A diagram calculus for KK and the noncommutative Riemann-Roch theorem

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Motivation:

- Developing some of the formalism for dealing with noncommutative spacetimes.

- Establishing a general formula for D-brane charges.

- Finding a version of Grothendieck-Riemann-Roch suited to the noncommutative world.

A classical formula

Let X be a compact spin^c manifold. Poincaré duality in ordinary cohomology says the usual cup-product pairing

$$(x, y) = \langle x \cup y, [X] \rangle,$$

is non-degenerate, while on K-theory, we have another pairing given by

$$([E], [F]) = \operatorname{Ind} \not{D}_{E \otimes F}.$$

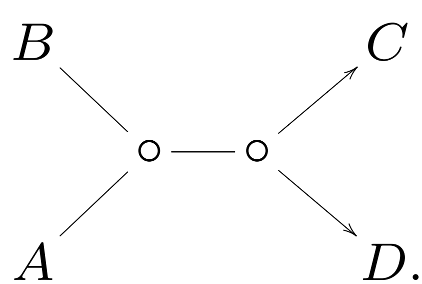
The Chern character gives an algebra isomorphism of $K^*(X) \otimes \mathbb{Q}$ with $H^*(X, \mathbb{Q})$, but under this isomorphism, the two pairings don't match. Since, by the Index Theorem,

$$\operatorname{Ind} \not{D}_{E \otimes F} = \langle \mathrm{Todd}(X) \cup \mathrm{Ch}(E \otimes F), [X] \rangle,$$

we can, however, get an isometry of pairings by correcting by the class $\sqrt{\mathrm{Todd}(X)}$. This simple observation, known to physicists from the Minasian-Moore formula for the Ramond-Ramond charge, is the origin of this paper.

The diagram calculus for KK

To make it easier to do certain calculations later, we represent classes in KK (or in similar bivariant theories, like Puschnigg's local bivariant cyclic cohomology) by diagrams (which we read from left to right). We have one "input" for each tensor factor in the first argument of KK, and one "output" for each tensor factor in the second argument of KK. For convenience, we can also add arrowheads pointing toward the outputs. The Kasparov product corresponds to concatenation of diagrams, except that one is only allowed to attach an input to a matching output. Thus, for example, an element of $KK(B \otimes A, C \otimes D)$ would be represented by a diagram like



The basic rule is that permutation of the input or output terminals may involve at most the switch of a sign. The nice thing about this formalism is that one can show that all associativity formulae in Kasparov theory correspond to the fact that one can concatenate in any order, except perhaps for signs (which go away in KK^0). For example, if $x \in KK^\bullet(B \otimes A, C)$, $y \in KK^\bullet(D, A)$, and $z \in KK^\bullet(E, B)$, then the associativity of the product gives

$$z \otimes_B (y \otimes_A x) = \pm y \otimes_A (z \otimes_B x),$$

even though when written this way, it looks as if the factors are out of order. But one can "prove" this graphically with the picture

$$\begin{tikzcd}
E \arrow[r, "z"] & B \arrow[dr, no head] & & \\
& & \circ \arrow[r, "x"] & C \\
D \arrow[r, "y"] & A \arrow[ur, no head] & &
\end{tikzcd}$$

Of course, a picture by itself is not a rigorous proof, but it can be made into one as follows. Here $\times$ is used to denote the "exterior" Kasparov product, and for simplicity we assume that all elements lie in KK_0, so that we don't have to worry about sign changes. On the one hand, we have

$$
\begin{aligned}
z \otimes_B (y \otimes_A x) &:= (z \times 1_D) \otimes_{B \otimes D} (y \otimes_A x) \\
&= (z \times 1_D) \otimes_{B \otimes D} \Big((1_B \times y) \otimes_{B \otimes A} x \Big) \\
&= \Big[(z \times 1_D) \otimes_{B \otimes D} (1_B \times y) \Big] \otimes_{B \otimes A} x.
\end{aligned}
$$

But on the other hand we have

$$
\begin{aligned}
y \otimes_A (z \otimes_B x) &:= (1_E \times y) \otimes_{E \otimes A} (z \otimes_B x) \\
&= (1_E \times y) \otimes_{E \otimes A} \Big((z \times 1_A) \otimes_{B \otimes A} x \Big) \\
&= \Big[(1_E \times y) \otimes_{E \otimes A} (z \times 1_A) \Big] \otimes_{B \otimes A} x.
\end{aligned}
$$

Now observe that

$$(z \times 1_D) \otimes_{B \otimes D} (1_B \times y) = z \times y = (1_E \times y) \otimes_{E \otimes A} (z \times 1_A).$$

Essentially everything applies also to products in bivariant cyclic homology, which has the same formal properties.

Application: the KK proof of Atiyah-Singer

Let M be a closed manifold, D an elliptic operator on M, with class $[D] \in KK(C(M), \mathbb{C})$, $\sigma(M)$ its symbol class in $KK(\mathbb{C}, C_0(T^*M))$, $[\bar{\partial}] \in KK(C_0(T^*M), \mathbb{C})$ the Dolbeault class. Let $[c] \in KK(\mathbb{C}, C(M))$ be the class of the map $\mathbb{C} \to C(M)$ by constant functions. Then (almost by definition) $\operatorname{Ind} D = [c] \otimes_{C(M)} [D]$. Atiyah-Singer says this is equal to $\sigma(D) \otimes_{C_0(T^*M)} [\bar{\partial}]$. The proof comes from a class $\Delta \in KK(C(M) \otimes C_0(T^*M), \mathbb{C})$ and the diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "{[c]}"] & C(M) \arrow[dr, dash] & & \\
& & \circ \arrow[r, "\Delta"] & \mathbb{C} \\
\mathbb{C} \arrow[r, "\sigma(D)"] & C_0(T^*M) \arrow[ur, dash] & &
\end{tikzcd}$$

Noncommutative Poincaré duality

Definition 1 *A pair of separable C^*-algebras (A, B) is said to be a strong Poincaré dual pair if $\exists \Delta \in KK_d(A \otimes B, \mathbb{C}) = K^d(A \otimes B)$ and $\exists \Delta^\vee \in KK_{-d}(\mathbb{C}, A \otimes B) = K_{-d}(A \otimes B)$ with the properties*

$$\begin{aligned} \Delta^\vee \otimes_B \Delta &= 1_A \in KK_0(A, A) \ , \\ \Delta^\vee \otimes_A \Delta &= (-1)^d \, 1_B \in KK_0(B, B) \ . \end{aligned}$$

The element Δ is called a fundamental K-homology class for the pair (A, B) and $\Delta^\vee$ is called its inverse. A separable C^*-algebra A is said to be a strong Poincaré duality algebra (strong PD algebra for short) if (A, A°) is a strong PD pair, where A° denotes the opposite algebra of A, i.e., the algebra with the same underlying vector space as A but with the product reversed.

The use of the opposite algebra in this definition is to describe A-bimodules as $(A \otimes A^\circ)$-modules.

Then product on the right with Δ and product on the left with $\Delta^\vee$ give inverse isomorphisms

$$K_i(A) \xrightarrow{\otimes_A \Delta} K^{i+d}(B)$$

and

$$K^i(B) \xrightarrow{\Delta^\vee \otimes_B} K_{i-d}(A).$$

One also gets Poincaré duality with coefficients in any auxiliary algebras (check this with the diagram calculus):

$$KK_i(C, A \otimes D) \cong KK_{i-d}(C \otimes B, D).$$

Sample application of the diagram calculus

Proposition 2 *Let* (A, B) *be a strong PD pair, and let* $\Delta \in K^d(A \otimes B)$ *be a fundamental class with inverse* $\Delta^\vee \in K_{-d}(A \otimes B)$*. Let*

$$\ell \in KK_0(A, A)$$

be an invertible element. Then

$$\ell \otimes_A \Delta \in K^d(A \otimes B)$$

is another fundamental class, with inverse

$$\Delta^\vee \otimes_A \ell^{-1} \in K_{-d}(A \otimes B).$$

Sketch of proof. The harder direction can be illustrated by the diagram

$$\begin{tikzcd}
 & & A \arrow[r, "\ell^{-1}"] & A & A \arrow[r, "\ell"] & A \arrow[dr, dash] & & \\
\mathbb{C} \arrow[r, dash, "\Delta^\vee"] & \circ \arrow[ur] \arrow[dr] & & & & & \circ \arrow[r, "\Delta"] & \mathbb{C} \\
 & & B \arrow[rrr, "1_B"] & & & B \arrow[ur, dash] & &
\end{tikzcd}$$

$\square$

Similarly, one gets a converse:

Proposition 3 *Let (A, B) be a strong PD pair, and let $\Delta_1, \Delta_2 \in K^d(A \otimes B)$ be fundamental classes with inverses $\Delta_1^\vee, \Delta_2^\vee \in K_{-d}(A \otimes B)$. Then $\Delta_1^\vee \otimes_B \Delta_2$ is an invertible element in $KK_0(A, A)$, with inverse given by $(-1)^d \Delta_2^\vee \otimes_B \Delta_1 \in KK_0(A, A)$.*

Corollary 4 *Let (A, B) be a strong PD pair. Then the moduli space of fundamental classes for (A, B) is isomorphic to the group of invertible elements in the ring $KK_0(A, A)$.*

Note: In the commutative case $A = C(X)$, the abelian group of units of $KK(C(X), C(X))$ is by UCT an extension of $\operatorname{Aut} K^\bullet(X)$ by

$$\operatorname{Ext}_{\mathbb{Z}}(K^\bullet(X), K^{\bullet+1}(X)).$$

When does this apply?

Note that the stable homotopy category of C^*-algebras has an involution gotten by sending $A \xrightarrow{f} B$ to $A^\circ \xrightarrow{f^\circ} B^\circ$, and this involution passes to the KK category. So if A is KK-equivalent to $C(X)$ for some compact space X, then A° is KK-equivalent to $C(X)^\circ = C(X)$ also, and hence we have:

Theorem 5 *Let A be a separable C^*-algebra satisfying the UCT for KK (i.e., KK-equivalent to a commutative C^*-algebra) with finitely generated K-theory. Then A is always part of a strong PD pair, and A is a strong PD algebra (i.e., we can take the other element of the pair to be A°) if and only if either* $\operatorname{rk} K_0(A) = \operatorname{rk} K_1(A)$ *(in this case we can take $d = 1$) or* $\operatorname{Tors} K_0(A) \cong \operatorname{Tors} K_1(A)$ *(in this case we can take $d = 0$).*

Proof. Without loss of generality, we can assume A abelian. Note that by the UCT,

$$\begin{cases} \operatorname{rk} K^j(A) = \operatorname{rk} K_j(A), \\ \operatorname{Tors} K_j(A) \cong \operatorname{Tors} K^{j+1}(A) \end{cases}$$

($j = 0, 1 \bmod 2$). So the condition for A to be a strong PD algebra is necessary to have an isomorphism $K_j(A) \to K^{j+d}(A)$. It remains to show that for A and B commutative, an isomorphism $K_j(A) \to K^{j+d}(B)$ can be implemented by a suitable Δ. By the KT and UCT, we can build Δ and $\Delta^\vee$ from knowledge of $K_*(A)$, one cyclic summand at a time. Alternatively, realize A as $C_0(X)$ for some (possibly noncompact) manifold X, take $B = C_0(T^*X)$, and construct Δ from the Dirac operator. When X is spin^c, B is KK-equivalent to $A = A^\circ$. $\square$

The upshot of this is that strong PD pairs are quite common.

The Todd class

Lemma 6 *Let A, B_1, B_2 be separable C^*-algebras such that (A, B_1) and (A, B_2) are both strong PD pairs. Then B_1 and B_2 are KK-equivalent.*

Let $\mathfrak{D}$ denote the class of all separable C^*-algebras A for which there exists another separable C^*-algebra B such that (A, B) is a strong PD pair. For any such A, we fix a representative of the KK-equivalence class of B and denote it by $\tilde{A}$. In general there is no canonical choice for $\tilde{A}$. If A is a strong PD algebra, the canonical choice $\tilde{A} := A^\circ$ will always be made.

In what follows we'll need a choice of a bivariant cyclic homology theory with a good multiplicative Chern character from KK. We can use Puschnigg's local bivariant cyclic cohomology, here denoted HL (we are not using Leibnitz homology!).

Moreover, if A and B are in the class of C^*-algebras for which the UCT holds for KK, then

$$HL_\bullet(A, B) \cong \mathrm{Hom}_{\mathbb{C}}(K_\bullet(A) \otimes_{\mathbb{Z}} \mathbb{C}, K_\bullet(B) \otimes_{\mathbb{Z}} \mathbb{C}).$$

If $K_\bullet(A)$ is finitely generated, this is also equal to $KK_\bullet(A, B) \otimes_{\mathbb{Z}} \mathbb{C}$.

By multiplicativity of the Chern character, KK-equivalence of algebras implies HL-equivalence, but not conversely, and each strong PD pair can also be made a PD pair for HL. However, frequently one does not want to take the HL fundamental class Ξ equal to $\mathrm{Ch}(\Delta)$. (See example below.)

Definition 7 *Let $A \in \underline{\mathfrak{D}}$, let $\Delta \in K^d(A \otimes \tilde{A})$ be a fundamental K-homology class for the pair $(A, \tilde{A})$ and let $\Xi \in HL^d(A \otimes \tilde{A})$ be a fundamental cyclic cohomology class. Then the Todd class of A is defined to be the class*

$$\mathsf{Todd}\Big(A\Big) := \Xi^\vee \otimes_{\tilde{A}} \mathsf{Ch}\left(\Delta\right)$$

in the ring $HL^0(A, A)$.

The Todd class is invertible with inverse given by

$$\mathsf{Todd}\Big(A\Big)^{-1} = (-1)^d \ \mathsf{Ch}\left(\Delta^\vee\right) \otimes_{\tilde{A}} \Xi \ .$$

Motivating example

Let $A = C(X)$, X a compact complex manifold. Then A is a strong PD algebra with Δ given by the Dolbeault operator on $X \times X$. We can identify HL with HP (usual periodic cyclic homology) in this case (after passage from A to the dense subalgebra $C^\infty(X)$), and so $HL^0(A, A)$ can be identified with $\operatorname{End} H^*(X, \mathbb{Q})$. The natural choice of Ξ and $\Xi^\vee$ comes from usual Poincaré duality in rational cohomology. Then $\mathrm{Todd}(A)$ is just cup product with the usual $\mathrm{Todd}(X) \in H^*(X, \mathbb{Q})$.

Another Example: Noncommutative Riemann Surfaces

Let π be the fundamental group of a closed Riemann surface Σ_g of genus $g \geq 1$. Then $H^2(\pi, \mathbb{T}) \cong H^2(\Sigma_g, \mathbb{T}) \cong \mathbb{T}$, so for any $\theta \in \mathbb{R}$, we have the twisted group algebra $A^g_\theta = C^*_r(\pi, \sigma_\theta)$ for the cocycle σ_θ defined by $\exp(2\pi i\theta)$. In the case of genus 1, this is the usual rotation algebra A_θ, and is simple if θ is irrational. A^g_θ is a strong PD algebra with $(A^g_\theta)^\circ = A^g_{-\theta} \cong A^g_\theta$.

There is a standard choice of fundamental classes, and thus of a Todd class, for A^g_θ, coming from the commuting digram

$$\begin{tikzcd}
K^\bullet(\Sigma_g) \arrow[r, "\cong"] \arrow[d, "\mathrm{Ch}"] & K_\bullet(A^g_\theta) \arrow[d, "\mathrm{Ch}"] \\
H^\bullet(\Sigma_g; \mathbb{C}) \arrow[r, "\cong"] & HL_\bullet(A^g_\theta).
\end{tikzcd}$$

Here the horizontal maps come from Baum-Connes assembly and the downward maps are isomorphisms after tensoring K-groups with $\mathbb{C}$.

K-oriented maps (d'après Connes-Skandalis)

Now we can define Gysin maps and, more generally, we can study K-oriented maps. If $f : A \to B$ is a morphism of C^*-algebras in a suitable category, a K-orientation is a functorial way of defining $f! \in KK(B, A)$. If A and B are strong PD algebras, then any morphism $f : A \to B$ is K-oriented, and $f!$ is determined as follows:

$$f! = (-1)^{d_A} \Delta_A^\vee \otimes_{A^\circ} [f^\circ] \otimes_{B^\circ} \Delta_B.$$

We can visualize this with the diagram

$$\begin{tikzcd}
 & & A & B \arrow[dr, dash] & & \\
\mathbb{C} \arrow[r, "\Delta_A^\vee"] & \circ \arrow[ur] \arrow[dr] & & & \circ \arrow[r, "\Delta_B"] & \mathbb{C} \\
 & & A^\circ \arrow[r, "f^\circ"] & B^\circ \arrow[ur, dash] & &
\end{tikzcd}$$

To check functoriality, observe that if A, B and C are strong PD algebras, and if $f : A \to B$, $g : B \to C$ are morphisms of C^*-algebras, then

$$\begin{aligned}
&\Big((-1)^{d_A}\Delta_A^\vee \otimes_{A^\circ} [f^\circ] \otimes_{B^\circ} \Delta_B\Big) \\
&\otimes_B \Big((-1)^{d_B}\Delta_B^\vee \otimes_{B^\circ} [g^\circ] \otimes_{C^\circ} \Delta_C\Big) \\
&= \Big((-1)^{d_A}\Delta_A^\vee \otimes_{A^\circ} [(g \circ f)^\circ] \otimes_{C^\circ} \Delta_C\Big).
\end{aligned}$$

by associativity of the Kasparov product and the basic relation

$$(-1)^{d_B}\Delta^\vee \otimes_B \Delta = 1_{B^\circ}.$$

(Again, use the diagram calculus!)

Theorem 8 *Suppose A and B are strong PD algebras with given HL fundamental classes. Then one has the Grothendieck-Riemann-Roch formula,*

$$\mathrm{Ch}(f!) = (-1)^{d_B}\,\mathrm{Todd}(B) \otimes_B f^{HL}! \otimes_A \mathrm{Todd}(A)^{-1}. \tag{1}$$

Proof. We will write out the right-hand side of (1) and simplify. In the notation of Definition 7, the Todd class of B is the class

$$\mathrm{Todd}\Big(B\Big) = \Xi_B^\vee \otimes_{\tilde{B}} \mathrm{Ch}\left(\Delta_B\right) \in HL^0(B,B)$$

and the inverse of the Todd class of A is the class

$$\mathrm{Todd}\Big(A\Big)^{-1} = (-1)^{d_A}\,\mathrm{Ch}\left(\Delta_A^\vee\right) \otimes_{\tilde{A}} \Xi_A \in HL^0(A,A).$$

Since A and B are strong PD algebras, then $f^H L!$ is determined as follows:

$$f^H L! = (-1)^{d_A}\; \Xi_A^\vee \otimes_{\tilde{A}} [(f^{HL})^\circ] \otimes_{\tilde{B}} \Xi_B,$$

where $[f^{HL}] = HL(f)$ denotes the class in $HL(A, B)$ of the morphism $(A \xrightarrow{f} B)$, and $[(f^{HL})^\circ]$ is defined similarly.

Therefore the right hand side of (1) is equal to

$$\begin{aligned}&(-1)^{d_B}\left(\Xi_B^\vee \otimes_{\tilde{B}} \mathsf{Ch}\left(\Delta_B\right)\right)\\&\otimes_B \left(\Xi_A^\vee \otimes_{\tilde{A}} [(f^{HL})^\circ] \otimes_{\tilde{B}} \Xi_B\right)\\&\otimes_A \left(\mathsf{Ch}\left(\Delta_A^\vee\right) \otimes_{\tilde{A}} \Xi_A\right),\end{aligned}$$

which by the associativity of the intersection product is equal to

$$\begin{aligned}&(-1)^{d_B}\left(\Xi_A^\vee \otimes_A \left(\mathsf{Ch}\left(\Delta_A^\vee\right) \otimes_{\tilde{A}} \Xi_A\right)\right)\\&\otimes_{\tilde{A}} [(f^{HL})^\circ] \otimes_{\tilde{B}} \left(\left(\Xi_B^\vee \otimes_{\tilde{B}} \mathsf{Ch}\left(\Delta_B\right)\right) \otimes_B \Xi_B\right).\end{aligned}$$

On the other hand,

$$f! = (-1)^{d_A}\ \Delta_A^\vee \otimes_{\tilde{A}} [f^\circ] \otimes_{\tilde{B}} \Delta_B.$$

Therefore the left hand side of (1) is equal to

$$(-1)^{d_A}\ \mathsf{Ch}(\Delta_A^\vee) \otimes_{\tilde{A}} \mathsf{Ch}[f^\circ] \otimes_{\tilde{B}} \mathsf{Ch}(\Delta_B).$$

By the functorial properties of the bivariant Chern character, one has

$$\mathsf{Ch}[f^\circ] = [(f^{HL})^\circ].$$

In order to prove the theorem, it therefore suffices to prove that

$$\left(\Xi_B^\vee \otimes_{\tilde{B}} \mathsf{Ch}\left(\Delta_B\right)\right) \otimes_B \Xi_B = (-1)^{d_B}\,\mathsf{Ch}(\Delta_B)$$

and

$$\Xi_A^\vee \otimes_A \left(\mathsf{Ch}\left(\Delta_A^\vee\right) \otimes_{\tilde{A}} \Xi_A\right) = (-1)^{d_A}\,\mathsf{Ch}(\Delta_A^\vee).$$

But both of these equalities also follow easily from the diagram calculus.

Symmetric fundamental classes

Definition 9 *A fundamental class Δ of a strong PD algebra A is said to be symmetric if $\sigma(\Delta)^\circ = \Delta \in K^d(A \otimes A^\circ)$ where*

$$\sigma : A \otimes A^\circ \longrightarrow A^\circ \otimes A$$

is the involution $x \otimes y^\circ \mapsto y^\circ \otimes x$ and σ also denotes the induced map on K-homology. In terms of the diagram calculus, Δ being symmetric implies that

$$\begin{tikzcd}
A \arrow[r, "x"] & A \arrow[dr, no head] & & & & A \arrow[r, "y"] & A \arrow[dr, no head] & & \\
& & \circ \arrow[r, "\Delta"] & \mathbb{C} & = & & & \circ \arrow[r, "\Delta"] & \mathbb{C} \\
A^\circ \arrow[r, "y^\circ"] & A^\circ \arrow[ur, no head] & & & & A^\circ \arrow[r, "x^\circ"] & A^\circ \arrow[ur, no head] & &
\end{tikzcd}$$

for all x and y.

The isometric pairing formula

Finally we get to an analogue of the classical isometry result that we started with:

Theorem 10 *Suppose that A satisfies the UCT for local cyclic homology, and that $HL_\bullet(A)$ is a finite dimensional vector space. If A has symmetric (even-dimensional) fundamental classes in both K-theory and in cyclic theory, then the modified Chern character*

$$\mathrm{Ch} \otimes_A \sqrt{\mathrm{Todd}(A)} : K_\bullet(A) \to HL_\bullet(A)$$

is an isometry with respect to the inner products

$$\langle \alpha, \beta \rangle = (\alpha \times \beta^\circ) \otimes_{A \otimes A^\circ} \Delta$$

and

$$(x, y) = (x \times y^\circ) \otimes_{A \otimes A^\circ} \Xi.$$