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C^*-algebras and Mackey's theory of group representations

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ABSTRACT. The subjects of C^*-algebras and of unitary representations of locally compact groups are both approximately 50 years old. While it was known from the start that these subjects are related, it was not originally appreciated just how close the relationship is, especially in the case of Mackey's theory of induced representations and of representations of group extensions. If G is a (locally compact) group, then the unitary representation theory of G is precisely that of its group C^*-algebra $C^*(G)$. And if G contains a normal subgroup N, then $C^*(G)$ may be identified with a "twisted crossed product" $C^*(G, C^*(N), \tau)$ of $C^*(N)$ by G/N. The aim of the "Mackey machine" is to describe the representation theory of such a crossed product in terms of knowledge of $C^*(N)$ and of the action of G on it by conjugation. For many years results were confined to describing the irreducible representations or primitive ideals of the crossed product as a set, or at best as a topological space. To do the latter is already usually difficult. However I will try to focus on cases where one can describe the "fine structure" of the Mackey machine, that is, the way the crossed product sits over its spectrum, and possible twisting in this structure.

I am honored to have been asked to participate in this Special Session on "C^*-Algebras: 1943–1993," especially since two of the originators of the subject of C*-algebras, Israel Gelfand and Irving Segal, are participating in this Session.

The year 1993 marks the fiftieth anniversary of two landmark papers in functional analysis, [**GeN1**], which effectively originated the study of C*-algebras, and [**GeR**], which marked the beginning of much of the theory of unitary representations of locally compact groups. Accordingly, this seems to be an appropriate time for reflection on the history of both subjects, and especially of the relationship between them. While this paper contains a discussion of this history, it is rather personal in taste and far from complete, so I hope the reader will forgive me for the lapses and inaccuracies which will undoubtedly be present.

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§1. Some Early History

The subject of group representations on complex vector spaces dates back to work of G. Frobenius and I. Schur starting around 1897, but was originally viewed as an area of algebra and confined to the study of finite groups. The famous work of Peter and Weyl [**PeW**] showed that the theory could be extended, with relatively minor changes, to compact groups, where it could be used for harmonic analysis in the same way that Fourier series are used for harmonic analysis on the circle. But the study of unitary representations of non-compact, non-commutative locally compact groups came much later, starting with the famous study by Wigner [**Wi**] of representations of the "inhomogeneous Lorentz group" or Poincaré group, the semidirect product $SO(3, 1) \ltimes \mathbb{R}^{3,1}$. This paper marked the first appearance (for non-compact groups) of what would later come to be known as the "Mackey machine." In particular, three features of Wigner's work are worthy of mention: the study of the orbits of a group on the dual of a normal subgroup, construction of irreducible representations by unitary induction, and the need to study **projective** unitary representations, here viewed as "double-valued" representations, in other words, representations of a double covering group. Wigner did not develop any of these topics from a general point of view; in fact, each of the three was the subject of a more complete study later, the first two by Mackey ([**M5**] and [**M3, M4**]) and the third by Bargmann [**Bar2**]. Furthermore, had Wigner wanted to systematically study representations of semi-direct product groups, it would have been natural for him to have begun with simpler examples, such as the Heisenberg group or the '$ax + b$' group of affine motions of the line, first studied by Gelfand and Naimark [**GeN2**] a few years later. Rather, it is fair to say that Wigner was more interested in possible physical applications of the Poincaré group than in the theory of unitary representations for its own sake, and he developed his tools in an *ad hoc* manner as he needed them.

The first substantial evidence that the unitary representations of locally compact groups were a subject worth developing systematically thus dates to the famous Gelfand-Raikov Theorem [**GeR**], published in 1943. Gelfand and Raikov showed, using the theory of positive-definite functions, that an arbitrary locally compact group has enough irreducible unitary representations to separate points. However, they did not indicate methods for actually constructing these representations. The first major developments in this direction were the classifications of the irreducible unitary representations of the '$ax+b$' group by Gelfand and Naimark [**GeN2**], of the Lorentz group by Gelfand-Naimark [**GeN3**] and Bargmann [**Bar1**], and of $SL(2, \mathbb{R})$ by Bargmann [**Bar1**]. Since my intention is not to give a complete history of the development of the subject of group representations, I will not go into further details; the interested reader can consult the historical sections of [**M6**] and of Kirillov's book [**Ki**], as well as the broader histories in [**M7**] and in [**Ho2**].

Instead, I would like to focus for a moment on the early history of the relationship between unitary representation theory and the theory of C*-algebras. In the same journal (*Mat. Sbornik*) and in the same year (1943) that the Gelfand-Raikov paper appeared, Gelfand and Naimark published their famous characterization [**GeN1**] of what later came to be known (largely through the influence of I. Segal) as **C*-algebras**. While the details of their axiomatization were later simplified, what is important about their work was the accomplishment of giving an abstract algebraic characterization of those Banach $*$-algebras, or Banach algebras with involution (they called these "normed $*$-rings" at the time) which are isometrically $*$-isomorphic to a norm-closed $*$-algebra of operators on a complex Hilbert space. Among the class of Banach algebras, C*-algebras are important for two reasons: they are general enough to serve as models for all phenomena that can be described in terms of Hilbert space operators, yet "rigid" enough to be in many cases classifiable. The key to this rigidity may with hindsight be found in the original Gelfand-Naimark characterization, but was first made explicit in the famous papers of Gelfand-Naimark [**GeN4**] and Segal [**S2**] describing the "GNS" construction of representations of C*-algebras from positive linear functionals, or **states**. In these two papers, new proofs were given of the Gelfand-Raikov Theorem, via the device of relating unitary representations of a locally compact group to representations of a suitable group algebra, first proposed in [**S1**]. The key observation is that if one fixes a left Haar measure dx on a locally compact group G, then $L^1(G)$ becomes a Banach algebra under convolution, which if G is finite is just the usual group ring $\mathbb{C}G$. Furthermore, $L^1(G)$ has a natural isometric (conjugate-linear) involution $f \mapsto f^*$, where

$$f^*(x) = \overline{f(x^{-1})}\Delta(x^{-1}),$$

Δ the modular function (the Radon-Nikodym derivative of left Haar measure with respect to right Haar measure, normalized to be 1 at the identity element).

It was undoubtedly clear to Gelfand and Naimark from the beginning that the subjects of C*-algebras and of unitary representations of locally compact groups are closely related. After all, several of the most interesting examples of von Neumann algebras (or in the original terminology, "rings of operators") constructed by Murray and von Neumann in the mid-30's came from unitary representations of groups. But full exploitation of C*-algebra techniques in describing unitary representations of groups was slow in coming. The proof of the Gelfand-Raikov Theorem in [**GeN4**], basically the same as the one later printed in Naimark's influential book [**N**], depends on the fact, already pointed out in [**S1**], that there is a natural bijection (preserving such features as irreducibility) between strongly continuous unitary representations of a locally compact group G and non-degenerate $*$-representations (representations sending the involution $*$ of the algebra to the adjoint operation on Hilbert space operators) of the L^1-algebra $L^1(G)$. Segal's proof in [**S2**] is similar, but already begins to rely implicitly on the rigidity properties of C*-algebras, stated as separate theorems in [**S3**]: any quotient of a C*-algebra by a closed two-sided ideal is again a C*-algebra, in fact in a unique way, and any (*-preserving) homomorphism of C*-algebras is automatically continuous with closed range. Thus Segal introduces what in modern terminology would be called $C_r^*(G)$, the reduced C*-algebra of a locally compact group G: the norm closure of the image of $L^1(G)$ in $\mathcal{B}(L^2(G))$, the bounded operators on $L^2(G)$, via the action of $L^1(G)$ on $L^2(G)$ by left convolution. The Gelfand-Raikov Theorem is then deduced from the general fact, as applied to $C_r^*(G)$, that any C*-algebra has a faithful family of irreducible $*$-representations.

Many of the early papers on unitary representations of locally compact groups, such as the papers of Mautner, Godement, and Segal on decomposition theory and Plancherel formulas, relied in some way on the correspondence between group representations and algebra representations. But as far as I can tell, the first published statement of the equivalence of unitary representations with representations of a certain C*-algebra associated with the group seems to be found in the following statement of Kaplansky in [**K2**], buried in the proof of Theorem 7 in that paper. The footnotes are my own parenthetical comments.

> Let G be a locally compact group, A its L^1-algebra[1], B the result of re-norming A by assigning to every element the sup of its norms in all possible $*$-representations[2], and C the completion of B in this new norm. Then C is a C*-algebra, and it is known that there is a 1-1 correspondence between $*$-representations[3] of C and (strongly continuous) unitary representations of G.

In modern terminology, the algebra C described by Kaplansky is called the **(full or universal) group C*-algebra**, $C^*(G)$. The algebra $C_r^*(G)$ that had been introduced by Segal is a quotient of $C^*(G)$; it was shown much later that the two coincide if and only if G is amenable (see [**P, Theorem 4.21**]). The assignment $G \rightsquigarrow C^*(G)$ is not quite a functor in the usual sense because of the technical complication that G does not sit inside $C^*(G)$ unless G is discrete, and thus in general a homomorphism $G \to H$ does not induce a $*$-homomorphism $C^*(G) \to C^*(H)$. However, one has functoriality under the class of group homomorphisms generated by quotient maps and embeddings of open subgroups. It is also convenient from the modern point of view (dating from [**J**]) to note that G naturally embeds in the **multiplier algebra** of $C^*(G)$, $M(C^*(G))$, which is the largest C*-algebra in which $C^*(G)$ can be embedded as a (closed two-sided) essential ideal. Thus the universal property of $C^*(G)$ shows that a homomorphism $G \to H$ induces a $*$-homomorphism $C^*(G) \to M(C^*(H))$.

The correspondence between $*$-representations of $C^*(G)$ and unitary representations of G is now easily described: any non-degenerate $*$-representation π of $C^*(G)$ extends canonically to a representation $\tilde{\pi}$ of $M(C^*(G))$ on the same

[1] Kaplansky writes L_1.

[2] Though not mentioned here, this sup is finite since all such representations are norm-decreasing.

[3] Assumed non-degenerate.

Hilbert space, and the associated unitary representation of G is then obtained by restricting $\tilde{\pi}$ to G. In the other direction, given a unitary representation π of G, it defines a non-degenerate $*$-representation of $L^1(G)$ by $f \mapsto \int f(x)\pi(x)\,dx$, which by the universal property of $C^*(G)$ must factor through $C^*(G)$. This correspondence is even better than the way Kaplansky described it; essentially **all** questions one can ask about the unitary representation theory of G are equivalent to questions about the structure of $C^*(G)$.

Around the year 1950, the structure theory of C*-algebras began to take on an independent flavor of its own, related to, but distinct from, the study of von Neumann algebras and direct integral decompositions of representations on the one hand (a subject that occupied many operator algebraists at the time), and the study of more general classes of Banach algebras (typified by L^1-algebras or function algebras) on the other hand. However, the subject was a bit slow in getting off the ground; up until the time of publication of Dixmier's book [**Di**], which brought the subject to the attention of a larger audience, the structure theory of C*-algebras was never studied by more than a handful of people at a time.

The first major development in this study of the structure theory of C*-algebras came in Kaplansky's paper [**K1**], in which the notions of CCR and GCR C*-algebras (later called **liminal** and **postliminal** algebras in [**Di**]) were first introduced. The postliminal algebras were later shown in a deep paper of Glimm ([**Gl1**]—see also [**Di**, §9]) to coincide with the **type I** C*-algebras, that is, with the C*-algebras all of whose factor representations are of Murray-von Neumann type I. However, at the time, the importance of Kaplansky's work was to point out that at least there are **some** type I C*-algebras with an interesting structure theory. At the same time, Kaplansky noticed that many C*-algebras can be described as algebras of continuous sections of "fields" or "bundles" of algebras, a point of view that was to be taken up again later in greater detail by Fell [**F2**] and by Dixmier and Douady ([**DiD**], [**Di**, §10]). And in [**K2**], he noted that liminal algebras must play an important role in unitary representation theory, as could be seen from the recently announced theorems of Harish-Chandra [**HC**] implying that the C*-algebras of semisimple Lie groups are liminal.[4]

At about this same time, George Mackey began to systematize the unitary representation theory of general locally compact groups, with the aim of finding an algorithm for classifying the irreducible unitary representations of a group G having a closed normal subgroup N, assuming that one already has sufficient information about N and G/N. Precedents for this work could already be seen in the papers of Frobenius and Schur, in an influential paper of Clifford [**Cl**] (working out the details of such an algorithm in the case of finite-dimensional

[4] A simplified version of (a slightly more restrictive version of) Harish-Chandra's theorem appears in [**Di**, **15.5.6**]. We will discuss a refinement below in §3, in connection with Research Problem 2.

representations of discrete groups), in the work of Stone and von Neumann on uniqueness of irreducible representations of the Heisenberg commutation relations [**vN**], and in the famous work of Wigner which we have already discussed [**Wi**]. Mackey's main idea was to give a means, known as **(unitary) induction**, for constructing a unitary representation $\operatorname{Ind}_H^G \pi$ of a locally compact group G out of a unitary representation π of a closed subgroup H, and to give a criterion, known as the **Imprimitivity Theorem**, for recognizing when a representation is induced from H. The construction was largely given in [**M1**], with further details (such as the "induction in stages" theorem $\operatorname{Ind}_H^G \operatorname{Ind}_K^H \pi \cong \operatorname{Ind}_K^G \pi$) in [**M4, I**]. Mackey's generalization in [**M2**] of the Stone-von Neumann Theorem basically amounted to a special case of the Imprimitivity Theorem with $H = \{1\}$. He noted incidentally that this result could be used to recover the classification of the irreducible unitary representations of the '$ax + b$' group by Gelfand and Naimark [**GeN2**]. Mackey then proceeded in [**M3**] and [**M4, I**] to show how to decompose tensor products of induced representations, or restrictions of induced representations to closed subgroups. Finally, in [**M4, I**] he gave an analogue of the Frobenius Reciprocity Theorem, which in the case of finite groups basically says that induction of representations from H to G and restriction of representations from G to H are adjoint functors. (Some of this work was duplicated in papers of Mautner such as [**Ma**], that were written at roughly the same time.) Mackey's methods were largely measure-theoretic; as a result he had to work hard to overcome technical difficulties that arise from dealing with equations that are only true almost everywhere. No C*-algebraic methods appear in Mackey's early papers (though there is plenty of use of direct integral decomposition of von Neumann algebras).

Mackey's algorithm for classifying the irreducible unitary representations of a group G having a "regularly embedded" closed type I normal subgroup N, known in the trade as the "Mackey machine," appears in his paper [**M5**]. Again Mackey's methods were largely measure-theoretic, which forced him to restrict attention to representations of second-countable groups on separable Hilbert spaces, though this is not much of a restriction since almost all cases of interest satisfy these conditions. We will describe the Mackey machine in informal terms, since another version (due primarily to Rieffel and Green) will be given in the next section. Suppose π is an irreducible unitary representation of G and N is a closed type I normal subgroup. The restriction $\pi|_N$ of π to N is then a unitary representation of N, though usually not irreducible. By the decomposition theory for representations of type I C*-algebras or groups, it splits up in an essentially unique way as a direct integral $\int^{\oplus} m_\pi(\rho)\rho \, d\mu_\pi(\rho)$ of irreducible representations of N, each occurring with a certain multiplicity, with respect to some measure μ_π on $\hat{N}$, the space of equivalence classes of irreducible unitary representations of N. Because N is normal in G, G acts on unitary representations of N by the formula $g \cdot \rho(n) = \rho(g^{-1}ng)$, and since clearly $g \cdot \pi|_N \cong \pi|_N$ for all $g \in G$, it follows that μ_π is G-quasi-invariant and the multiplicity function m_π

is G-invariant. Furthermore, since π was assumed irreducible, it is easy to see that the measure μ_π is ergodic and that the multiplicity function m_π is constant almost everywhere. Mackey calls the class of μ_π a **quasi-orbit**. In cases where the action of G on $\hat{N}$ is nice enough (this is the "regular embedding" condition), μ_π will be supported on a single G-orbit, say $G \cdot \rho$, and we may assume that

$$\pi|_N \cong m_\pi \int_{G/G_\rho}^{\oplus} \dot{g} \cdot \rho \, d\dot{g},$$

where m_π is now a constant and $d\dot{g}$ is a quasi-invariant measure on the homogeneous space G/G_ρ. This basically verifies that the hypothesis of Mackey's Imprimitivity Theorem applies to π, and thus π is induced from an irreducible unitary representation of G_ρ whose restriction to N is a multiple of ρ. One can also go in the other direction: given $\rho \in \hat{N}$, one can compute its stabilizer G_ρ in G (for the action of G on $\hat{N}$), and if σ is an irreducible unitary representation of G_ρ whose restriction to N is a multiple of ρ, then $\mathrm{Ind}_{G_\rho}^G \sigma$ will be an irreducible unitary representation of G "lying over" the G-orbit of ρ. In this way, $\hat{G}$ is partitioned into "fibers" over the various orbits of G on $\hat{N}$, and this concludes the first part of the Mackey machine.

The second part of the Mackey machine gives a mechanism for determining all irreducible unitary representations of G_ρ whose restriction to N is a multiple of ρ, and in particular for showing that this set is non-empty for any $\rho \in \hat{N}$. This is where the theory of projective representations comes in. By definition of G_ρ, if $g \in G_\rho$, then $g \cdot \rho \cong \rho$, where "$\cong$" denotes unitary equivalence. In other words, if $g \in G_\rho$, there is a unitary operator U_g on the Hilbert space $\mathcal{H}_\rho$ where ρ is acting, such that $\rho(g^{-1}ng) = U_g^{-1}\rho(n)U_g$ for all $n \in N$. It is also clear from the fact that ρ is a representation that if $g \in N$, we may take $U_g = \rho(g)$. Since ρ is irreducible, Schur's Lemma says that U_g is uniquely defined up to a scalar, and thus for any g and g' in G, $U_{gg'}$ must agree with $U_g U_{g'}$ up to a scalar. Thus $g \mapsto U_g$ is a projective unitary representation of G_ρ on $\mathcal{H}_\rho$ which extends ρ on N.

In general, there is a cohomological obstruction, the **Mackey obstruction**, to our being able to choose the U_g's so that $U_{gg'} = U_g U_{g'}$, in other words, to our being able to extend ρ to an actual unitary representation of G_ρ on $\mathcal{H}_\rho$. This obstruction is defined as a certain member of a cohomology group $H^2(G_\rho/N, \mathbb{T})$, the second cohomology of the **"little group"** G_ρ/N with coefficients in the circle group, where since G_ρ is a second-countable locally compact group one uses a version of group cohomology for such topological groups, cohomology with Borel cochains. More precisely, Mackey shows that one can choose the U_g's to be constant on cosets of N and to vary "measurably", so that $U_{gg'} = \omega(\dot{g}, \dot{g}')U_g U_{g'}$ for some Borel measurable function $\omega : (G_\rho/N) \times (G_\rho/N) \to \mathbb{T}$. (Here $\dot{g}$ denotes the N-coset of g.) The function ω satisfies the cocycle identity, and its cohomology class $[\omega]$ is independent of the choice of the U_g's. One can extend ρ to an actual unitary representation of G_ρ on $\mathcal{H}_\rho$ if and only if this

Mackey obstruction vanishes. When G is a semidirect product $N \rtimes H$ and N is abelian, so that $\mathcal{H}_\rho$ is one-dimensional and ρ is a one-dimensional character, we actually have $\rho(g^{-1}ng) = \rho(n)$ for all $n \in N$ and $g \in H_\rho$, so one may choose U_g to be identically 1 on $H_\rho \cong G_\rho/N$ and the Mackey obstruction vanishes. But in general, any class in $H^2(G_\rho/N,\, \mathbb{T})$ can occur as a Mackey obstruction for some group extension.

The main result of the second part of the Mackey machine is then that the irreducible unitary representations of G_ρ whose restriction to N are multiples of ρ are in natural bijection with the $[\bar\omega]$-dual of the little group G_ρ/N, where $\bar\omega$ is the conjugate or inverse cocycle to ω. The idea is that if σ is an irreducible projective unitary representation of G_ρ/N satisfying $\sigma(\dot g\dot g') = \bar\omega(\dot g,\, \dot g')\sigma(\dot g)\sigma(\dot g')$ for $g,\, g' \in G_\rho$, then

$$\begin{aligned} U_{gg'} \otimes \sigma(\dot g\dot g') &= \omega(\dot g,\, \dot g')U_gU_{g'} \otimes \bar\omega(\dot g,\, \dot g')\sigma(\dot g)\sigma(\dot g') \\ &= U_gU_{g'} \otimes \sigma(\dot g)\sigma(\dot g') = \left(U_g \otimes \sigma(\dot g)\right)\left(U_{g'} \otimes \sigma(\dot g')\right), \end{aligned}$$

and thus $U \otimes \sigma$ is an ordinary unitary representation of G_ρ on $\mathcal{H}_\rho \otimes \mathcal{H}_\sigma$, whose restriction to N is equivalent to $\rho \otimes 1_{\mathcal{H}_\sigma}$. Since one can show the cocycle dual of any locally compact group is always non-empty for any 2-cocycle (by the analogue of the Gelfand-Raikov Theorem for projective representations), it follows that the fiber of $\hat G$ over $G \cdot \rho \hookrightarrow \hat N$ is non-empty.

This brings us to the end of the first fifteen years or so from the date of the papers [**GeN1**] and [**GeR**]. By this time, the subjects of C*-algebras and of unitary group representations were both flourishing, the latter somewhat more so than the former. From the work of Gelfand, Graev, and Naimark in Russia and of Harish-Chandra in the U. S. (which we have only casually touched on here), the main outlines of the unitary representation theory of semisimple Lie groups had begun to emerge, and Dixmier had embarked on a similar study of the unitary representation theory of nilpotent Lie groups. Numerous examples were known of groups with type II or type III unitary representations. And Mackey's theory of induced representations and of representations of group extensions was fully developed. On the C*-algebra side, it was known that any locally compact group has a group C*-algebra $C^*(G)$ whose representation theory is fully equivalent to the unitary representation theory of G, but few non-trivial examples were known in which the structure of $C^*(G)$ was understood.

§2. Some C^*-Algebraic Machinery

A more complete synthesis of the subjects of C*-algebras and of unitary group representations began to emerge in the 1960's, largely through the impetus provided by the work of Glimm [**Gl1**], identifying type I C*-algebras with Kaplansky's GCR algebras, and by two influential papers of Fell ([**F1**] and [**F2**]). One of Fell's important contributions was to demonstrate that there is a topological, as well as a measure-theoretic, side to unitary representation theory, and that $C^*(G)$ provides a very convenient language for understanding this aspect of representation theory. Let us state the key result of [**F1**] as it applies to group representations. Suppose G is a locally compact group and S is a subset of $\hat{G}$, the set of equivalence classes of irreducible unitary representations of G. We say that $\pi \in \hat{G}$ lies in the closure of S or is **weakly contained** in S if one (or equivalently, all) matrix coefficient of π can be approximated uniformly on compacta by matrix coefficients of the representations in S. On the other hand, we may identify π with an irreducible $*$-representation $\tilde{\pi}$ of $C^*(G)$, and S with a collection $\tilde{S}$ of such representations. Then $\tilde{\pi}$ and $\tilde{S}$ have kernels which are closed 2-sided ideals of $C^*(G)$ (with $\ker \tilde{\pi}$ a **primitive ideal**, that is, the kernel of an irreducible representation; it makes no difference whether one considers algebraically irreducible representations on complex vector spaces or topologically irreducible representations on Hilbert spaces, because of an important theorem of Kadison ([**K**], [**Di, 2.9.6–7**])). Fell's theorem states that π lies in the closure of S if and only if $\ker \tilde{S} \subseteq \ker \tilde{\pi}$. Thus approximation of matrix coefficients, which seems to be a property of harmonic analysis on G, turns out to be closely related to the ideal structure of $C^*(G)$, and in particular to the Jacobson topology (the analogue of the Zariski topology for non-commutative rings) on primitive ideals. In particular, $\hat{G}$ has a natural structure as a locally quasi-compact topological space, which is usually not Hausdorff. It is quasi-compact (that is, compact but not necessarily Hausdorff) if and only if G is discrete, and discrete if and only if G is compact (at least if the group is second-countable [**B**]). As a consequence of Glimm's Theorem [**Gl1**], it is T_0 (that is, no two distinct points have the same neighborhoods) if and only if it is type I, in which case it may be identified with $\operatorname{Prim} G$, the space of primitive ideals of $C^*(G)$. If G is not type I, $\operatorname{Prim} G$ is still a locally quasi-compact T_0 topological space, and tends to be a more useful object than $\hat{G}$.

The idea of viewing $\hat{G}$ or $\operatorname{Prim} G$ as a topological space makes it possible to look for a realization of $C^*(G)$ as an algebra of sections of a bundle (or continuous field) of algebras over this space or some Hausdorff space closely related to it. Fell began to carry out this program in [**F2**], where he both gave an explicit description of $C^*(SL(2, \mathbb{C}))$ in this way, and also introduced machinery that would be needed for tackling this problem in general. The paper [**F2**] contains the definition of **continuous-trace** C*-algebras, which are the basic building blocks out of which type I C*-algebras are composed (in a sense made precise in [**Di, 4.5.6–7**]). Any continuous-trace C*-algebra has a Hausdorff dual. Fell also showed that if a C*-algebra A is n-homogeneous for some finite n, in the sense that all its irreducible representations are n-dimensional, then A is a continuous-trace algebra and consists of the continuous sections vanishing at infinity of some locally trivial bundle over $\hat{A}$ with fibers $M_n(\mathbb{C})$ and structure group $\operatorname{Aut}(M_n(\mathbb{C})) = PU(n) = U(n)/\mathbb{T} \cong SU(n)/\mu_n$. (Here $\mathbb{T}$ is the group of scalar matrices with diagonal entries of absolute value 1, and μ_n is the subgroup of scalar matrices with diagonal entries that are n-th roots of unity.) Thus such algebras over a fixed locally compact Hausdorff space can be completely classified; up to isomorphisms fixing X pointwise, they are[5] classified by the Čech cohomology group $H^1\left(X, \underline{PU(n)}\right)$, where $\underline{PU(n)}$ denotes the sheaf of germs of continuous $PU(n)$-valued functions on X.

The program begun by Fell in **[F2]** was then carried much further by Dixmier and Douady in **[DiD]**. They extended Fell's classification of n-homogeneous C*-algebras to a classification of general continuous-trace algebras A over a paracompact locally compact Hausdorff space X, up to a suitable equivalence relation which we'll describe. By Fell's theory, the algebra A can be realized as the algebra of continuous sections vanishing at infinity of some continuous field $\mathcal{A}$ of elementary C*-algebras over X. (A C*-algebra is called **elementary** if it is isomorphic to $\mathcal{K}(\mathcal{H})$, the compact operators on some Hilbert space $\mathcal{H}$.) The simplest possibility for $\mathcal{A}$ is the continuous field of elementary C*-algebras attached to a continuous field of Hilbert spaces over X. Dixmier and Douady showed there is a unique obstruction $\delta(\mathcal{A}) \in H^3(X, \mathbb{Z})$ to being able to write $\mathcal{A}$ in this way, and that if in addition X is second-countable and finite-dimensional, then "$\aleph_0$-homogeneous" continuous-trace algebras over X are precisely classified by the invariant δ in $H^3(X, \mathbb{Z})$. If A is n-homogeneous and determined by the sheaf cohomology class $c \in H^1\left(X, \underline{PU(n)}\right)$, then $\delta(\mathcal{A})$ can be described as $\beta_n \circ \partial_n(c)$, where ∂_n is the connecting map in the sheaf cohomology exact sequence

$$H^1(X, \mu_n) \to H^1\left(X, \underline{SU(n)}\right) \to H^1\left(X, \underline{PU(n)}\right) \xrightarrow{\partial_n} H^2(X, \mu_n)$$

and β_n is the Bockstein homomorphism in the exact sequence

$$\begin{aligned} H^2(X, \mathbb{Z}) \xrightarrow{n} H^2(X, \mathbb{Z}) &\xrightarrow{\text{"reduction mod } n\text{"}} H^2(X, \mathbb{Z}/n) \\ &= H^2(X, \mu_n) \xrightarrow{\beta_n} H^3(X, \mathbb{Z}) \xrightarrow{n} H^3(X, \mathbb{Z}). \end{aligned}$$

In particular, $\delta(\mathcal{A})$ is an n-torsion class in this case.

It still remained to recast Mackey's theory of induced representations in C*-algebraic terms. The mechanism for doing this was worked out by Rieffel in an important series of papers **[Ri1–4]** in the early 1970's, following major simplifications in Mackey's Imprimitivity Theorem due to Blattner **[Bl]**. (A still more elegant proof of the Imprimitivity Theorem has been given in **[Ø]**.) Rieffel's theory made it possible to completely strip measure theory out of Mackey's theory, and to view unitary induction as a process for relating $C^*(H)$ to $C^*(G)$, when H is a closed subgroup of G. One important by-product of Rieffel's work was the definition of a new equivalence relation on C*-algebras, called **(strong) Morita equivalence** **[Ri3]**. Two C^*-algebras are called strongly Morita equivalent if, roughly speaking, their $*$-representation theories are identical. (The adjective "strong" is to avoid confusion with a weaker equivalence relation, also discussed in **[Ri3]**, that matches up representations but doesn't preserve weak containment. The weaker equivalence relation will not be of any interest to us,

[5] Assuming X is paracompact, which is automatic if X is locally compact Hausdorff and second-countable.

so hereafter we drop the adjective "strong.") Since the purpose of introducing $C^*(G)$ is to understand the unitary representation theory of G, which is equivalent to the $*$-representation theory of $C^*(G)$, the natural goal in studying unitary representations of a group G from Rieffel's pont of view is thus to classify $C^*(G)$ up to Morita equivalence.

To get a feeling for Morita equivalence of C*-algebras, it is good to keep a few examples in mind. The simplest is the Morita equivalence between $\mathcal{K}(\mathcal{H})$, the compact operators on any Hilbert space $\mathcal{H}$, and the complex numbers $\mathbb{C}$. While $\mathcal{K}(\mathcal{H})$ is highly non-commutative if $\mathcal{H}$ is infinite-dimensional, from the point of view of representation theory it is no different from $\mathbb{C}$, since up to equivalence it has one and only one irreducible representation [**Di, 4.1.5**]. A more general example comes from taking any C*-algebra A and a projection (self-adjoint idempotent) $p \in M(A)$. Then pAp is called a "corner" of A, and there is a Morita equivalence between this "corner" and the ideal ApA. The case of $\mathcal{K}(\mathcal{H})$ and $\mathbb{C}$ is the special case of this construction with $A = \mathcal{K}(\mathcal{H})$, p a projection in A of rank one. It was shown in [**BrGR**] that two separable C*-algebras A and B are Morita-equivalent if and only if they are **stably isomorphic**, that is, if and only if $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$, where $\mathcal{K} = \mathcal{K}(\mathcal{H})$ with $\dim \mathcal{H} = \aleph_0$. However, in many cases, Morita equivalence is more useful than stable isomorphism, in that often one has a canonical Morita equivalence between two algebras but no **canonical** stable isomorphism. Still one more example of Morita equivalence is useful to keep in mind: it was shown by P. Green in the mid-1970's (in unpublished lecture notes) that the class of continuous-trace algebras is closed under Morita equivalence, that a C*-algebra is Morita-equivalent to a commutative C*-algebra if and only if it is of continuous trace with vanishing Dixmier-Douady invariant, and that the Morita equivalence classes of continuous-trace algebras A over a paracompact locally compact Hausdorff space X are (if one requires all Morita equivalences to be "over X") in one-to-one correspondence with $H^3(X, \mathbb{Z})$ via the Dixmier-Douady invariant.

To describe Rieffel's version of the Imprimitivity Theorem in terms of Morita equivalence, we need one other C*-algebraic construction, that of the **crossed product**. This construction was already standard in pure algebra, and was introduced into the theory of C*-algebras it seems first by Turumaru [**Tu**], and then in various forms by Doplicher-Kastler-Robinson [**DKR**], Effros and Hahn [**EH**], Glimm [**Gl2**], Guichardet [**Gu**], Leptin [**L**], Takesaki [**Take**], and Zeller-Meier [**ZM**] (perhaps even others!). Suppose A is a C*-algebra, G is a locally compact group, and one has a (strongly continuous) homomorphism $\alpha : G \to \mathrm{Aut}(A)$, which one can think of as an action of G on A. (When $A = C_0(X)$ is commutative, α amounts to giving X the structure of a G-space.) Then one can define a crossed product algebra, written $A \rtimes G$ or $C^*(G, A)$ (or if one needs to keep track of α, $A \rtimes_\alpha G$ or $C^*(G, A, \alpha)$), whose (non-degenerate) $*$-representations are in natural one-to-one correspondence with **covariant pairs** (σ, π), consisting of a (non-degenerate) $*$-representation π of A together with a unitary representation

σ of G on the same Hilbert space, satisfying the "covariance condition"

$$\sigma(g)\pi(a)\sigma(g)^{-1} = \pi(\alpha(g)(a))$$

for all $g \in G$ and $a \in A$. When $A = \mathbb{C}$, α is necessarily trivial and the crossed product is just $C^*(G)$. When G acts by automorphisms of another locally compact group N and α is the induced action of G on $A = C^*(N)$, then there is a natural isomorphism between $A \rtimes G$ and $C^*(N \rtimes G)$, the group C*-algebra of the semidirect product of G and N. When $A = C_0(X)$ is abelian (the case studied in [**Gl2**] and in [**EH**]), the crossed product is sometimes called the **transformation group algebra** $C^*(G, X)$.

Rieffel noticed that a system of imprimitivity (in Mackey's sense) for a unitary representation σ of G amounts to writing σ as half of a covariant pair (σ, π) with π a representation of $C_0(G/H)$ for some closed subgroup H of G. (The action of G on $C_0(G/H)$ is the obvious one induced by the left action of G on G/H.) By the Imprimitivity Theorem, any representation with such a system of imprimitivity is an induced representation from H. Thus the Imprimitivity Theorem can be rephrased as a construction of a Morita equivalence between $C^*(H)$ and the crossed product $C_0(G/H) \rtimes G$, which Rieffel calls the "imprimitivity algebra." Unitary induction is then the composite of two operations:

$$\begin{aligned}
&\text{unitary representations of } H \approx \text{representations of } C^*(H)\\
&\qquad\xrightarrow{\text{Morita equivalence}} \text{representations of } C_0(G/H) \rtimes G\\
&\qquad\qquad\xrightarrow{\text{restriction to } G} \text{unitary representations of } G.
\end{aligned}$$

A still more general C*-algebraic version of Mackey's theory based upon Rieffel's ideas was given in [**Gr2**], so to avoid repeating ourselves, we skip ahead in the chronology and describe Green's "ultimate version" of the Mackey machine. A more complete survey of this work may be found in [**Gr4**], and a different but related approach to an "abstract Mackey machine" may be found in [**FD**], especially the last two chapters. First we need Green's notion of a **twisted crossed product** or **twisted covariance algebra**. Roughly speaking, this bears the same relationship to a general group extension that an ordinary (or untwisted) crossed product bears to a split extension or semidirect product. Namely, suppose one has an action α of G on A as above, and also a closed normal subgroup N_τ of G. A **twisting map** in Green's sense is a (strongly continuous) homomorphism τ from N_τ to the unitary group of $M(A)$, such that

$$\tau(n)a\tau(n)^{-1} = \alpha(n)(a), \qquad \tau(gng^{-1}) = \alpha(g)\left(\tau(n)\right),$$

for all $n \in N_\tau$, $a \in A$, and $g \in G$. The twisted crossed product $C^*(G, A, \tau)$ is then the quotient of $C^*(G, A)$ who representations correspond to covariant pairs (σ, π) with σ a unitary representation of G, π a representation of A, and satisfying the "twisting condition"

$$\sigma(n)\pi(a) = \pi(\tau(n)a)$$

for all $n \in N_\tau$, $a \in A$. Green showed that if $A = C^*(N_\tau)$ and $\tau : N_\tau \to M(C^*(N_\tau))$ is the usual embedding, then $C^*(G, A, \tau)$ is naturally isomorphic to $C^*(G)$. Similarly he pointed out that one may construct a universal C*-algebra for projective representations of a group G with Mackey obstruction $[\omega] \in H^2(G, \mathbb{T})$ as such a twisted crossed product $C^*(\tilde{G}, \mathbb{C}, \tau)$, where $\tilde{G}$ is a central extension of G by $N_\tau = \mathbb{T}$ corresponding to the given cohomology class, and $\tau : \mathbb{T} \to \mathbb{T} = U(1)$ is the identity map.

One should think of $C^*(G, A, \tau)$ as $A \rtimes_\tau (G/N_\tau)$, that is, as a crossed product of A by G/N_τ "with a twist," and in fact a recent theorem of Packer and Raeburn [**PR1**], improved by Echterhoff [**Ec**], shows that $C^*(G, A, \tau)$ is always Morita-equivalent to an ordinary crossed product $A' \rtimes (G/N_\tau)$, with A' Morita-equivalent to A. This device makes it possible to reduce almost all questions about twisted crossed products to the case of ordinary crossed products.

Now we can describe the Rieffel-Green version of the Mackey machine. Following Green, we state the main results for twisted crossed products $C^*(G, A, \tau)$; the reader should keep in mind that the main case of interest is computing $C^*(G)$ when $A = C^*(N)$ and the representation theory of $N = N_\tau$ is known (the situation of the original Mackey machine). However, phrasing things in greater generality has a noticeable advantage: we can study $C^*(G)$ "locally" by replacing $A = C^*(N)$ by one of its G-invariant subquotients; this in effect restricts attention to representations of G "lying over" the G-invariant locally closed subset Prim(A) of Prim(N). We do not need to assume to begin with either that A is type I or that some "regular embedding" condition is satisfied, though as we shall see, the results are a little more precise if these hypotheses are satisfied.

First we review the main tools at our disposal: as before they are "restriction" and "induction" of representations. Restriction is easy: given a closed subgroup H of G with $N_\tau \subseteq H$, and given a covariant pair (σ, π) satisfying the "twisting condition" as above, it is clear that $(\sigma|_H, \pi)$ is a covariant pair of representations of (H, A) also satisfying the "twisting condition," and so we get a representation of $C^*(H, A, \tau)$. Conversely, one can induce representations of $C^*(H, A, \tau)$ by using a Morita equivalence of this algebra with the "imprimitivity algebra" $C^*(G, C_0(G/H) \otimes A, 1 \otimes \tau)$, followed by a kind of "restriction" (forgetting the $C_0(G/H)$ factor). In understanding how induction works in concrete terms, it is sometimes useful to know about a later paper of Green [**Gr3**] that showed that a choice of a Baire cross-section for G/H in G gives an isomorphism of the imprimitivity algebra with $C^*(H, A, \tau) \otimes \mathcal{K}(L^2(G/H))$.

To start the Mackey machine, suppose we are given an irreducible representation of $C^*(G, A, \tau)$. Then we can restrict it to $C^*(N_\tau, A, \tau) = A$, but since we are not assuming type I-ness and are trying to avoid measure theory, we need a substitute for decomposing this restricted representation as a direct integral. The substitute is merely to take the kernel of this representation, which gives us a closed G-invariant subset of Prim(A), which only depends of the kernel P of the irreducible representation we started with. Under very mild hypotheses (which are automatic if A is separable), this G-invariant subset must be the closure of a single G-orbit. Hence we replace Mackey's measure-theoretic quasi-orbits with **topological quasi-orbits**, by defining an equivalence relation $\sim$ on $\mathrm{Prim}(A)$ by

$$x \sim y \Longleftrightarrow \overline{G \cdot x} = \overline{G \cdot y}.$$

The quotient $\mathrm{Prim}(A)/\sim$ (with the quotient topology) is a T_0-space, called the **quasi-orbit space** for the action of G on $\mathrm{Prim}(A)$ (this was introduced by Effros and Hahn [**EH**]). Thus P restricted to A "lives" on a single quasi-orbit, and $\mathrm{Prim}\, C^*(G, A, \tau)$ "fibers" over the space of these quasi-orbits. If each orbit of G on $\mathrm{Prim}(A)$ is locally closed (open in its closure), then the quasi-orbit space is the same as the orbit space and we say A is regularly embedded. In this case, the Mackey machine works as before: $\mathrm{Prim}\, C^*(G, A, \tau)$ "fibers" over $\mathrm{Prim}(A)/G$ and the fiber over any orbit $G \cdot I$ consists of kernels of representations induced from the stabilizer $H = G_I$ of I. In fact, induction gives a Morita equivalence from $C^*(H, B, \tau)$ to a subquotient of $C^*(G, A, \tau)$, where B is the subquotient of A corresponding to the locally closed point I in $\mathrm{Prim}(A)$. When $B = \mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, which will happen if for instance A is type I, then the second step of the Mackey machine goes through as well, and $C^*(H, B, \tau)$ is Morita-equivalent to a crossed product $\mathcal{K}(\mathcal{H}) \rtimes H/N_\tau$, or alternatively to a twisted group C*-algebra $C^*(H/N_\tau, [\omega])$.

Even without regular embedding, when A is separable, G is second-countable, and G/N_τ is amenable, for example solvable, a (somewhat expanded version of a) conjecture of Effros and Hahn, proved in [**GoR**], identifies those $P \in \mathrm{Prim}\, C^*(G, A, \tau)$ that "live" over a single quasi-orbit as being the kernels of representations induced from $C^*(H, A, \tau)$, where H is the stabilizer in G of some point in the quasi-orbit. (In general, this fails if G/N_τ is not amenable.) In particular, if G/N_τ is amenable and acts freely on $\mathrm{Prim}(A)$, and if every orbit is dense, then $C^*(G, A, \tau)$ is a simple C*-algebra. The Effros-Hahn Conjecture thus provides a quite reasonable substitute for the first step of the Mackey machine even when A is neither type I not regularly embedded. One word of caution: since a quasi-orbit is usually bigger than an orbit, the stabilizers of different primitive ideals in the same quasi-orbit are not necessarily conjugate to one another.

When A is type I and regularly embedded, the Mackey machine locally reduces the structure of $C^*(G, A, \tau)$ to that of certain twisted group C*-algebras $C^*(H/N_\tau, [\omega])$. The structure of these thus becomes a problem of fundamental importance. The one case that is fairly well understood is that of a twisted group C*-algebra $C^*(H, [\omega])$ with H abelian. The basic results about this case are due to Baggett and Kleppner [**BaK**], with later refinements by Kleppner (unpublished) and Green [**Gr2**, §7]. Suppose ω is a measurable 2-cocycle on the locally compact abelian group H, which we can assume is normalized to satisfy $\omega(x, x^{-1}) = 1$ for all $x \in H$. Then we can define a homomorphism $h_\omega : H \to \hat{H}$ ($\hat{H}$ is the Pontyagin dual of H) by

$$h_\omega(x) : y \mapsto \omega(x, y)\omega(y, x)^{-1},$$

and h_ω is continuous and only depends on the cohomology class $[\omega]$ of ω. Let $S_\omega = \ker h_\omega$. The main results of Baggett and Kleppner are that any ω-representation of H becomes an ordinary representation when restricted to S_ω, and that in this way $\operatorname{Prim} C^*(H, [\omega])$ can be identified with $\widehat{S_\omega}$. Furthermore, $C^*(H, [\omega])$ is type I if and only if h_ω has closed range and is a homeomorphism onto its image. (Because of the open mapping theorem, this second condition is automatic if H is second-countable.) The work of Kleppner and of Green adds a few other details in the non-type I case: the simple quotients of $C^*(H, [\omega])$ are all isomorphic to one another, and have a unique trace (up to scalar multiples), so that the quasi-equivalence classes of traceable factor representations of $C^*(H, [\omega])$ are in natural bijection with $\operatorname{Prim} C^*(H, [\omega])$ and with $\widehat{S_\omega}$.

In working with crossed products, there is one technique which is often quite useful which we haven't mentioned yet, that of duality. This was first introduced in the C*-algebra context by Takai [**Taka**], after the same idea had been brilliantly applied to von Neumann algebra crossed products by Takesaki. For simplicity we deal only with the case of an ordinary crossed product $A \rtimes G$ with G abelian; the same idea was applied by Green to twisted crossed products $C^*(G, A, \tau)$ with G/N_τ abelian, although nowadays we could almost reduce this case to the former one using the result of [**Ec**]. So let G be an abelian locally compact group acting on A via $\alpha : G \to \operatorname{Aut}(A)$. The crossed product $A \rtimes_\alpha G$ is spanned by products $a \cdot b$ with $a \in A$, $b \in C^*(G) \cong C_0(\hat{G})$, where we think of both A and $C^*(G)$ as sitting in $M(A \rtimes G)$. (However such products $a \cdot b$ actually live in the crossed product itself.) Then there is a unique **dual action** $\hat{\alpha} : \hat{G} \to \operatorname{Aut}(A \rtimes G)$ satisfying $\hat{\alpha}(\gamma)(a \cdot b) = (a \cdot (\lambda_\gamma b))$ for $\gamma \in \hat{G}$, where λ is the translation action of $\hat{G}$ on $C^*(G) \cong C_0(\hat{G})$. The Takai duality theorem then states that $(A \rtimes_\alpha G) \rtimes_{\hat{\alpha}} \hat{G}$ is Morita-equivalent to A, and that $\hat{\hat{\alpha}}$ is in a suitable sense Morita-equivalent to the original action α. Takai's theorem may be viewed as a generalization of Mackey's version [**M2**] of the Stone-von Neumann Theorem, which amounts to the case where $A = \mathbb{C}$.

With the Dixmier–Douady theory and the C*-algebraic Mackey machine in place, it became feasible by the mid-1970's to say something substantial about $C^*(G)$ for quite a number of locally compact groups G. We will not attempt to be exhaustive here, but we give a few representative examples of interesting results. As far as semisimple Lie groups were concerned, the structure of $C^*(G)$ was worked out for a number of groups of real-rank one [**BoM1, BoM2**], following the example of $SL(2, \mathbb{C})$ worked out previously by Fell [**F2**]. In addition, even though $C^*(G)$ never has Hausdorff dual when G is a non-compact semisimple Lie group, Miličič showed that the continuous-trace property does not fail too badly for $C^*(G)$ [**Mi**]. Pukanszky [**Pu2**] discovered some deep facts about the C*-algebras of general connected Lie groups. Using a version of the non-type I Mackey machine, he gave an algorithm for computing $\operatorname{Prim} G$ for any connected Lie group G, even if it is not of type I, from which one can deduce that every point in $\operatorname{Prim} G$ is locally closed. He also managed to show that for each primitive ideal of $C^*(G)$, there is one and only one quasi-equivalence class of traceable factor representations of $C^*(G)$ having that ideal as kernel. This quasi-equivalence class contains an irreducible representation if and only if the primitive subquotient of $C^*(G)$ attached to the primitive ideal is $\mathcal{K}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$, in which case the primitive ideal is "type I," in the sense that all representations with that kernel are type I. Alternative proofs of Pukanszky's results were given by Green [**Gr2**], using his Mackey machine for twisted crossed products. Similar results were obtained by Howe [**Ho1**] for certain classes of nilpotent locally compact groups. (A historical note: Howe's theorem actually came first, and motivated Pukanszky to try to prove something similar.) It was already known from earlier work of Guichardet [**Gu**] that such results failed for many solvable groups. However, these results basically only looked at the structure of $C^*(G)$ "one primitive ideal at a time." The global structure of $C^*(G)$ was usually far too complicated to be described completely, except for some very simple low-dimensional examples such as those considered in [**Ro1**], and even the topology on $\operatorname{Prim} G$ was only known in a limited number of cases. The big success in this direction was for connected (and let's say for simplicity, simply connected) nilpotent Lie groups G, where the orbit method of Kirillov (see [**Ki**]) gives a bijection between $\mathfrak{g}^*/G$ and $\hat{G}$, where $\mathfrak{g}^*$ is the dual of the Lie algebra of G, on which G acts by the dual of the adjoint action. If the orbit space $\mathfrak{g}^*/G$ is given the quotient topology, it is fairly easy to see that the bijection between $\hat{G}$ and $\mathfrak{g}^*/G$ is a Borel isomorphism. To show that the bijection is a homeomorphism is much harder and was only accomplished by I. Brown [**Br**]. A much simpler argument was later given by K. Joy [**Joy**]. For exponential solvable Lie groups, Kirillov theory as extended by Bernat and others again gives a bijection from $\mathfrak{g}^*/G$ to $\hat{G}$, and Pukanszky [**Pu1**] had shown that this map is continuous, but the problem of showing that its inverse is continuous remained open.

§3. Group C^*-Algebras and "Fine Structure" of the Mackey Machine: Current Directions and Open Problems

Within the last 15 years or so, the study of Mackey theory from the point of view of C*-algebras has exploded into a large industry, thanks to the development of much of the machinery described in §2 above. Since it is always dangerous to write of an ongoing enterprise in historical terms, I will not attempt to give a chronology or even to survey everything that is going on. Instead I will focus on three (closely related) major research problems, discuss some of the partial results about them, and indicate what sorts of issues need to be studied in the future.

RESEARCH PROBLEM 1. *Describe as completely as possible the structure of a crossed product $A \rtimes G$ or twisted crossed product $C^*(G, A, \tau)$, at least when A has continuous trace and G has a single "orbit type" on $\hat{A}$.*

MOTIVATION. The classical Mackey machine is usually quite efficient in computing $\operatorname{Prim} G$ *as a set*, when G is a locally compact group containing a regularly embedded type I normal subgroup N. However, as we have seen, a complete description of unitary representation theory of G (including its "topological" aspects) requires a complete description of $C^*(G) = C^*(G, C^*(N), \tau)$ as a C*-algebra. In most cases, this is likely to be too complicated to be understandable. However, as a first step toward understanding $C^*(G)$ as a C*-algebra, one can try to understand it *locally* or *semi-locally*, that is, to describe the subquotient of $C^*(G)$ sitting over some locally closed Hausdorff G-invariant subset of $\hat{N}$ on which the action of G is not too wild. Such a subset of $\hat{N}$ corresponds to an algebra A with Hausdorff spectrum, of which the simplest example is a continuous-trace algebra. Similarly, since "jumps" in the stability groups of points in $\hat{A}$ tend to lead to crossed products with quite complicated non-Hausdorff primitive ideal spaces, it is natural to restrict attention first to the case where there is a single "orbit type" and the stability groups vary continuously.

RECENT AND CURRENT RESULTS. As far as I know, the first significant result on the global structure of a crossed product $A \rtimes G$ when G has only a single orbit type on $\hat{A}$ was due to Phil Green [**Gr1, Theorem 14**], for the case where $A = C_0(X)$ is abelian and G acts freely and properly on $X = \hat{A}$. Green's result was as nice as could be hoped for: in this case, $C_0(X) \rtimes G$ is Morita-equivalent to $C_0(X/G)$, and even isomorphic to $C_0(X/G) \otimes \mathcal{K}(L^2(G))$ if X satisfies some mild conditions and G is not finite. When G is finite and acts freely on X, the quotient map $X \to X/G$ is a covering map, and Green's reasoning shows that $C_0(X) \rtimes G$ is isomorphic to the algebra of continuous sections vanishing at infinity of the bundle of full matrix algebras associated to the vector bundle $X \times_G \mathbb{C}G$ over X/G. However, Green did not indicate whether or not this bundle can be non-trivial. Assuming X (and hence X/G) is paracompact, we can describe things this way: the covering map $X \to X/G$ is classified by an element of $H^1(X/G, G)$ (non-abelian sheaf cohomology). Then the class of the vector bundle $X \times_G \mathbb{C}G$ over X/G is described by the image of this class in $H^1\left(X/G, \underline{U(\mathbb{C}G)}\right)$ under the map of sheaves $G \to \underline{U(\mathbb{C}G)}$ induced by the left regular representation of G, $\lambda_G : G \to U(\mathbb{C}G)$. Finally, the class of the bundle of matrix algebras associated to $C_0(X) \rtimes G$ is given by the image of this class in $H^1\left(X/G, \underline{PU(\mathbb{C}G)}\right)$. From the sheaf cohomology exact sequence (with $n = |G|$)

$$H^2(X/G, \mathbb{Z}) \cong H^1(X/G, \underline{\mathbb{T}}) \to H^1\left(X/G, \underline{U(n)}\right) \to H^1\left(X/G, \underline{PU(n)}\right),$$

we see that this class will be trivial only if it has a lifting to

$$H^2(X/G, \mathbb{Z}) \cong H^1(X/G, \underline{\mathbb{T}}),$$

in other words, only if the vector bundle $X \times_G \mathbb{C}G$ is a direct sum of n identical line bundles, which will rarely be the case. (It's not even true if $X = S^2$ and $G = \mathbb{Z}/2$ acting by the antipodal map, since the induced 2-plane bundle on $\mathbb{RP}^2$

is the direct sum of a trivial and a non-trivial line bundle.) However, at least we know that the Dixmier-Douady invariant, which is the image of our sheaf cohomology class in $H^3(X/G, \mathbb{Z})$, vanishes.

Not long after Green's work was published, Phillips and Raeburn [**PhR1**] showed that even when $\mathbb{Z}$ acts on a continuous-trace algebra A and acts trivially on $\hat{A}$, the classical Mackey machine is insufficient to describe the dual topology of $A \rtimes \mathbb{Z}$. Indeed, since $\mathbb{Z}$ is the stabilizer of each point in $\hat{A}$, each irreducible representation of A "extends" to a representation of $\mathbb{Z}$, and since $H^2(\mathbb{Z}, \mathbb{T}) = 0$, the Mackey obstructions vanish. Thus the Mackey machine says $(A \rtimes \mathbb{Z})\hat{}$ fibers over $\hat{A}$, with each fiber a copy of $\hat{\mathbb{Z}} = \mathbb{T}$. One might guess from this that $(A \rtimes \mathbb{Z})\hat{} \cong \hat{A} \times \mathbb{T}$, but this is not necessarily the case. In fact, Phillips and Raeburn [**PhR1**] showed that every principal $\mathbb{T}$-bundle over $\hat{A}$ can arise as the dual topology for some $A \rtimes \mathbb{Z}$ with $\mathbb{Z}$ acting trivially on $\hat{A}$, even **locally unitarily**. (An action of a group G on a C*-algebra is called **unitary** if it is implemented by a homomorphism $G \to U(M(A))$. Such actions yield "trivial" crossed products. More interesting are actions which are **pointwise** or **locally** unitary, that is, which restrict to unitary actions on all the simple quotients of A or on a family of ideals of A corresponding to an open covering of $\hat{A}$, respectively.)

Soon after this, in an unpublished manuscript dating from 1981, A. Wassermann generalized Green's work on free transformation groups to the case of twisted crossed products $C^*(G, X, \omega)$, where G is a compact group acting freely on the locally compact space X and $\omega \in H^2(G, \mathbb{T})$. In Green's notation, such a crossed product would be written $C^*(\tilde{G}, C_0(X), \tau)$, where $\tilde{G}$ is the extension of G by $N_\tau = \mathbb{T}$ determined by ω and τ is the identity map $\mathbb{T} \to \mathbb{T}$. This time, still another phenomenon appears: the twist τ influences the topological nature of the crossed product. More precisely, $C^*(G, X, \omega)$ is a continuous-trace algebra over the quotient space X/G, with Dixmier-Douady class given by the image of ω under the composite

$$H^2(G, \mathbb{T}) \xrightarrow{\cong} H^3(G, \mathbb{Z}) \xrightarrow{\cong} H^3(BG, \mathbb{Z}) \xrightarrow{c^*} H^3(X/G, \mathbb{Z}),$$

where BG is the classifying space for G and $c : X/G \to BG$, which is well-defined up to homotopy, is the classifying map for the principal G-bundle $X \to X/G$.

Meanwhile, Phillips and Raeburn [**PhR2**] generalized their work on single automorphisms of continuous-trace algebras (actions of $\mathbb{Z}$) to the case of locally unitary actions of arbitrary abelian locally compact groups. The result is basically that any such action $G \to \mathrm{Aut}(A)$, with A a continuous-trace algebra over X (separable, say), defines an obstruction in $H^1\left(X, \underline{\hat{G}}\right)$, such that $(A \rtimes G)\hat{}$ is the total space of the corresponding principal $\hat{G}$-bundle over X. In fact, the $\hat{G}$-action on $(A \rtimes G)\hat{}$ comes from the dual action on the crossed product.

Putting all these results together showed by the mid-1980's that understanding our Research Problem, even in the simplest cases where the crossed product is a continuous-trace algebra, involves a considerable amount of bundle theory and topology not apparent in the classical formulation of the Mackey machine.

It is this combination of techniques which I have called the "fine structure of the Mackey machine." In the last few years, quite a number of people have been working in this area, notably Echterhoff, Gootman, Lazar, Olesen, Packer, Raeburn, Williams, and the author. With apologies to those whose work I am skipping over, I will present just a few representative results.

By way of background, it is useful to know how to classify actions of a second-countable locally compact group G on a separable continuous-trace algebra A with spectrum X. The natural equivalence relation here is **exterior equivalence**, which means that two actions differ by a 1-cocycle on G with values in $U(M(A))$, the unitary group of the multiplier algebra. If actions α and β are exterior equivalent, then the corresponding crossed products $A \rtimes_\alpha G$ and $A \rtimes_\beta G$ are isomorphic, in fact in a $\hat{G}$-equivariant way if G is abelian [**RaR, §0**]. Obviously, any action of G on A determines an action of G on X by homeomorphisms. Under a very mild condition on X (automatic if X is compact), two actions α and β of G on A inducing the same action of G on X are exterior equivalent if and only if a certain obstruction in $H^2(G, C(X, \mathbb{T}))$ vanishes [**RaR, Corollary 0.13**]. (Here the action of G on $C(X, \mathbb{T})$ is induced by the action on X.) When $A = C_0(X) \otimes \mathcal{K}$, one even can show that $H^2(G, C(X, \mathbb{T}))$ parameterizes the exterior equivalence classes of actions [**HORR, Proposition 3.1**]. Up to Morita equivalence, the crossed product for the action given by $[\omega] \in H^2(G, C(X, \mathbb{T}))$ is what Packer and Raeburn [**PR1, PR2**] call $C_0(X) \times_{\tau, \omega} G$. In [**PR2, Theorem 3.9**], they present a generalization of Wassermann's theorem: if G acts freely on X with Hausdorff quotient X/G, and if $X \to X/G$ is a locally trivial principal G-bundle (these conditions are automatic if G is compact), then $C_0(X) \times_{\tau, \omega} G$ is a continuous-trace algebra with spectrum X/G and with Dixmier-Douady class given by the image of $[\omega]$ under a certain homomorphism

$$\delta : H^2(G, C(X, \mathbb{T})) \cong H^2\left(G, H^0(X, \underline{\mathbb{T}})\right) \to H^2(X/G, \underline{\mathbb{T}}) \cong H^3(X/G, \mathbb{Z}).$$

A special case of this formula appeared in different form in [**RaR, Theorem 1.5**].

When G has non-trivial isotropy groups for its action on X, the study of $A \rtimes G$ is much more complicated, because of the need to consider the projective representations of the isotropy groups associated to the Mackey obstructions. Even when a Mackey obstruction is trivial, there may be no canonical way to trivialize it, and this leads to difficulties. The only good results available are for certain special situations with G abelian. (Commutativity of G is used both in order to apply Takai duality and in order to apply the Baggett-Kleppner results on projective representations.) Some cases where all the Mackey obstructions are trivial are treated in [**RaR**] and [**RaW**], and cases with non-trivial Mackey obstructions are considered in [**EcR**].

Perhaps the most interesting case studied in [**RaR**] is one which one might have guessed would be trivial, namely that where $G = \mathbb{R}$. (This group is unusual in that $H^2(H, \mathbb{T}) = 0$ for all closed subgroups H of $\mathbb{R}$, so at least one doesn't have to worry about Mackey obstructions.) By [**RaR, Theorem 4.1**], $H^2(\mathbb{R}, C(X, \mathbb{T})) = 0$ under very mild conditions on X (automatic if X has the homotopy type of a compact metric space), so an action of $\mathbb{R}$ on a continuous-trace algebra A with such a spectrum X is determined by its action on X up to exterior equivalence. Furthermore, again under mild conditions on X, given any stable separable continuous-trace algebra A with spectrum X, any action of $\mathbb{R}$ on X can be realized by an (essentially unique) action on A. However, the structure of the crossed product can be complicated. For example, if every point in X has isotropy group $\mathbb{Z}$, so that the action of $\mathbb{R}$ on X comes from a principal $\mathbb{T} = \mathbb{R}/\mathbb{Z}$-bundle over the quotient space $T = X/\mathbb{R}$, then $A \rtimes \mathbb{R}$ is a continuous-trace algebra whose spectrum is **another** principal $\mathbb{T}$-bundle over T, this time with the action of $\mathbb{T}$ coming from the dual action to the action of $\mathbb{Z}$. The class of the bundle $(A \rtimes G)\hat{} \to T$ is the class in $H^2(T, \mathbb{Z})$ which is the image of the Dixmier-Douady class of A under the Gysin map $H^3(X, \mathbb{Z}) \to H^2(T, \mathbb{Z})$ ("integration over the fibers"). Similarly, since everything is symmetric under Takai duality, the image of the Dixmier-Douady class of $A \rtimes G$ under the Gysin map $H^3((A \rtimes G)\hat{}, \mathbb{Z}) \to H^2(T, \mathbb{Z})$ is the characteristic class of the bundle $X \to T$ [**RaR, Theorem 4.12**]. It is possible for all the bundles and Dixmier-Douady classes to be simultaneously non-zero. Various generalizations of this result (which, however, are somewhat less explicit) may be found in [**RaW**] for the case of actions of abelian groups G on A for which the stabilizers G_x of points $x \in X$ are not necessarily constant but at least "vary continuously." For example, by [**RaW, Proposition 6.8**], if $\mathbb{R}$ acts on X in a "locally trivial" way with Hausdorff quotient $X/\mathbb{R}$ and continuous stabilizers, then $C_0(X) \rtimes \mathbb{R}$ is a continuous-trace algebra whose spectrum is $((X/\mathbb{R}) \times \mathbb{R})/\sim$, where the equivalence relation $\sim$ is defined by

$$(\mathbb{R}\cdot x,\, s) \sim (\mathbb{R}\cdot y,\, t) \iff \mathbb{R}\cdot x = \mathbb{R}\cdot y \text{ and } s - t \in \mathbb{R}_x.$$

Furthermore, the Dixmier-Douady class of $C_0(X) \rtimes \mathbb{R}$ is trivial if and only if the quotient map $X \to X/\mathbb{R}$ has a continuous global cross-section.

To conclude this review of recent results on the fine structure of the Mackey machine, we briefly summarize some of the results from [**EcR**] on the case of actions of abelian groups G with non-zero Mackey obstructions. The results are mostly limited to the case where G acts on a continuous-trace algebra A over X with constant stability groups $G_x = N$ for the action on X, with the quotient map $q : X \to X/G$ a principal G/N-bundle, and with constant Mackey obstruction class $[\omega] \in H^2(N, \mathbb{T})$. Let $S = S_\omega \subseteq N$ in the sense of Baggett and Kleppner, and suppose (this is often automatic) that the action of S on A is locally unitary. Then by [**HORR, Theorem 1**], the natural map $(A \rtimes N)\hat{} \to (A \rtimes S)\hat{}$ is a homeomorphism, and by [**EcR, Theorem 5**], $\mathrm{Prim}(A \rtimes G)$ is a locally trivial $\hat{S}$-bundle over X/G (via the dual action of $\hat{S}$), and the pull-back of this bundle via q is the bundle $(A \rtimes S)\hat{} \to \hat{A} = X$ defined by Phillips and Raeburn. (Recall we are assuming the action of S is locally unitary.) By [**EcR,**

Theorem 5], $A \rtimes G$ will be a continuous-trace algebra if and only if ω is type I and the action of G/N on $(A \rtimes N)^\wedge \cong (A \rtimes S)^\wedge$ is proper. Finally, when this is the case, one can often give a formula for the Dixmier-Douady class of $A \rtimes G$. One surprising consequence is that even when $G = N$ (so that G acts trivially on X), and even though natural map $(A \rtimes G)^\wedge \to (A \rtimes S)^\wedge$ is a homeomorphism, the algebras $A \rtimes G$ and $A \rtimes S$ are often **not** Morita-equivalent. Thus in some sense the Mackey obstructions introduce a "global twisting" of the crossed product.

OPEN CASES. Clearly we are still quite far from a complete solution of Research Problem 1. However, when G is abelian, the action of G on $X = \hat{A}$ is proper, and the Mackey obstructions $[\omega] \in H^2(N, \mathbb{T})$ are constant and type I, we largely understand the situation. To complete the analysis, we need to deal with a number of more complicated cases. One of these is what happens when ω is not type I, so that $C^*(N, \omega)$ is a non-type C*-algebra with primitive ideal space $\hat{S}$. In this case one would expect the crossed product to be an algebra of sections of a locally trivial bundle over Prim$(A \rtimes G)$ whose fibers are Morita-equivalent to $C^*(N, \omega)$, but the invariants that classify such bundles up to Morita equivalence have not been fully worked out. More importantly, we need additional machinery for dealing with the case where G is not abelian. Several difficulties now arise: there is usually no simple description of $C^*(G_x, \omega)$, and Takai duality needs to be replaced by a more complicated duality theory for coactions. The beginnings of the application of such a duality theory to the case where G is compact but not abelian can be found in [**GoL**]. However, much more needs to be done in the general case.

RESEARCH PROBLEM 2. *If G is a locally compact group, the "operator-valued Fourier transform" of a function on G is an element of $C^*(G)$. Can this be "twisted"? In particular, if A is a subquotient of $C^*(G)$ which is a "homogeneous" algebra over a Hausdorff space X, can the associated bundle be non-trivial? stably non-trivial?*

MOTIVATION. Although computing $C^*(G)$ "locally" as in Research Problem 1 has its technical advantages, we do not not want to totally disregard the "global" aspects of the structure of $C^*(G)$. For instance, in doing harmonic analysis on a locally compact group G, the map that sends $f \in L^1(G)$ to its image in $C^*(G)$, viewed as an algebra of sections of some field of generically simple C*-algebras, should be viewed as the "operator-valued Fourier transform" of the function. By its very nature, this is a global object. Fell observed in his study of $C^*(SL(2, \mathbb{C}))$ in [**F2**] that the fields of C*-algebras that arise in the case of the Lorentz group are of a fairly trivial sort. However, one would like to know whether this must always be the case. If $C^*(G)$ has a subquotient A which is the algebra of continuous sections of some non-trivial bundle whose fibers are elementary C*-algebras (in the type I case) or simple C*-algebras (in the non-type I case), then this means that a proper understanding of harmonic analysis on G requires an understanding of the topology of this bundle.

Of course, if the subquotient A is at least Morita-equivalent to an algebra of sections of a *trivial* bundle whose fibers are simple C*-algebras, then this means that the topology of the bundle disappears from the point of view of Morita equivalence. By Dixmier-Douady theory [**DiD**], a separable continuous-trace algebra is Morita-equivalent to an abelian C*-algebra if and only if its Dixmier-Douady invariant vanishes. And a separable continuous-trace algebra with finite-dimensional spectrum X which is homogeneous, that is, whose irreducible representations are all of the same dimension, is always the algebra of continuous sections of a locally trivial bundle of elementary C*-algebras. The Dixmier-Douady invariant is the obstruction to "stable" triviality of this bundle (triviality after everything is tensored with $\mathcal{K}$).

RECENT AND CURRENT RESULTS. From the discussion above of the "fine structure of the Mackey machine," it is easy to give examples of group C*-algebras of (disconnected) groups that involve non-trivial bundles. For example, suppose $G = \mathbb{R}^n \rtimes K$, where K is a finite group acting orthogonally on $\mathbb{R}^n$. Since $\mathbb{R}^n \setminus \{0\} \cong S^{n-1} \times (0, \infty)$ is a K-invariant open set with K-fixed complement, it is easy to see that $C^*(G)$ has the structure of an extension

$$0 \to C_0((0, \infty)) \otimes (C(S^{n-1}) \rtimes K) \to C^*(G) \to C^*(K) \to 0.$$

By Green's theorem, if K acts freely on the sphere S^{n-1}, then $C(S^{n-1}) \rtimes K$ is usually the algebra of sections of a non-trivial bundle over S^{n-1}/K, though at least the algebra is Morita-equivalent to an abelian C*-algebra. But if K doesn't act freely, $C(S^{n-1}) \rtimes K$ will often be a continuous-trace algebra with non-zero Dixmier-Douady invariant. An example with K the quaternion group of order 8 is given in [**RaR, Example 4.9**].

If G is a type I central extension of $\mathbb{Z}^n$ by $\mathbb{T}$, $C^*(G)$ is a direct sum of algebras $C^*(\mathbb{Z}^n, \omega)$ for type I cocycles ω, often called rational non-commutative tori. The type I condition forces S_ω to be a subgroup of finite index in $\mathbb{Z}^n$, and $C^*(\mathbb{Z}^n, \omega)$ is the algebra of continuous sections of a bundle of matrix algebras over T^n. The bundles that arise are often non-trivial, but the Dixmier-Douady invariant always vanishes (see [**Ri5**, §3] and [**De**] for the case $n = 2$, [**EcR**] for a proof of vanishing of the Dixmier-Douady invariant in the general case).

A more peculiar example of serious "twisting," mentioned in [**RaR, Example 4.20**], comes from taking G to be the universal cover of $\mathbb{C}^n \rtimes \mathbb{T}$, $n \geq 2$ (with $\mathbb{T}$ acting on $\mathbb{C}^n$ by scalar multiplication). Then as in the example $\mathbb{R}^n \rtimes K$ treated above, $C^*(G)$ has the structure of an extension

$$0 \to C_0((0, \infty)) \otimes (C(S^{2n-1}) \rtimes \mathbb{R}) \to C^*(G) \to C^*(\mathbb{R}) \to 0.$$

From this and the fact that $S^{2n-1}/\mathbb{T} = \mathbb{CP}^{n-1}$, one can see from the theory of [**RaR**] that $C^*(G)$ has an essential ideal which is a continuous-trace algebra with spectrum

$$(0, \infty) \times \mathbb{CP}^{n-1} \times \mathbb{T},$$

whose Dixmier-Douady invariant is a generator of $H^3(\mathbb{CP}^{n-1} \times S^1, \mathbb{Z}) \cong \mathbb{Z}$.

To what extent is "twisting" in $C^*(G)$ a prevalent phenomenon? An easy result that excludes this (at least modulo torsion) in many cases of interest is the following:

THEOREM. *Suppose G is a locally compact group with a "large" compact subgroup K, in the sense that the restriction of any irreducible unitary representation of G to K can only contain each "K-type" $\sigma \in \hat{K}$ with finite multiplicity. (This condition is satisfied if G is a finitely connected semisimple real Lie group with finite center, or if G is a "motion group" (a semidirect product of a vector group by a compact group), or if G is a reductive p-adic linear group.) Then $C^*(G)$ has an exhaustion by ideals each Morita-equivalent to a C^*-algebra all of whose representations are finite-dimensional. And for any continuous-trace subquotient of $C^*(G)$ (with compact spectrum), the Dixmier-Douady invariant is rationally trivial.*

PROOF. Basically by the Peter-Weyl Theorem, $C^*(K)$ is a (C^*-algebraic) direct sum of finite-dimensional matrix algebras indexed by the $\sigma \in \hat{K}$. The embedding of $C^*(K)$ into the multiplier algebra of $C^*(G)$ therefore gives a family p_σ, $\sigma \in \hat{K}$, of mutually orthogonal projections in $M(C^*(G))$ summing (in the "strict" topology) to 1, such that $p_\sigma C^*(G) p_\sigma$ gives the action of G on the σ-isotypic component of any unitary representation space of G. This being a "corner" of $C^*(G)$, it is Morita-equivalent to the ideal $C^*(G)p_\sigma C^*(G)$ of $C^*(G)$, whose irreducible representations are those unitary representations of G containing the "K-type" σ. However, since K is large, each irreducible representation of $p_\sigma C^*(G)p_\sigma$ is finite-dimensional. Thus if we index the irreducible representations of K as $\sigma_1, \sigma_2, \ldots$, we have an increasing chain

$$C^*(G)p_{\sigma_1}C^*(G) \subseteq C^*(G)p_{\sigma_1}C^*(G) + C^*(G)p_{\sigma_2}C^*(G) \subseteq \cdots$$

of ideals which exhaust $C^*(G)$, each Morita-equivalent to an algebra all of whose representations are finite-dimensional.

Now suppose A is a continuous-trace subquotient of $C^*(G)$ with compact spectrum X. By compactness, X must be contained in the dual of one of the ideals in the above sequence, which says that A is a subquotient of one of the ideals. Thus A is Morita-equivalent to an algebra A' all of whose representations are finite-dimensional. Since the continuous-trace property is preserved by Morita equivalence, A' has continuous trace, and thus the dimensions of the irreducible representations of A' cannot "jump." In other words, over each connected component of X, A' must be n-homogeneous for some n. As we saw earlier, the Dixmier-Douady invariant of an n-homogeneous algebra has exponent n, and so is rationally trivial. So the Dixmier-Douady invariant of A is rationally trivial.

REMARK. This does not quite prove that the Dixmier-Douady invariant of an **arbitrary** continuous-trace subquotient of $C^*(G)$ must be rationally trivial. The problem is that if X is non-compact and X is exhausted by an increasing family of open sets U_i, it is possible to have a class in $H^3(X, \mathbb{Z})$ which is rationally non-trivial but which is rationally trivial when pulled to $X \cap U_i$ for any i, because of the "$\varprojlim^1$ exact sequence"

$$0 \to \varprojlim{}^1 H^2(X \cap U_i, \mathbb{Z}) \to H^3(X, \mathbb{Z}) \to \varprojlim H^3(X \cap U_i, \mathbb{Z}) \to 0.$$

However, it seems unlikely that this would cause trouble for the sorts of spaces X that would arise from the duals of semisimple Lie groups, since one would expect them to have finite homotopy type (on each connected component).

This theorem suggests that it should be possible to compute $C^*(G)$ quite explicitly (up to Morita equivalence) when G is a reductive real or p-adic group, modulo the problem which is not yet settled of completely parameterizing the unitary representations. Some progress along these lines has indeed been made recently, and more can be expected as our understanding of the unitary dual improves. For example, Wassermann [**Wa**] has computed $C^*_r(G)$ for connected reductive linear Lie groups G: the result is that

$$C^*_r(G) \simeq \bigoplus_{P,\,\omega} C_0(\hat{A}/W'_\omega) \rtimes R_\omega,$$

where $\simeq$ denotes Morita equivalence, and the sum is over conjugacy classes of cuspidal parabolic subgroups $P = MAN$ of G and discrete series representations ω of the semisimple group M. Here W'_ω is a finite Weyl group and the "R-group" R_ω is a finite elementary abelian 2-group. From the point of view of structure theory of C*-algebras, the structure of $C^*_r(G)$ thus reduces to that of crossed products of abelian C*-algebras by finite groups. Thus, even though G has no interesting normal subgroups, the fine structure of the Mackey machine (even for the special cases discussed above under Research Problem 1) has quite a bit to say about the structure of $C^*_r(G)$.

Similar results about $C^*_r(G)$ for reductive linear p-adic groups G have been obtained by Plymen and his co-workers, in [**Pl**] for the case of $GL(n)$ (for which $C^*_r(G)$ is Morita-equivalent to an abelian C*-algebra), and in [**LPl**] and [**PlL**] for the general case.

However, when G is reductive, computing $C^*(G)$ should be much harder than computing $C^*_r(G)$, since the latter only reflects the structure of the "tempered" part of the dual, which is the part easiest to describe. Nevertheless, there is evidence that from a C*-algebraic point of view, $C^*(G)$ should be rather simple in structure. For example, Vogan in [**V**] has determined the irreducible unitary representations of $GL(n)$ over $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ quite explicitly, and the method of proof reduces the structure of the unitary dual to the determination of the spherical unitary dual for $GL(k)$'s with $k \leq n$. (A representation is **spherical** if its restriction to the maximal compact subgroup K contains the trivial representation of K.) However, since an irreducible representation of G can only contain the trivial representation 1_K of K at most once, the ideal $C^*(G)p_{1_K}C^*(G)$ (in the notation of the proof of the above theorem), which is the universal C*-algebra for the spherical unitary representations of G, is Morita-equivalent to an abelian

C*-algebra, and thus is "totally untwisted." Similarly, Tadić [**Tad**] has computed the dual topology of $GL(n)$ over a p-adic field and has obtained results similar to those in the archimedean case.

Roughly speaking, the group C*-algebras of exponential solvable, or even of nilpotent Lie groups, are much more complicated from an operator-theoretic point of view than those of reductive groups, and their duals are much further from being Hausdorff. Nevertheless, it is known how to find an explicit composition series for $C^*(G)$ with continuous-trace subquotients (see [**Pe**] and [**Cu**]), and in cases where one has been able to compute them, all the Dixmier-Douady invariants seem to vanish. This motivates the following:

CONJECTURE. *If G is a connected nilpotent (even exponential solvable) or semisimple Lie group, any continuous-trace subquotient of $C^*(G)$ has trivial Dixmier-Douady invariant.*

OPEN CASES. I actually know of very few cases where the conjecture above has been verified, and there are very few cases where one has an "explicit" description of $C^*(G)$. It would be helpful to have a "calculation" of $C^*(G)$ at least for those semisimple groups for which the unitary dual is now known.

RESEARCH PROBLEM 3. *For what non-type I groups G is there a good description of* $\mathrm{Prim}(G)$*?*

MOTIVATION. Our understanding of the C*-algebras of non-type I groups is for the most part rather poor, and in most cases a complete description of $C^*(G)$ or of $\hat{G}$ is out of reach. Thus the first step in understanding the unitary representation theory of a non-type I group should be to describe the irreducible unitary representations up to weak equivalence, that is, to describe $\mathrm{Prim}\, G$. (Two representations are said to be **weakly equivalent** if each is weakly contained in the other.) In addition, for purposes of studying harmonic analysis and character theory on G, one should try to understand the space $\overset{\frown}{G}_{\mathrm{norm}}$ of quasi-equivalence classes of traceable factor representations, i.e., the space of "characters" of $C^*(G)$ ([**Gu**] or [**Di, §6.7**]). When G is second-countable, $C^*(G)$ is separable and it is known that each prime ideal of $C^*(G)$ is primitive [**Di, 3.9.1**], hence the kernel of each factor representation of $C^*(G)$ is a primitive ideal. Thus "taking the kernel" defines a map $\overset{\frown}{G}_{\mathrm{norm}} \to \mathrm{Prim}\, G$, which for type I groups is a bijection. The extent to which this map is a bijection for a group that is not type I measures to some extent how well character theory works for G.

RECENT AND CURRENT RESULTS. As we mentioned earlier, Pukanszky [**Pu2**] gave an algorithm for computing $\mathrm{Prim}\, G$ for any connected Lie group G, and showed that the natural map $\overset{\frown}{G}_{\mathrm{norm}} \to \mathrm{Prim}\, G$ is a bijection in this case. Howe [**Ho1**] basically did the same for finitely generated torsion-free discrete nilpotent groups, and also gave a description of $\mathrm{Prim}\, G$ closely related to Kirillov's orbit method for nilpotent Lie groups. Howe's work led to the conjecture that for any nilpotent locally compact group, $\operatorname{Prim} G$ is a T_1-space (i.e., all primitive ideals of $C^*(G)$ are maximal), and the map $\widehat{G}_{\text{norm}} \to \operatorname{Prim} G$ is a bijection. There is now substantial evidence for this. Poguntke [**Po**] showed that $\operatorname{Prim} G$ is a T_1-space for any discrete nilpotent group, and his result has been extended by Carey and Moran [**CaM**] and by Ludwig [**Lu**] to non-discrete nilpotent locally compact groups containing a compactly generated open normal subgroup. Furthermore, Howe's original "orbit method" approach has now been extended to nilpotent algebraic linear groups over $\mathbb{Q}$ (with the discrete topology) by Pfeffer [**Pf**], who showed the map $\widehat{G}_{\text{norm}} \to \operatorname{Prim} G$ is a bijection for such groups. One interesting byproduct of this work (originally proved by Howe, with a shorter proof in [**GoR, Note added in proof**]) is that for any second-countable nilpotent locally compact group, every irreducible unitary representation is weakly equivalent to a monomial representation (a representation induced from a one-dimensional representation of some subgroup).

The representation theory of discrete nilpotent groups of course suggests the general question of what one can say about unitary representation theory for discrete groups in general. The foundations of unitary representation theory for discrete groups were developed by Thoma [**Th1, Th2**], who showed among other things that if G is discrete, G is type I if and only if G contains an abelian subgroup of finite index. (When this happens, all irreducible representations of G are finite-dimensional and the structure of $C^*(G)$ can be computed using the techniques which we described earlier.) Thus most interesting discrete groups are not type I. Thoma's approach depended heavily on the study of the subset $\widehat{G}_{\text{fin}}$ of $\widehat{G}_{\text{norm}}$ represented by **finite** factor representations (or finite characters). Thoma computed this set in a number of cases, and using some of his results and techniques, Hauenschild [**Hd1, Hd2**] has computed the map $\widehat{G}_{\text{fin}} \to \operatorname{Prim} G$ in the cases of $GL(\infty, \mathbb{F}_q)$ and $S(\infty)$ (the infinite symmetric group). The map is a bijection in the first case, and neither an injection nor a bijection in the second case. In both cases, $\operatorname{Prim} G$ as a topological space is somewhat bizarre; for example, $\operatorname{Prim} GL(\infty, \mathbb{F}_q)$ is a countable space consisting of a single dense point (corresponding to the 0-ideal of $C^*(G)$), together with $q-1$ copies of the natural numbers $\mathbb{N}$ with the (very non-Hausdorff) topology defined by the order structure. Thus there is one open point, there are $q-1$ closed points, and all remaining points are locally closed but neither open nor closed. These two groups are examples of inductive limits $G = \varinjlim G_n$ of compact groups G_n, for which Strătilă and Voiculescu [**SV**] have developed a general C*-algebraic approach to unitary representations of G, even when G is not locally compact. With this technology, Boyer [**Bo**] has explicitly computed $\widehat{G}_{\text{norm}}$ and $\operatorname{Prim} G$ for $G = U(\infty)$, and again found that the natural map from the former to the latter is neither injective nor surjective.

Finally, the unitary representation theory of $G = GL(n, K)$, where K is an infinite field and G is given the discrete topology, has been studied by Howe and Rosenberg [**HoR**]. They found that the only maximal ideals of $C^*(G)$ are the kernels of one-dimensional characters and of representations induced from one-dimensional characters of the center. Using this fact together with a result of Kirillov, I showed in [**Ro2**] that the map $\overset{\cap}{G}_{\text{norm}} \to \operatorname{Prim} G$ is a bijection when $G = GL(n, \bar{\mathbb{F}}_p)$. Also, the map $\overset{\cap}{G}_{\text{norm},\,r} \to \operatorname{Prim} C^*_r(G)$ is a bijection when $G = GL(n, K)$, K any infinite field.

OPEN CASES. Clearly we have far to go in the understanding of the unitary representation theory of most non-type I groups. While for groups "chosen at random" it is unlikely that there would be much of a decent theory, groups which are "algebraic" in some sense ought to have an interesting and somewhat understandable representation theory. Thus nilpotent groups and groups such as $G = GL(n, K)$, where K is an infinite field, provide interesting test cases. In the case of general nilpotent groups, perhaps it may now be feasible to prove the conjecture that every primitive ideal is maximal and is associated with a unique trace. In the case of $G = GL(n, K)$, we understand the maximal ideals and the finite characters, but not the non-maximal primitive ideals; we also do not know if there are any infinite characters. In fact, for most discrete groups, while Thoma's theory provides some information about finite characters, we usually know nothing about infinite characters.

Except for a brief reference to inductive limit groups such as $U(\infty)$, I have not touched here on the unitary representation theory of groups which are not locally compact, but in some cases (such as the unitary groups of C*-algebras themselves!), this theory may also be related to structure theory of C*-algebras and some aspects of Mackey theory. Infinite-dimensional groups are playing an increasing role in mathematics and physics, and so I'm sure this relationship will be an interesting subject of future study.

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