to reduce the fundamental group, as $(M^{10} \times S^2)/\sigma$, σ a suitable free involution, and as $M^{10} \times \mathbb{RP}^3$. ∎

Finally, we explain the meaning of Conjecture 0.1 in the non-spin case. If M is a manifold whose universal cover $\tilde{M}$ has a spin structure, then the sections of any flat vector bundle over M may be identified with a suitable space of vector-valued functions on $\tilde{M}$. As such, there is a Dirac operator acting on them (after tensoring with the spinor bundle on $\tilde{M}$). For some vector bundles, the Dirac operator will map this space back into itself, and thus there is an associated KO_*-valued index of the twisted Dirac operator which will be an obstruction to a metric of positive scalar curvature on M. The meaning of the Conjecture is that these should be the only obstructions. So far we have only paltry evidence for the Conjecture, but it should be possible to test it by using the 2-connected bordism class of the clasifying map $M \to B\pi$, as introduced in [KS1], [KS2].

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