The G-Signature Theorem Revisited

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Dedicated to Mel Rothenberg on his 65th birthday

ABSTRACT. In a strengthening of the G-Signature Theorem of Atiyah and
Singer, we compute, at least in principle (modulo certain torsion of exponent
dividing a power of the order of G), the class in equivariant K-homology of
the signature operator on a G-manifold, localized at a prime idea of $R(G)$, in
terms of the classes in non-equivariant K-homology of the signature operators
on fixed sets. The main innovations are that the calculation takes (at least
some) torsion into account, and that we are able to extend the calculation to
some non-smooth actions.

1. Introduction

Let G be a finite group. In studying actions on G on closed manifolds M^n,[1] one
of the most important tools, which comes from analysis of the signature operator,
is the G-Signature Theorem of Atiyah and Singer ([**1**], §6). Suppose M is oriented
and the G-action preserves the orientation. If we fix a G-invariant Riemannian
metric on M, then we can construct the signature operator D_M (or simply D, if
M is understood), a G-invariant elliptic first-order differential operator acting on
$\bigwedge T^*_{\mathbb{C}} M$, given simply by $d + d^*$ (exterior differentiation plus its adjoint with respect
to the metric) together with a $\mathbb{Z}/2$-grading of $\bigwedge T^*_{\mathbb{C}} M$ determined by the Hodge
$*$-operator (which in turn depends on the orientation and the metric). Suppose
further that n, the dimension of M, is even. Then by the formalism of Kasparov
theory ([**10**] — see also [**19**] for a quick summary and [**3**] and [**6**] for good detailed
expositions), D determines an equivariant K-homology class $[D] \in K_0^G(M)$ which
is independent of the choice of metric. If E is any G-vector bundle on M, then
a choice of a connection on E enables us to define the signature operator D_E
with coefficients in E, which again has a class $[D_E] \in K_0^G(M)$ independent of the

[1]For the moment we assume the manifold and the action to be smooth, though we will relax
this later on.

connection. If $c \colon M \to$ point is the "collapse" map, then the G-index of D_E is simply $G\text{-Ind}\, D_E = c_*([D_E]) \in K_0^G(\text{point}) = R(G)$, and in turn this is just the pairing $\langle [D_E],\, [E] \rangle$, where $[E] \in K_G^0(M)$ is the equivariant K-theory class of the bundle E. The Atiyah-Singer G-Signature Theorem computes this in terms of characteristic class data of M and E.

However, this is only part of the story. Computing $G\text{-Ind}\, D_E$ for all possible G-vector bundles amounts to computing the image of $[D]$ under the index map

(1.1) $$G\text{-Ind} \colon\ K_*^G(M) \to \text{Hom}_{R(G)}(K_G^{-*}(M),\, R(G)).$$

However, as pointed out in [16] and [17] (based in part on unpublished work of M. Bökstedt), the map of equation (1.1) fits inside a short exact universal coefficient sequence

(1.2)
$$0 \to \text{Ext}_{R(G)}^1(K_G^{1-*}(M),\, R(G)) \to K_*^G(M) \xrightarrow{\ G\text{-Ind}\ } \text{Hom}_{R(G)}(K_G^{-*}(M),\, R(G)) \to 0,$$

and there are canonical isomorphisms

(1.3)
$$\text{Hom}_{R(G)}(N,\, R(G)) \cong \text{Hom}_{\mathbb{Z}}(N,\, \mathbb{Z}), \qquad \text{Ext}_{R(G)}^1(N,\, R(G)) \cong \text{Ext}_{\mathbb{Z}}^1(N,\, \mathbb{Z}),$$

for any $R(G)$-module N. Since $R(G)$ is a finitely generated free $\mathbb{Z}$-module, and $K_G^{-*}(M)$ is finitely generated over $R(G)$ (since M has the G-homotopy type of a finite G-CW-complex), it follows that the kernel of the index map $G\text{-Ind}$ of equation (1.1) is precisely the $\mathbb{Z}$-torsion in $K_*^G(M)$. Thus the Atiyah-Singer theorem only computes $[D] \in K_0^G(M)$ modulo torsion.

However, even in the non-equivariant setting, the torsion part of the K-homology class of the signature operator is a very interesting invariant of a manifold [23]. The purpose of this paper is therefore to get more precise information on the K-homology class $[D] \in K_0^G(M)$.

2. Bordism invariance

While the signature operator on a manifold M with even dimension $n = 2k$ is usually described using the de Rham complex, to describe the signature operator for manifolds of all dimensions it is convenient to use an equivalent approach using Clifford algebras ([15], Ch. II, Example 6.2). By means of the usual identification of the exterior algebra and Clifford algebra (as vector spaces, of course, not as algebras), we can view D_M as being given by the Dirac operator on $\text{Cliff}_{\mathbb{C}} T^* M$, the complexified Clifford algebra bundle of the cotangent bundle (with connection and metric coming from the Riemannian connection and metric), with grading operator τ given by the "complex volume element" ([15], pp. 33–34 and 135–137), a parallel section of $\text{Cliff}_{\mathbb{C}} T^* M$ which in local coordinates is given by $i^k e_1 \cdots e_n$, where $e_1, \ldots, e_n$ are a local orthonormal frame for the cotangent bundle. When the dimension $n = 2k + 1$ of M is odd, $\tau = i^{k+1} e_1 \cdots e_n$ acting on $\text{Cliff}_{\mathbb{C}} T^* M$ by Clifford multiplication still satisfies $\tau^2 = 1$, but the Dirac operator commutes with τ. Furthermore, if σ is the usual grading operator on $\text{Cliff}_{\mathbb{C}} T^* M$ (which is $(-1)^p$ on products $e_{i_1} \cdots e_{i_p}$), then τ and the Dirac operator both anticommute with σ. So in this case we consider the Dirac operator on $\text{Cliff}_{\mathbb{C}} T^* M$, with the grading given by σ, but with the extra action of the Clifford algebra $C_1 = \text{Cliff}_{\mathbb{C}} \mathbb{R}$, where the odd generator of C_1 acts by τ. So for n odd, the signature operator gives a class in the K-homology group $K_1(M)$. (If M is non-compact and we use a complete

Riemannian metric, the K-homology class of the signature is still well-defined and independent of the metric, but lives in *locally finite K-homology*.)

Now suppose that a finite group G acts on M^n, preserving the orientation. If we choose a G-invariant Riemannian metric on M, the signature operator becomes G-equivariant, and defines a class $[D_M]$ living in $K_0^G(M)$ if n is even, $K_1^G(M)$ if n is odd. In this paper we will be interested in computing $[D_M]$ as precisely as possible, including torsion information.

An important fact in this context, observed for example in [**18**] or in [**23**], is that if M^n is the boundary of a compact manifold with boundary W^{n+1}, then if n is even, $[D_M]$ is the image of $[D_{\mathrm{int}\,W}]$ under the boundary map $K_1^G(W, \partial W) \to K_0^G(\partial W = M)$. However, if n is odd, the image of $[D_{\mathrm{int}\,W}]$ under the boundary map $K_0^G(W, \partial W) \to K_1^G(\partial W = M)$ is *twice* $[D_M]$. Nevertheless, $[D_M]$ will in this case be the boundary of an operator in $K_0^G(W, \partial W)$ if W admits a G-invariant nonvanishing vector field pointing, say, inward on $\partial W = M$, or in other words if M bounds in the sense of equivariant *Reinhart bordism* as studied in [**14**] and [**33**]. (The point is that the vector field can be used to get a further splitting of $\mathrm{Cliff}_{\mathbb{C}} T^* W$.) So we can summarize this information in the following:

THEOREM 2.1 (Bordism Invariance, cf. [**23**]). *Let M^n be a closed oriented manifold, and suppose a finite group G acts smoothly on M, preserving the orientation. Suppose given a map $f\colon M \to X$. Then if n is even, $f_*([D_M]) \in K_0^G(X)$ only depends on the bordism class of f in $\Omega_n^G(X)$.*

PROOF. It clearly suffices to show that $f_*([D_M]) = 0$ when $M = \partial W$ and f extends to a map $g\colon W \to X$. We use the commutative diagram

$$
\begin{array}{ccc}
K_1^G(W, \partial W) & \xrightarrow{\ g_*\ } & K_1^G(X, X) = 0 \\
{\scriptstyle \partial}\big\downarrow & & {\scriptstyle \partial}\big\downarrow \\
K_0^G(M) & \xrightarrow{\ f_*\ } & K_0^G(X)
\end{array}
$$

and observe that $[D_{\mathrm{int}\,W}]$ maps to $f_*([D_M])$ going down and then across, and to 0 going across and then down. $\qquad\square$

If n is odd, the proof of Theorem 2.1 only gives bordism invariance up to a factor of 2. However, we assume in addition that G is of odd order, then in some cases the results of [**14**] and [**33**] can be used replace oriented G-bordism by oriented Reinhart G-bordism, and then the argument will go through.

We should point out that using Theorem 2.1, one can get an approach to a refinement of the Atiyah-Singer Theorem using localization in equivariant bordism. Note that $\Omega_0^G = A(G)$, the Burnside ring of G, and one can localize at prime ideals of $A(G)$ as explained in the last chapter of [**32**]. Results along these lines, at least philosophically related to what we shall do in Section 3 using Kasparov's KK^G-theory, may be found in [**12**] and [**13**].

3. Localization and KK

We shall rely on the Localization Theorem of Segal (Proposition 4.1 of [**27**]) as well as its dual formulation for K-homology (see for example [**22**], Theorem 2.4). We briefly review how this works.

To compute $[D]$ in the $R(G)$-module $K_0^G(M)$, it suffices to compute its image $[D]_{\mathfrak{p}}$ in the localizations $K_0^G(M)_{\mathfrak{p}}$ of $K_0^G(M)$ with respect to prime (or even just

maximal) ideals $\mathfrak{p}$ of $R(G)$. Every such ideal has a *support*, a conjugacy class (H) of cyclic subgroups H of G, with the property that $\mathfrak{p}$ is the inverse image of a prime ideal of $R(H)$ under the restriction map $R(G) \to R(H)$, and such that if $\mathfrak{p}$ is also the inverse image of a prime ideal of $R(J)$ for some other subgroup J of G, then H is conjugate to a subgroup of J. Let $M^{(H)}$ denote the union of the fixed sets $M^{gHg^{-1}}$ as g runs over the elements of G. The Localization Theorem states:

THEOREM 3.1 (Localization Theorem [**27**]). *Let M be a compact G-space with the G-homotopy type of a finite G-CW complex. Let $\mathfrak{p}$ be a prime ideal of $R(G)$ with support (H). Then the inclusion $M^{(H)} \hookrightarrow M$ induces an isomorphism on K-homology and K-cohomology localized at $\mathfrak{p}$.*

We will also need some properties of Kasparov's bivariant K-theory in the equivariant setting, in other words KK^G. If X and Y are *locally* compact G-spaces, then $C_0(X)$ and $C_0(Y)$ (the continuous $\mathbb{C}$-valued functions vanishing at infinity) are abelian C^*-algebras with G-actions, so $KK^G(C_0(X), C_0(Y))$ is defined in [**10**]; for simplicity, we denote this by $KK^G(X, Y)$. This can be extended to a $\mathbb{Z}/2$-graded bivariant theory $KK_i^G(X, Y)$. The bivariant groups subsume both K-homology and K-cohomology since $KK_i^G(\text{point}, Y) = K_G^{-i}(Y)$, equivariant K-theory with compact supports as defined in [**27**], and $KK_i^G(X, \text{point}) = K_i^G(X)$, *locally finite* equivariant K-homology for locally compact spaces. There is an associative bilinear Kasparov product:

$$\otimes_X \colon KK_i^G(Z, X) \times KK_j^G(X, Y) \to KK_{i+j}^G(Z, Y).$$

If E is a G-vector bundle over a compact G-space X, then E corresponds to a finitely generated projective module over $C(X)$ with compatible G-action. Since $C(X)$ is commutative, we may view this as a (Kasparov) $C(X)$-bimodule, which gives a class

$$(3.1) \qquad\qquad [[E]] \in KK^G(X, X),$$

which is in fact the Kasparov product $[E] \otimes_X [\Delta_X]$ of $[E] \in K_G^0(X)$ with the KK^G-class of the diagonal map $\Delta_X \colon X \to X \times X$ (see [**3**], Lemma 24.5.3). The cup-product in K_G^* may then be expressed in this language since

$$[E] \cup [E'] = [E] \otimes_X [[E']] = [E'] \otimes_X [[E]].$$

We can also generalize Theorem 3.1 as:

THEOREM 3.2 (Localization in KK^G). *Let $\mathfrak{p}$ be a prime ideal of $R(G)$ with support (H), and let X and Y be locally compact G-spaces which each have the G-homotopy type of $W_1 \setminus W_2$, (W_1, W_2) some finite G-CW pair. Then the inclusions $X^{(H)} \hookrightarrow X$ and $Y^{(H)} \hookrightarrow Y$ induce isomorphisms*

$$KK_i^G(X^{(H)}, Y^{(H)})_{\mathfrak{p}} \cong KK_i^G(X, Y)_{\mathfrak{p}}.$$

PROOF. By the long exact sequences for the pairs $(X, X^{(H)})$ and $(Y, Y^{(H)})$, it is enough to show that $KK_i^G(X, Y)_{\mathfrak{p}} = 0$ if $X^{(H)} = \emptyset$ or $Y^{(H)} = \emptyset$. Equivariant homotopy invariance, the long exact sequences, and inductions on cells reduce everything to the case of a single equivariant cell in each variable, and then by Bott periodicity, we just need to show that if K and J are subgroups of G,

$$KK_*^G(G/K, G/J)_{\mathfrak{p}} = 0$$

whenever H is not subconjugate to both K and J. But by equivariant Poincaré duality ([**11**], §4), we may move the (0-dimensional) G-manifold G/K across to the other side, obtaining that

$$KK_*^G(G/K, G/J) \cong K_G^*((G/K) \times (G/J)).$$

But $(G/K) \times (G/J)$ only has H-fixed points if both G/K and G/J do, and we conclude using Theorem 3.1 (or the fact from [**26**] on which it is based, that $R(K)_{\mathfrak{p}} \neq 0$ if and only if H is subconjugate to K). $\qquad\square$

Now let's return to the situation of Section 1. For any G-invariant open subset U of M, we have a restriction map $K_*^G(M) \to K_*^G(U)$ sending the class $[D_M]$ to the class $[D_U]$ of the signature operator on U (with respect to some complete G-invariant metric on U — see the introductions to [**20**] and [**21**] for more details and references). If U is an nice open neighborhood of $M^{(H)}$, the Localization Theorem again says that the restriction map $K_*^G(M)_{\mathfrak{p}} \to K_*^G(U)_{\mathfrak{p}}$ is an isomorphism, so that $[D_M]_{\mathfrak{p}}$ may be identified with $[D_U]_{\mathfrak{p}}$. Passing to the limit over smaller and smaller G-invariant neighborhoods U of $M^{(H)}$, we obtain:

THEOREM 3.3. ([**22**], Theorem 2.6. However, the result was not stated correctly there when H is not normal in G; see also [**21**], Theorem 2.9.) *Let M be an oriented closed G-manifold, where G is a finite group acting smoothly on M and preserving the orientation, and let H be a cyclic subgroup of G. Then $[D]_{\mathfrak{p}}$ is a sum of terms coming from the various components F_i of $M^{(H)}/G = M^H/N$, where $N = N_G(H)$ is the normalizer of H in G. The contribution from F_i only depends on the germ of $G \cdot \overline{F}_i$ in M as a G-space. (Here if F_i is a component of M^H/N, $\overline{F}_i$ is its preimage in M^H, which might be disconnected.)*

If H is not normal in G, then $M^{(H)}$ can fail to be a manifold, and Theorem 3.3 is of only limited usefulness. Hereafter we will ignore this situation, and assume H is normal in G. (Even when this is not the case, we can obtain some useful information by replacing G by the normalizer of H or even something smaller; see Theorem 3.4 below.) In fact, if G is abelian or a quaternion group, then every cyclic subgroup of G is normal, so we can replace $M^{(H)}$ by the manifold M^H in Theorem 3.3. Note that even when H is normal in G, M^H may still be disconnected, and the G-action on it may permute the components. However, if F is a component of M^H, and if G' is the (setwise) stabilizer of F in G, then $G \cdot F$ is the disjoint union of $|G/G'|$ components, and the contribution of $G \cdot F$ to $[D_M]_{\mathfrak{p}} \in K_*^G(M)_{\mathfrak{p}}$ may be identified with the class in $K_*^{G'}(U)_{\mathfrak{q}}$ of the signature operator on some small G'-invariant tubular neighborhood U of F, where $\mathfrak{q}$ is the prime idea of $R(G')$ corresponding to $\mathfrak{p} \lhd R(G)$. (See the beginning of the proof of Theorem 2.12 in [**21**].) To avoid cluttering up the notation, we thus replace G by G' and assume that H is a cyclic normal subgroup of G, that F is a component of M^H, that G acts on F, and that U is a G-invariant tubular neighborhood of F. We want to compute the class $[D_U]_{\mathfrak{p}} \in K_*^G(U)_{\mathfrak{p}} \cong K_*^G(F)_{\mathfrak{p}}$ in terms of the class $[D_F]_{\mathfrak{p}} \in K_*^G(F)_{\mathfrak{p}}$ (or a twisted analogue, if F is not orientable) and the normal bundle of F. This will require looking at the signature operator along the fibers of a vector bundle, in the case of our specific situation. First it will be convenient to point out certain facts about "change of group" in equivariant K-(co)homology.

THEOREM 3.4 (see [**25**]). *Let G be a finite group, let X be a locally compact G-space, and let $r\colon K_*^G(X) \to \bigoplus_{S \subseteq G \text{ cyclic}} K_*^S(X)$ be the direct sum of the restriction*

maps from G-equivariant K-homology to S-equivariant K-homology, as S runs over the cyclic subgroups of G. Then the kernel of r is torsion of exponent dividing the order of G. If $\mathfrak{p}$ is a prime ideal of $R(G)$ with support (H), then modulo torsion of exponent dividing the order of G, $r_{\mathfrak{p}} \colon K_*^G(X)_{\mathfrak{p}} \to \bigoplus_{H \subseteq S \text{ cyclic}} K_*^S(X)_{\mathfrak{p}}$ is injective.

PROOF. This is proved in [**25**] (in the dual situation of K-cohomology, for the much harder case of compact Lie groups, but without the statement about the exponent of the torsion). In our particular situation the proof is easy once one makes use of Artin's Theorem on induced characters ([**28**], §II.9.4), which asserts that for any $\chi \in R(G)$, $|G|\chi$ is an integral linear combination of characters induced from cyclic subgroups. We only need this for $\chi = 1$, the trivial representation. Write $|G| = \sum_S \operatorname{Ind}_S^G \chi_S$, where S runs over the cyclic subgroups of G and with $\chi_S \in R(S)$. Then construct a map

$$s \colon \bigoplus_{S \subseteq G \text{ cyclic}} K_*^S(X) \to K_*^G(X)$$

by sending $c \in K_*^S(X)$ to $\operatorname{Ind}_S^G(\chi_S \cdot c)$, where Ind_S^G denotes the composite

$$K_*^S(X) \cong K_*^G(G/S \times X) \xrightarrow{(\operatorname{proj}_2)_*} K_*^G(X).$$

Then by construction, $s \circ r$ is multiplication by $|G|$, and so the kernel of r is torsion of exponent dividing $|G|$.

The final statement about the localized case follows now from [**26**], Proposition 3.7, which asserts that $R(S)_{\mathfrak{p}} = 0$ unless a conjugate of S contains H, together with the fact that in the above construction, we really only needed one cyclic subgroup in each conjugacy class of cyclic subgroups (since all conjugate subgroups induce the same representations of G). $\qquad\square$

An immediate application of Theorem 3.4 is that, at the expense of killing some torsion of exponent dividing $|G|$, we can always restrict attention to cyclic groups, thereby bypassing the problem we mentioned earlier about $M^{(H)}$ not always being a manifold.

LEMMA 3.5. *Let G be a finite abelian group, let $\mathfrak{p}$ be a prime ideal of $R(G)$, and let H be the cyclic subgroup which is its support. Then there is a unique prime ideal of $R(H)$, say $\mathfrak{q}$, which pulls back to $\mathfrak{p}$ under restriction $r \colon R(G) \to R(H)$, and $R(H)_{\mathfrak{p}} = R(H)_{\mathfrak{q}}$. Furthermore, if the residual characteristic p of $\mathfrak{p}$ is either 0 or relatively prime to $|G/H|$, then r induces an isomorphism $R(G)_{\mathfrak{p}} \overset{\cong}{\Rightarrow} R(H)_{\mathfrak{q}}$.*

PROOF. Since G is abelian, $R(G) = \mathbb{Z}\widehat{G}$ and $R(H) = \mathbb{Z}\widehat{H}$, where $\widehat{G}$ and $\widehat{H}$ are the dual groups. Since $\widehat{G} \twoheadrightarrow \widehat{H}$, $R(H) \cong R(G)/I$, where I is the kernel of r. By [**26**], Proposition 3.3(i), $N_G(H)/Z_G(H)$ acts transitively on the prime ideals of $R(H)$ pulling back to $\mathfrak{p}$, so in the abelian case there is only one such ideal, say $\mathfrak{q}$, and $\mathfrak{p}/I = \mathfrak{q}$. We have $R(H)_{\mathfrak{p}} = R(H)_{\mathfrak{q}}$ by [**4**], Ch. II, §2.2, Proposition 6. Furthermore, the character $\chi \in R(G)$ of $\operatorname{Ind}_H^G 1_H$ takes the value $|G/H|$ on H and vanishes off of H, so that if p is 0 or relatively prime to $|G/H|$, then $\chi \notin \mathfrak{p}$ and χ annihilates I, so that $I_{\mathfrak{p}} = 0$ ([**4**], Ch. II, §2.2, Corollary 2) and r induces an isomorphism $R(G)_{\mathfrak{p}} \overset{\cong}{\Rightarrow} R(H)_{\mathfrak{q}}$ ([**4**], Ch. II, §2.5, Proposition 11). $\qquad\square$

LEMMA 3.6. *Let F be a closed oriented G-manifold, with G preserving the orientation, and suppose that a normal subgroup N of G acts trivially on F (so*

that the G-action on F comes from an action of G/N). Then the class of the G-equivariant signature operator of F in $K_*^G(F)$ is the image of the (G/N)-equivariant class of the signature operator in $K_*^{G/N}(F)$, under the "inflation" map of $R(G/N)$-modules $K_*^{G/N}(F) \to K_*^G(F)$.

PROOF. Obvious.   □

Now we're ready for the key step in the calculation. First, a few relevant reminders concerning Kasparov Theory. Suppose $E \xrightarrow{p} B$ is a fibration with smooth manifold fibers, and we are given a differential operator D_E on E which only involves differentiation along the fibers, and which is elliptic when restricted to each fiber. (Note that it is not essential that E itself be a manifold, as the natural domain of D_E consists of continuous functions on E which are smooth in the fiber directions.) Then the "elliptic operator along the fibers" D_E commutes with multiplication by functions pulled back from the base B and defines a Kasparov class in $KK_*(E, B)$, or in the corresponding equivariant KK-group if everything commutes with the action of a finite group. In fact, this is a special case of what is done in [**5**], and is the set-up for proving the index theorem for families using KK-theory. (See also [**6**], §4.8.)

Secondly, we need to review something about the calculation of Kasparov products, as developed for example in [**5**], Appendix A, in [**6**], §5, and in [**3**], §18. Suppose one has classes in $KK(E, B)$ and in $KK(B, \mathbb{C})$, represented by Kasparov bimodules $(\mathcal{H}_1, T_1)$ and $(\mathcal{H}_2, T_2)$. Thus $\mathcal{H}_1$ is a $\mathbb{Z}/2$-graded Hilbert B-module with an action of E, T is an odd B-linear operator on $\mathcal{H}_1$ "approximately commuting" with the action of E, and similarly for the Hilbert space $\mathcal{H}_2$ and the operator T_2. Then the Kasparov product $[\mathcal{H}_1, T_1] \otimes_B [\mathcal{H}_2, T_2]$ is represented by the $\mathbb{Z}/2$-graded Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes_B \mathcal{H}_2$, together with an operator T which is a T_2-connection in the sense of [**5**], Appendix A. In the situation where $E \xrightarrow{p} B$ is a smooth manifold fiber bundle, T_1 comes from an elliptic differential operator D_E of order 1 along the fibers as above, and T_2 comes from an elliptic differential operator D_B of order 1 on B, then T comes from the elliptic operator $D_E \otimes 1 + 1 \otimes p^* D_B$ on E.

THEOREM 3.7. *Let G be a finite abelian group, let $\mathfrak{p}$ be a prime ideal of $R(G)$, and let H be the cyclic subgroup which is its support. Let M^n be a smooth compact G-manifold equipped with a G-invariant orientation, and let $[D_M] \in K_*^G(M)$ be the class of the signature operator on M. Observe that G permutes the components F of M^H. Then $[D_M]_{\mathfrak{p}} \in K_*^G(M)_{\mathfrak{p}} \cong K_*^G(M^H)_{\mathfrak{p}}$ is a sum over orbits $G \cdot F$ for this action. Let G' be the (setwise) stabilizer in G of some component F, and let $\mathfrak{q}$ be the prime idea of $R(G')$ corresponding to $\mathfrak{p} \lhd R(G)$. The contribution of $G \cdot F$ to $[D_M]_{\mathfrak{p}}$ is the image in $K_*^G(G \cdot F)_{\mathfrak{p}} \cong K_*^{G'}(F)_{\mathfrak{q}}$ of the Kasparov product of $[D_F] \in K_*^{G'/H}(F)$, "inflated" from $K_*^{G'/H}$ to $K_*^{G'}$ and then localized at $\mathfrak{q}$, and of the class in $KK^{G'}(F, F)_{\mathfrak{q}}$ of the "signature operator along the fibers" on the normal bundle to F in M. (In case F is non-orientable, both the signature operator on F and the signature operator along the fibers of the normal bundle must be twisted by the real line bundle determined by $w_1(F)$.)*

PROOF. By the Localization Theorem 3.1, the contribution of $G \cdot F$ to $[D_M]_{\mathfrak{p}}$ is the same as the class of $[D_U]_{\mathfrak{p}}$, where U is a G-invariant tubular neighborhood of $G \cdot F$, or as the class of $[D_{U'}]_{\mathfrak{p}}$, where U' is a G'-invariant tubular neighborhood of F. We may identify U' with the total space E of the normal bundle $E \xrightarrow{p} F$

to F, equipped with a G'-invariant metric. Fixing a G'-invariant connection on E enables us to identify $\bigwedge T^*E$ with $p^*(\bigwedge E^*) \otimes p^*(\bigwedge T^*F)$, and the signature operator on E with $D_{\mathrm{fib}} \otimes 1 + 1 \otimes p^*D_F$, where D_{fib} is the signature operator along the fibers. When F is not orientable, then E will not be either, and both D_F and D_{fib} have to be taken with coefficients in the flat real line bundle determined by $w_1(F)$. Also, it's understood that we have to choose compatible orientations on the two factors, as discussed in [**1**], p. 581. So by the above description of the Kasparov product, $[D_E]$ is the Kasparov product of $[D_{\mathrm{fib}}]$ and $[D_F]$. (Note: Since G preserves the orientation of M, F has to have even codimension. Thus $[D_{\mathrm{fib}}]$ lives in $KK_0^{G'}$, not $KK_1^{G'}$. This is important since the signature operator class on a product is the Kasparov product of the signature classes on the factors *provided* that the manifolds aren't both odd-dimensional [**23**]; in the exceptional case there is a factor of 2 because of the way one keeps track of the gradings on the Clifford algebras. Since the fibers of E have even dimension we don't have a problem here.) Finally, since H acts trivially on F, we may apply Lemma 3.6 to view F as a (G'/H)-manifold. □

As long as H is non-trivial and acts effectively, Theorem 3.7 reduces the calculation of $[D_M]_{\mathfrak{p}}$ down to the situation of smaller manifolds F and smaller groups G'/H, provided we can compute the contribution of the signature operator along the fibers. We do this calculation next. Here the manifold structure of F becomes irrelevant, and all we care about is the G-vector bundle E. (Note that for simplicity of notation, we have converted G' to G.)

PROPOSITION 3.8. *Let F be a compact G-space, where G is a finite group, and let $\mathfrak{p}$ be a prime ideal of $R(G)$ with support a central cyclic subgroup $H = \langle g_0 \rangle$ that acts trivially on F. Let E be a real G-vector bundle over F of even dimension $2k$, and assume that E can be given a G-invariant orientation. Note that E splits as a direct sum of isotypical subbundles for the various irreducible real representations of H. Assume that the trivial representation of H doesn't occur in this decomposition. Since H is cyclic, the remaining irreducible representations of H are all two-dimensional and of complex type, with the exception of the "sign" representation $G \twoheadrightarrow \{\pm 1\}$ if $|H|$ is even. So we may write E as a direct sum of oriented even dimensional subbundles $E(-1)$ and $E(e^{i\theta_j})$, where $0 < \theta_j < \pi$. Here g_0 acts on $E(-1)$ by multiplication by -1; for $0 < \theta_j < \pi$, $E(e^{i\theta_j})$ has a complex structure and g_0 acts on it by complex multiplication by $e^{i\theta_j}$. Let $E_c(e^{i\theta_j})$ denote $E(e^{i\theta_j})$ viewed as a complex vector bundle. Since H is central, the decomposition of E into the $E(e^{i\theta_j})$'s is preserved by G. Furthermore, since the complex structure on $E_c(e^{i\theta_j})$ comes from the action of g_0, this is a complex G-vector bundle and not just a complex H-vector bundle. Let D_E^{fiber} be the signature operator along the fibers of E, for a G-invariant Euclidean metric on E. (So on each fiber of E, D looks like the signature operator on $\mathbb{C}^k$, where g_0 acts on $\mathbb{C}^k$ with eigenvalues -1 and/or $e^{i\theta_j}$, $0 < \theta_j < \pi$.) Then D_E^{fiber} defines a class $[D_E^{\mathrm{fiber}}] \in KK^G(E, F)$. Localized at $\mathfrak{p}$, this class lives in $KK^G(E, F)_{\mathfrak{p}} \cong KK^G(F, F)_{\mathfrak{p}}$, and may be identified with $[[\mathcal{E}]] = [\mathcal{E}] \otimes_F [\Delta_F]$ (notation of equation (3.1)), where $\mathcal{E}$ is the cup product of classes $[\mathcal{E}(e^{i\theta_j})]$, $0 < \theta_j < \pi$, and $[\mathcal{E}(-1)]$. Furthermore, we have*

$$[\mathcal{E}(e^{i\theta_j})] = \left[\bigwedge E_c(e^{i\theta_j})\right] / \left(\left[\bigwedge{}^{\mathrm{even}} E_c(e^{i\theta_j})\right] - \left[\bigwedge{}^{\mathrm{odd}} E_c(e^{i\theta_j})\right]\right)$$

for $\theta_j < \pi$, and also for $\theta_j = \pi$ if $E(-1)$ has a G-invariant complex structure. If $E(-1)$ has a G-invariant spin^c structure, we have a similar formula:

$$[\mathcal{E}(-1)] = [\mathcal{S}(E(-1))] / \left([\mathcal{S}^+(E(-1))] - [\mathcal{S}^-(E(-1))] \right),$$

where $\mathcal{S}(E(-1))$ is the complex spinor bundle for the spin^c structure on $E(-1)$, and $\mathcal{S}^\pm$ are the half-spinor bundles. Finally, if $E(-1)$ does not have a G-invariant spin^c structure, the formula is the same, but must be interpreted in the sense of twisted coefficients.

PROOF. Let $\mathcal{X}$ denote the continuous field of Hilbert spaces over F, whose fiber over $x \in F$ is $L^2\left(\bigwedge T^*_{\mathbb{C}} E_x \right)$, with the $\mathbb{Z}/2$-grading of [**1**], p. 575. Since D_E^{fiber} commutes with multiplication by functions on the base F, and is self-adjoint, local, and elliptic along the fibers, and since G acts isometrically, the pair $(\mathcal{X}, D_E^{\text{fiber}})$ satisfies the conditions for an unbounded Kasparov module in the sense of [**2**], giving a class $[D_E^{\text{fiber}}] \in KK^G(E, F)$. Since $+1$ is not an eigenvalue of the action of g_0 on E, $E^{(H)} = F$ and we have an isomorphism

$$KK^G(E, F)_{\mathfrak{p}} \cong KK^G(F, F)_{\mathfrak{p}}$$

by Theorem 3.2. But on an even-dimensional spin^c manifold, the signature operator may be expressed as the Dirac operator with coefficients in the dual of the complex spinor bundle ([**15**], §II.6.2). Suppose first that $E(-1)$ also has a G-invariant complex structure, so that E is the underlying real G-vector bundle of a complex G-vector bundle E_c, the direct sum of $E_c(-1)$ and the $E_c(e^{i\theta_j})$'s. Then

$$\text{Cliff}_{\mathbb{C}} E^* \cong \bigwedge(E_c^*) \otimes_{\mathbb{C}} \bigwedge(E_c) \cong \left(\bigotimes_{0 < \theta_j \leq \pi} \bigwedge E_c(e^{i\theta_j}) \right) \otimes \left(\bigotimes_{0 < \theta_j \leq \pi} \bigwedge E_c(e^{i\theta_j})^* \right),$$

with the first factor identified as the complex spinor bundle and the second factor identified with its dual. So $[D_E^{\text{fiber}}]$ is the class of the Dirac operator along the fibers, with coefficients in $\bigwedge(E_c)$. But the Dirac operator along the fibers gives the inverse of the Thom isomorphism $\tau \in KK^G(F, E)$ in equivariant K-theory, so that we have the formula

$$\tau \otimes_E [D_E^{\text{fiber}}] = [[\bigwedge(E_c)]] \in KK^G(F, F).$$

When we localize at $\mathfrak{p}$ and restrict to the fixed-point set (which is just the zero-section of E), τ is multiplication by $\bigwedge_{-1}(E_c) = [\bigwedge^{\text{even}}(E_c)] - [\bigwedge^{\text{odd}}(E_c)]$ ([**27**], §3). So

$$[D_E^{\text{fiber}}] = [[\bigwedge(E_c)/\bigwedge_{-1}(E_c)]],$$

which in turn splits into pieces corresponding to the various rotation angles θ_j, as claimed. (The division makes sense since $\bigwedge_{-1}(E_c)$ is a unit in equivariant K-theory localized at $\mathfrak{p}$.)

Now consider the case where $E(-1)$ does not have a complex structure (or at least one that is G-invariant). We proceed as before, except that we need twisted coefficients in the Thom isomorphism ([**10**], §5, Theorem 8) for the factor associated to $E(-1)$. Note that in the case of a G-invariant spin^c structure on $E(-1)$, $\mathcal{S}^+(E(-1))$ substitutes for $\bigwedge^{\text{even}} E_c(-1)$, and $\mathcal{S}^-(E(-1))$ substitutes for $\bigwedge^{\text{odd}} E_c(-1)$. $\qquad \square$

EXAMPLE 3.9. To give a very simple example, suppose G is abelian, $\mathfrak{p}$ is a prime ideal with support H, and F is a component of M^H whose normal bundle is stably equivariantly trivial (i.e., a G-invariant tubular neighborhood of F is stably just a product of F with a representation space V of G whose restriction to H is non-trivial on a generator of H). As we've seen, V must have a G-invariant complex

structure, so we can think of V as the realification of a complex representation V_c, and we choose the orientation of F so that the orientation on $F \times V_c$ agrees with the orientation of M. Then our formula for the contribution of G to $[D_M]_\mathfrak{p}$ reduces simply to

$$\left[\bigwedge(V_c)/\bigwedge_{-1}(V_c)\right] \cdot [D_F].$$

Here $\bigwedge(V_c)$ and $\bigwedge_{-1}(V_c)$ are viewed as elements of $R(G)$. While we can't divide them in $R(G)$, the fact that each constituent of V_c is non-trivial on a generator of H means that $\bigwedge_{-1}(V_c)$ does not lie in the prime ideal $\mathfrak{p}$, so the division makes sense in $R(G)_\mathfrak{p}$, the coefficient ring for the localized theory. To see this, first observe that if we write V_c as a sum of irreducible characters $\chi_i \colon G \to U(1)$, then $\bigwedge_{-1}(V_c) = \prod_i(1 - \chi_i)$. So we just need to show that if χ is a one-dimensional representation of G which is non-trivial on a generator g of H, then $1 - \chi \notin \mathfrak{p}$. If the residual characteristic of $\mathfrak{p}$ is 0, then $\mathfrak{p}$ is just the ideal consisting of virtual representations of G whose characters vanish at g, so by assumption on χ, $\chi(g) \neq 1$, i.e., $1 - \chi \notin \mathfrak{p}$. And if the residual characteristic of $\mathfrak{p}$ is finite, say p, then the order of H is relatively prime to p ([**26**], Proposition 3.5), which since $\chi(g) \neq 1$ will force $\chi|_H$ to map to an element other than 1 in the finite field $R(H)/\mathfrak{q}$ of characteristic p. (Here $\mathfrak{q}$ is as in Lemma 3.5.) Thus again $1 - \chi \notin \mathfrak{p}$ in this case. $\qquad\square$

We put everything together to show that the equivariant K-homology class of the signature operator can contain quite complicated torsion information not preserved under G-homotopy equivalences, even when M is a very simple manifold and G is cyclic of prime order.

THEOREM 3.10. *Let p be an odd prime. Then there exist actions of $G = \mathbb{Z}/p$ on odd spheres $M = S^{2k+1}$ for which the equivariant K-homology class $[D_M] \in K_1^G(M)$ contains "arbitrarily complicated torsion information." More precisely, let q be a prime (which may or may not be equal to p). Then there exist actions of $G = \mathbb{Z}/p$ on spheres $M_1 = M_2 = S^{2k+1}$ and G-homotopy equivalences $M_1 \overset{h}{\to} M_2$ such that $[D_{M_2}] - h_*([D_{M_1}])$ is torsion of order as large a power of q as one wants.*

PROOF. (*Sketch*) First suppose $q = p$. Then take M_1 and M_2 to be free linear G-spheres such that the lens spaces $L_1 = M_1/G$ and $L_2 = M_2/G$ are homotopy equivalent but not diffeomorphic. Note that under the $\mathbb{Z}$-module isomorphism $K_*^G(M_j) \cong K_*(L_j)$, $[D_{M_j}]$ corresponds to $[D_{L_j}] \in K_1(L_j)$. This class is computable (see [**23**], §2) and is not homotopy-invariant. (The idea of the calculation is to write the signature operator as a Dirac operator with coefficients in the dual of the spinor bundle of the cotangent bundle, as in the proof of Proposition 3.8.) In fact, given $r \geq 1$, we can choose k sufficiently large (depending on p and r) so that there is a homotopy equivalence $h \colon L_1 \to L_2$ between lens spaces of dimension $2k + 1$ for which $[D_{L_2}] - h_*([D_{L_1}])$ has order p^r in $K_1(L_2)$, giving us the example we want when we pull back to the universal covers.

Next, suppose $q \neq p$, and this time consider lens spaces L_1 and L_2 of dimension $2j + 1$, each with fundamental group $\mathbb{Z}/q^r$, which are homotopy equivalent but have different signature operator classes, just as above. Then L_1 and L_2 are $\mathbb{Z}/p$-homology spheres with rationally trivial stable normal bundles, so by "converse Smith theory" it is known that they can be realized fixed sets of semifree actions of G on spheres $M_1 = M_2 = S^{2k+1}$, provided that $j < k$ are in an appropriate range (k roughly equal to jp). (See [**9**], §5, and [**34**], §6.) In this way one can get an equivariant homotopy equivalence $h \colon M_1 \to M_2$. We can then compute

$[D_{M_2}] - h_*([D_{M_1}])$, localized at a prime ideal $\mathfrak{q}$ of residual characteristic q supported on all of G, using Theorem 3.7. We obtain a difference between $[D_{L_2}]$, twisted by some normal characteristic classes, and something similar for L_1, transported over to L_2 via h. One can arrange for $[D_{M_2}] - h_*([D_{M_1}])$ to involve large q-primary torsion. $\square$

4. Comparison with the Atiyah-Singer Theorem

Let us check that the formula for the localization of $[D_M]$ derived in Section 3 agrees with the Atiyah-Singer *G*-Signature Theorem ([**1**], Theorem 6.12) in the case when the dimension of M is even. To see this, let $g \in G$ and let H be the cyclic subgroup it generates. Let $\mathfrak{p}$ be the prime ideal of $R(G)$ consisting of virtual representations whose characters vanish at g, so that $R(G)/\mathfrak{p} \hookrightarrow \mathbb{C}$ via evaluation of characters at g. Clearly $\mathfrak{p}$ has support H in the sense of [**26**]. We may compute $\mathrm{Sign}(g, X)$ (in the sense of [**1**]) by mapping $[D_M]_{\mathfrak{p}}$ to $R(G)_{\mathfrak{p}}$ via the collapse map $c \colon M \to$ point, and then mapping to the residue field $R(G)_{\mathfrak{p}}/\mathfrak{p}_{\mathfrak{p}}$ (a subfield of $\mathbb{C}$, in fact a number field). We get a sum of terms coming from the components F of M^H, and for purposes of computing $\mathrm{Sign}(g, X)$, we may as well assume $G = H$. Then Theorem 3.7 and Proposition 3.8 apply to this situation. So the contribution of F to $[D_M]_{\mathfrak{p}}$ is thus effectively the Kasparov product of the non-equivariant *K*-homology class $[D_F]$ of the signature operator on F and of the class in $KK^H(F, F)$ of the signature operator on the fibers of the normal bundle. The Chern character of the former is the Poincaré dual of the Atiyah-Singer *L*-class $\mathcal{L}(F)$, or in other words the factor A_1 in [**1**], p. 581, and of the latter is the Chern character of the virtual H-bundle described in Proposition 3.8. The second factor accounts for the factors B_1 and C_1^θ in [**1**], p. 581. Consider for example the contribution of $E_c(e^{i\theta})$, which in the Atiyah-Singer notation is $N^g(\theta)$, when $0 < \theta < \pi$. If this splits into complex line bundles with first Chern classes x_j, then since g acts by $e^{in\theta}$ on the n-th tensor power of one of these line bundles, we get from $[[\bigwedge(E_c(e^{i\theta}))/\bigwedge_{-1}(E_c(e^{i\theta}))]]$ a contribution of

$$\prod_j \left(\frac{1 + e^{x_j} e^{i\theta}}{1 - e^{x_j} e^{i\theta}} \right) = \prod_j \left(\frac{e^{(x_j + i\theta)/2} + e^{-(x_j + i\theta)/2}}{e^{-(x_j + i\theta)/2} - e^{(x_j + i\theta)/2}} \right),$$

which up to a sign $((-1)^{s(\theta)})$ is

$$\prod_j \coth \left(\frac{x_j + i\theta}{2} \right),$$

just as on p. 581 of [**1**].

5. Extension to the non-smooth case, Concluding remarks

The results of Section 3 can be interpreted as an inductive algorithm for computing the class in equivariant *K*-homology of the signature operator D_M of a smooth closed *G*-manifold M, modulo perhaps the loss of some torsion of order dividing a power of the order of G, on the basis of two ingredients:

(1) the (non-equivariant) *K*-homology classes of the signature operators on certain submanifolds, namely, the connected components F of the fixed sets for cyclic subgroups $H \subseteq G$. (In case F is non-orientable, we use the

signature operator with coefficients in the real line bundle determined by
$w_1(F)$.)

(2) certain characteristic classes in equivariant K-theory for the normal bundles E of these submanifolds F, as given in Proposition 3.8. (If F is not orientable, then the normal bundle isn't orientable either, and we replace it by its tensor product with the real line bundle determined by $w_1(F)$, which now *is* orientable.)

Before going on to the non-smooth case, let us review this algorithm. By Theorem 3.4, if we are prepared to accept the loss of some torsion of order dividing the order of G, we can always restrict to subgroups and reduce to the case where G is abelian, in fact cyclic. Then since

$$K_*^G(M) \to \bigoplus_{\mathfrak{p} \text{ maximal in } R(G)} K_*^G(M)_{\mathfrak{p}}$$

is injective ([**4**], Ch. II, §3.3, Theorem 1), it is no loss of generality to localize at a maximal ideal $\mathfrak{p}$ of $R(G)$, say with support H. If $H = \{1\}$, then $M^H = M$ and localization doesn't do much; however, if the residual characteristic p of $\mathfrak{p}$ is prime to $|G|$, then by Lemma 3.5, $R(G)_{\mathfrak{p}} \cong \mathbb{Z}_{(p)}$ and we loose nothing by forgetting G entirely. If $H = \{1\}$ and the residual characteristic p of $\mathfrak{p}$ does divide $|G|$, then $R(G)_{\mathfrak{p}} \to \mathbb{Z}_{(p)}$ is not an isomorphism, but the restriction map $K_*^G(M)_{\mathfrak{p}} \to K_*(M)_{(p)}$ only kills some p-primary torsion. So assume $H \neq \{1\}$. If H fixes all of M, then we can apply Lemma 3.6 (with $N = H$), and deduce that the class $[D_M] \in K_*^G(M)_{\mathfrak{p}}$ comes from the class $[D_M] \in K_*^{G/H}(M)$, which only involves the action of the smaller group G/H. Otherwise, choose some component F of M^H, which is now a submanifold of smaller dimension, and apply Theorem 3.7. This computes the contribution of F to $[D_M]_{\mathfrak{p}}$ in terms of the ingredients (1) and (2) above.

There is hope for carrying out all or most of the same program when M is only a Lipschitz manifold and the action of G is Lipschitz and locally linear, using the Lipschitz signature operator and its KK-class as constructed in [**30**], [**31**], [**7**], and [**8**]. The Lipschitz locally linear category of group actions was studied to some extent in [**22**] and in [**24**], and as explained in [**24**], is quite close to the topological locally linear category. (Since PL manifolds have a canonical Lipschitz structure, the discussion here includes the PL locally linear case. However, the construction of the PL signature operator class in [**29**] is much easier than in the Lipschitz case.)

The first steps of the program, involving restriction to cyclic subgroups and localization at prime ideals of $R(G)$, go through with almost no change, thanks to [**22**], which enables us to localize the Lipschitz signature operator in a G-invariant neighborhood of some fixed set component F. The problem is that even in the PL locally linear category, this neighborhood can be identified with a block bundle over F, not in general with a vector bundle, and it is not clear if one can split the signature operator as in Theorem 3.7. So we conclude with the following question:

QUESTION 5.1. *Suppose M is a locally linear Lipschitz G-manifold, equipped with a G-invariant orientation, with G a finite abelian group, and let $[D_M]$ be the class in $K_*^G(M)$ of the Lipschitz signature operator on M. Let $\mathfrak{p}$ be a prime ideal of $R(G)$, and let H be the cyclic subgroup which is its support. Let F be a component of M^H (a locally flatly embedded topological submanifold of M). Then is it possible as in Theorem 3.7 to split the contribution of $G \cdot F$ to $[D_M]_{\mathfrak{p}}$ as the Kasparov*

product of $[D_F]$, *the signature operator class on* F, *and of a term corresponding to the "signature operator along the fibers"? If so, how can one compute the latter?*

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