Examples and applications of noncommutative geometry and K-theory

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Escuela de Invierno Luis Santaló
Buenos Aires, August 2–6, 2010

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Plan of the Lectures

1. Introduction to Kasparov's KK-theory.
2. K-theory and KK-theory of crossed products.
3. The universal coefficient theorem for KK and some of its applications.
4. A fundamental example in noncommutative geometry: topology and geometry of the irrational rotation algebra.
5. Applications of the irrational rotation algebra in number theory and physics.

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Part I

Introduction to Kasparov's KK-theory

What is KK?

KK-theory is a bivariant version of topological K-theory, due to Gennadi Kasparov, defined for C^*-algebras, with or without a group action. It can be defined for either real or complex algebras, but in this course we will stick to separable complex algebras for simplicity. For such algebras A and B, an abelian group $KK(A, B)$ is defined, with the property that $KK(\mathbb{C}, B) = K(B) = K_0(B)$ if the first algebra A is just the scalars.

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A class in $KK(A, B)$ gives rise to a map $K(A) \to K(B)$, but also to a natural family of maps $K(A \otimes C) \to K(B \otimes C)$ for all C. I.e., it gives a natural tranformation from the functor $K(A \otimes \_)$ to the functor $K(B \otimes \_)$. Here $\otimes$ is the completed (minimal) tensor product.

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This is almost the definition — for A and B nice enough, any such natural transformation comes from a *KK* element.

Why KK?

Let's take A and B are commutative. Thus $A = C_0(X)$ and $B = C_0(Y)$, where X and Y are locally compact Hausdorff. We will abbreviate $KK(C_0(X), C_0(Y))$ to $KK(X, Y)$. We want $KK(\mathbb{C}, C_0(Y)) = KK(\text{pt}, Y) = K(Y)$, the K-theory of Y with compact support, the Grothendieck group of complexes of vector bundles over Y that are exact off a compact set, or the reduced K-theory $\widetilde{K}(Y_+)$ of the one-point compactification Y_+ of Y.

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The Thom isomorphism theorem asserts that if $p\colon E \to X$ is a complex vector bundle, there is a natural isomorphism $\beta_E\colon K(X) \to K(E)$. The map β_E can be described by the formula $\beta_E(a) = p^*(a) \cdot \tau_E$. Here $p^*(a)$ is the pull-back of $a \in K(X)$ to E, and τ_E is the Thom class, which has compact support in the fiber directions. β_E can be described by a class in $KK(X, E)$, though one can also just use simple vector bundle theory to define it.

Connection with elliptic operators

Bu how do we prove that β_E is an isomorphism? The simplest way would be to construct an inverse map $\alpha_E\colon K(E) \to K(X)$. As Atiyah recognized, α_E uses elliptic operators, in fact the family of Dolbeault operators along the fibers of E. We want a class α_E in $KK(E, X)$ corresponding to this family of operators, and the Thom isomorphism theorem is a Kasparov product calculation, the fact that α_E is a KK inverse to the class $\beta_E \in KK(X, E)$. Atiyah also noticed it's enough to prove that α_E is a one-way inverse to β_E, or in other words, in the language of Kasparov theory, that $\beta_E \otimes_E \alpha_E = 1_X$. This comes down to an index calculation, which because of naturality comes down to the single calculation $\beta \otimes_{\mathbb{C}} \alpha = 1 \in KK(\mathrm{pt}, \mathrm{pt})$ when X is a point and $E = \mathbb{C}$, which amounts to the Riemann-Roch theorem for $\mathbb{CP}^1$.

How to think about KK?

The example of Atiyah's class $\alpha_E \in KK(E, X)$, based on a family of elliptic operators over E parametrized by X, shows that one gets an element of the bivariant K-group $KK(X, Y)$ from a family of elliptic operators over X parametrized by Y. The element that one gets should be invariant under homotopies of such operators. Hence Kasparov's definition of $KK(A, B)$ is based on a notion of homotopy classes of generalized elliptic operators for the first algebra A, "parametrized" by the second algebra B (and thus commuting with a B-module structure).

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- $\phi(a)(T^2 - 1) \in \mathcal{K}(\mathcal{H})$ $\forall a \in A$ (ellipticity),
- $[\phi(a), T] \in \mathcal{K}(\mathcal{H})$ $\forall a \in A$ (pseudolocality).

Comments on the definition

If $B = C_0(Y)$, a Hilbert B-module is equivalent to a continuous field of Hilbert spaces over Y. In this case, $\mathcal{K}(\mathcal{H})$ is the continuous fields of compact operators, while $\mathcal{L}(\mathcal{H})$ consists of strong-$*$ continuous fields of bounded operators. In general, a Hilbert B-module means a right B-module equipped with a B-valued inner product $\langle \_, \_ \rangle_B$, right B-linear in the second variable, satisfying $\langle \xi, \eta \rangle_B = \langle \eta, \xi \rangle_B^*$ and $\langle \xi, \xi \rangle_B \geq 0$, with equality only if $\xi = 0$. Such an inner product gives rise to a norm on $\mathcal{H}$: $\|\xi\| = \|\langle \xi, \xi \rangle_B\|_B^{1/2}$, and we require $\mathcal{H}$ to be complete with respect to this norm. The C^*-algebra $\mathcal{L}(\mathcal{H})$, consists of bounded adjointable B-linear operators a on $\mathcal{H}$, i.e., with an adjoint a^* such that $\langle a\xi, \eta \rangle_B = \langle \xi, a^*\eta \rangle_B$ for all ξ, $\eta \in \mathcal{H}$. Inside $\mathcal{L}(\mathcal{H})$ is the ideal of B-compact operators $\mathcal{K}(\mathcal{H})$. This is the closed linear span of the "rank-one operators" $T_{\xi,\eta}$ defined by $T_{\xi,\eta}(\nu) = \xi \langle \eta, \nu \rangle_B$.

Examples

The simplest kind of Kasparov bimodule is associated to a homomorphism $\phi\colon A \to B$. In this case, we simply take $\mathcal{H} = \mathcal{H}_0 = B$, viewed as a right B-module, with the B-valued inner product $\langle b_1, b_2 \rangle_B = b_1^* b_2$, and take $\mathcal{H}_1 = 0$ and $T = 0$. In this case, $\mathcal{L}(\mathcal{H}) = M(B)$ (the multiplier algebra of B, the largest C^*-algebra containing B as an essential ideal), and $\mathcal{K}(\mathcal{H}) = B$. So ϕ maps A into $\mathcal{K}(\mathcal{H})$, and even though $T = 0$, the condition that $\phi(a)(T^2 - 1) \in \mathcal{K}(\mathcal{H})$ is satisfied for any $a \in A$.

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One special case which is especially important is the case where $A = B$ and ϕ is the identity map. The above construction then yields a distinguished element $1_A \in KK(A, A)$, which will play an important role later.

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In applications to index theory, Kasparov A-B-bimodules typically arise from elliptic (or hypoelliptic) pseudodifferential operators. Kasparov bimodules also arise from quasihomomorphisms.

The equivalence relation

There is a natural associative addition on Kasparov bimodules, obtained by taking the direct sum of Hilbert B-modules and the block direct sum of homomorphisms and operators. Then we divide out by the equivalence relation generated by addition of degenerate Kasparov bimodules (those for which for all $a \in A$, $\phi(a)(T^2 - 1) = 0$ and $[\phi(a), T] = 0$) and by homotopy. (A homotopy of Kasparov A-B-bimodules is just a Kasparov A-$C([0,1], B)$-bimodule.) Then it turns out that $KK(A, B)$ is actually an abelian group, with inversion given by reversing the grading, i.e., reversing the roles of $\mathcal{H}_0$ and $\mathcal{H}_1$, and interchanging F and F^*. It is not really necessary to divide out by degenerate bimodules, since if $(\mathcal{H}, \phi, T)$ is degenerate, then $C_0((0,1], \mathcal{H})$ (along with the action of A and the operator which are given by ϕ and T at each point of $(0,1]$) is a homotopy from $(\mathcal{H}, \phi, T)$ to the 0-module.

Relation with K-theory

An interesting exercise is to consider what happens when $A = \mathbb{C}$ and B is a unital C^*-algebra. Then if $\mathcal{H}_0$ and $\mathcal{H}_1$ are finitely generated projective (right) B-modules and we take $T = 0$ and ϕ to be the usual action of $\mathbb{C}$ by scalar multiplication, we get a Kasparov $\mathbb{C}$-B-bimodule corresponding to the element $[\mathcal{H}_0] - [\mathcal{H}_1]$ of $K_0(B)$. With some work one can show that this gives an isomorphism between the Grothendieck group $K_0(B)$ of usual K-theory and $KK(\mathbb{C}, B)$. By considering what happens when one adjoins a unit, one can then show that there is still a natural isomorphism between $K_0(B)$ and $KK(\mathbb{C}, B)$, even if B is nonunital.

Morita equivalence

Suppose A and B are Morita equivalent in the sense of Rieffel. That means we have an A-B-bimodule X with the following special properties:

1. X is a right Hilbert B-module and a left Hilbert A-module.
2. The left action of A is by bounded adjointable operators for the B-valued inner product, and the right action of B is by bounded adjointable operators for the A-valued inner product.
3. The A- and B-valued inner products on X are compatible in the sense that if $\xi, \eta, \nu \in X$, then ${}_A\langle \xi, \eta\rangle \nu = \xi \langle \eta, \nu\rangle_B$.
4. The inner products are "full," in the sense that the image of ${}_A\langle \_, \_\rangle$ is dense in A, and the image of $\langle \_, \_\rangle_B$ is dense in B.

Under these circumstances, X defines classes in $[X] \in KK(A, B)$ and $[\widetilde{X}] \in KK(B, A)$ which are inverses to each other (with respect to the product discussed below).

The product

The hardest aspect of Kasparov's approach to KK is to prove that there is a well-defined, functorial, bilinear, and associative product $\otimes_B\colon\ KK(A, B) \times KK(B, C) \to KK(A, C)$. There is also an external product $\boxtimes\colon\ KK(A, B) \times KK(C, D) \to KK(A \otimes C, B \otimes D)$, where $\otimes$ denotes the completed *minimal* or *spatial* C^*-tensor product.

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$\boxtimes\colon\ KK(A, B) \times KK(C, D) \to KK(A \otimes C, B \otimes D)$, where $\otimes$ denotes the completed *minimal* or *spatial* C^*-tensor product. The external product is built from the usual product using dilation (external product with 1). We can dilate a class $a \in KK(A, B)$ to a class $a \boxtimes 1_C \in KK(A \otimes C, B \otimes C)$, by taking a representative $(\mathcal{H}, \phi, T)$ for a to the bimodule $(\mathcal{H} \otimes C, \phi \otimes 1_C, T \otimes 1)$. Similarly, we can dilate a class $b \in KK(C, D)$ (on the other side) to a class $1_B \boxtimes b \in KK(B \otimes C, B \otimes D)$. Then

$$a \boxtimes b = (a \boxtimes 1_C) \otimes_{B \otimes C} (1_B \boxtimes b) \in KK(A \otimes C, B \otimes D),$$

and this is the same as $(1_A \boxtimes b) \otimes_{A \otimes D} (a \boxtimes 1_D)$.

More on the products

The Kasparov products include all the usual cup and cap products relating K-theory and K-homology. For example, the cup product in ordinary topological K-theory for a compact space X,
$\cup\colon K(X) \times K(X) \to K(X)$, is a composite of two products:

$$a \cup b = (a \boxtimes b) \otimes_{C(X \times X)} \Delta,$$

where $\Delta \in KK(C(X \times X), C(X))$ is the class of the diagonal map $X \to X \times X$.

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where $\Delta \in KK(C(X \times X), C(X))$ is the class of the diagonal map $X \to X \times X$. Suppose we have classes represented by $(\mathcal{E}_1, \phi_1, T_1)$ and $(\mathcal{E}_2, \phi_2, T_2)$, where $\mathcal{E}_1$ is a right Hilbert B-module, $\mathcal{E}_2$ is a right Hilbert C-module, $\phi_1\colon A \to \mathcal{L}(\mathcal{E}_1)$, $\phi_2\colon B \to \mathcal{L}(\mathcal{E}_2)$, T_1 essentially commutes with the image of ϕ_1, and T_2 essentially commutes with the image of ϕ_2. It is clear that we want to construct the product using $\mathcal{H} = \mathcal{E}_1 \otimes_{B,\phi_2} \mathcal{E}_2$ and $\phi = \phi_1 \otimes 1\colon A \to \mathcal{L}(\mathcal{H})$. The main difficulty is getting the correct operator T. In fact there is no canonical choice; the choice is only unique up to homotopy, and is defined using the Connes-Skandalis notion of a connection.

Cuntz's approach

Joachim Cuntz noticed that all Kasparov bimodules come from a quasihomomorphism $A \rightrightarrows D \trianglerighteq B$, a formal difference of two homomorphisms $f_\pm \colon A \to D$ which agree modulo an ideal isomorphic to B. Thus $a \mapsto f_+(a) - f_-(a)$ is a linear map $A \to B$. Suppose for simplicity (one can always reduce to this case) that $D/B \cong A$, so that $f_\pm$ are two splittings for an extension $0 \to B \to D \to A \to 0$. Then for any *split-exact* functor F from C^*-algebras to abelian groups (meaning it sends split extensions to short exact sequences — an example would be $F(A) = K(A \otimes C)$ for some coefficient algebra C), we get an exact sequence

$$0 \longrightarrow F(B) \longrightarrow F(D) \underset{(f_-)_*}{\overset{(f_+)_*}{\leftrightarrows}} F(A) \longrightarrow 0.$$

Thus $(f_+)_* - (f_-)_*$ gives a well-defined homomorphism $F(A) \to F(B)$, which we might well imagine should come from a class in $KK(A, B)$.

Cuntz's universal construction

A quasihomomorphism $A \rightrightarrows D \trianglerighteq B$ factors through a universal algebra qA. Start with the *free product* C^*-algebra $QA = A * A$, the completion of linear combinations of words in two copies of A. There is an obvious morphism $QA \twoheadrightarrow A$ obtained by identifying the two copies of A. The kernel of $QA \twoheadrightarrow A$ is called qA, and if $0 \to B \to D \underset{f_-}{\overset{f_+}{\rightleftarrows}} A \to 0$ is a quasihomomorphism, we get a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & qA & \longrightarrow & QA & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & D & \longrightarrow & A & \longrightarrow & 0, \end{array}$$

with the first copy of A in QA mapping to D via f_+, and the second copy of A in QA mapping to D via f_-. In this way $KK(A, B)$ turns out to be simply the set of homotopy classes of $*$-homomorphisms from qA to $B \otimes \mathcal{K}$.

Higson's approach

Higson proposed making an additive category **KK** whose objects are the separable C^*-algebras, and where the morphisms from A to B are given by $KK(A, B)$. Associativity and bilinearity of the Kasparov product, along with properties of the special elements $1_A \in KK(A, A)$, ensure that this is indeed an additive category.

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1. Matrix stability. If A is an object in **KK** (that is, a separable C^*-algebra) and if e is a rank-one projection in $\mathcal{K} = \mathcal{K}(\mathcal{H})$, $\mathcal{H}$ a separable Hilbert space, then the homomorphism $a \mapsto a \otimes e$, viewed as an element of $\text{Hom}(A, A \otimes \mathcal{K})$, is an equivalence in **KK**, i.e., has an inverse in $KK(A \otimes \mathcal{K}, A)$.

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2. Split exactness. KK takes splits short exact sequences to split short exact sequences (in either variable).

Part II

K-theory and *KK*-theory of crossed products

Equivariant Kasparov theory

G will be a second-countable locally compact group. A G-C^*-algebra will mean a C^*-algebra A with a jointly continuous action of G on A by $*$-automorphisms. If G is compact, making KK-theory equivariant is straightforward. We just require all algebras and Hilbert modules to be equipped with G-actions, we require $\phi\colon A \to \mathcal{L}(\mathcal{H})$ to be G-equivariant, and we require the operator $T \in \mathcal{L}(\mathcal{H})$ to be G-invariant. We get groups $KK^G(A, B)$ for (separable, say) G-C^*-algebras A and B, and the same argument as before shows that $KK^G(\mathbb{C}, B) \cong K_0^G(B)$, equivariant K-theory. In particular, $KK^G(\mathbb{C}, \mathbb{C}) \cong R(G)$, the representation ring of G. For example, if G is compact and abelian, $R(G) \cong \mathbb{Z}[\widehat{G}]$, the group ring of the Pontrjagin dual. If G is a compact connected Lie group with maximal torus T and Weyl group $W = N_G(T)/T$, then $R(G) \cong R(T)^W \cong \mathbb{Z}[\widehat{T}]^W$. The properties of the Kasparov product all go through, and product with $KK^G(\mathbb{C}, \mathbb{C})$ makes all KK^G-groups into modules over the ground ring $R(G)$.

The case of noncompact groups

When G is noncompact, the definition and properties of KK^G are considerably more subtle, and were worked out by Kasparov. The problem is that in this case, topological vector spaces with a continuous G-action are very rarely completely decomposable, and there are rarely enough G-equivariant operators to give anything useful. Kasparov's solution was to work with G-continuous rather than G-equivariant Hilbert modules and operators; rather remarkably, these still give a useful theory with all the same formal properties as before. The KK^G-groups are again modules over the commutative ring $R(G) = KK^G(\mathbb{C}, \mathbb{C})$, though this ring no longer has such a simple interpretation as before, and in fact, is not known for most connected semisimple Lie groups.

Functorial properties

A few functorial properties of the KK^G-groups will be needed below, so we just mention a few of them. First of all, if H is a closed subgroup of G, then any G-C^*-algebra is by restriction also an H-C^*-algebra, and we have restriction maps $KK^G(A, B) \to KK^H(A, B)$. To go the other way, we can "induce" an H-C^*-algebra A to get a G-C^*-algebra $\mathrm{Ind}_H^G(A)$, defined by

$$\mathrm{Ind}_H^G(A) = \{f \in C(G, A) \mid f(gh) = h \cdot f(g) \quad \forall g \in G, h \in H, \\ \|f(g)\| \to 0 \text{ as } g \to \infty \mod H\}.$$

The induced action of G on $\mathrm{Ind}_H^G(A)$ is just left translation. An imprimitivity theorem due to Green shows that $\mathrm{Ind}_H^G(A) \rtimes G$ and $A \rtimes H$ are Morita equivalent. If A and B are H-C^*-algebras, we then have an induction homomorphism

$$KK^H(A, B) \to KK^G(\mathrm{Ind}_H^G(A), \mathrm{Ind}_H^G(B)).$$

Basic properties of crossed products

If A is a G-C^*-algebra, one can define two new C^*-algebras, called the full and reduced crossed products of A by G, which capture the essence of the group action. These are easiest to define when G is discrete and A is unital. The full crossed product $A \rtimes_\alpha G$ (we often omit the α if there is no possibility of confusion) is the universal C^*-algebra generated by a copy of A and unitaries u_g, $g \in G$, subject to the commutation condition $u_g a u_g^* = \alpha_g(a)$, where α denotes the action of G on A. The reduced crossed product $A \rtimes_{\alpha,r} G$ is the image of $A \rtimes_\alpha G$ in its "regular representation" π on $L^2(G, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space on which A acts faithfully, say by a representation ρ. Here A acts by $(\pi(a)f)(g) = \rho(\alpha_{g^{-1}}(a))f(g)$ and G acts by left translation.

More general crossed products

In general, the full crossed product is defined as the universal C^*-algebra for covariant pairs of a $*$-representation ρ of A and a unitary representation π of G, satisfying the compatibility condition $\pi(g)\rho(a)\pi(g^{-1}) = \rho(\alpha_g(a))$. It may be constructed by defining a convolution multiplication on $C_c(G, A)$ and then completing in the greatest C^*-algebra norm. The reduced crossed product $A \rtimes_{\alpha,r} G$ is again the image of $A \rtimes_\alpha G$ in its "regular representation" on $L^2(G, \mathcal{H})$.

More general crossed products

In general, the full crossed product is defined as the universal C^*-algebra for covariant pairs of a $*$-representation ρ of A and a unitary representation π of G, satisfying the compatibility condition $\pi(g)\rho(a)\pi(g^{-1}) = \rho(\alpha_g(a))$. It may be constructed by defining a convolution multiplication on $C_c(G, A)$ and then completing in the greatest C^*-algebra norm. The reduced crossed product $A \rtimes_{\alpha,r} G$ is again the image of $A \rtimes_\alpha G$ in its "regular representation" on $L^2(G, \mathcal{H})$. For example, $\mathbb{C} \rtimes G$ is the group C^*-algebra $C^*(G)$, and $\mathbb{C} \rtimes_r G$ is $C^*_r(G)$, the image of $C^*(G)$ in the left regular representation on $L^2(G)$. The natural map $C^*(G) \twoheadrightarrow C^*_r(G)$ is an isomorphism if and only if G is amenable. When the action α is trivial, then $A \rtimes G$ is the maximal tensor product $A \otimes_{\max} C^*(G)$ while $A \rtimes_r G$ is the minimal tensor product $A \otimes C^*_r(G)$. Again, $A \otimes_{\max} C^*(G)$ to $A \otimes C^*_r(G)$ is an isomorphism if and only if G is amenable.

More about crossed products

When A and the action α are arbitrary, the natural map $A \rtimes_\alpha G \twoheadrightarrow A \rtimes_{\alpha,r} G$ is an isomorphism if G is amenable, but also more generally if the action α is amenable in a certain sense. For example, if X is a locally compact G-space, the action is automatically amenable if it is proper, whether or not G is amenable.

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When X is a locally compact G-space, the crossed product $C_0(G) \rtimes G$ often serves as a good substitute for the "quotient space" X/G in cases where the latter is badly behaved. Indeed, if G acts freely and properly on X, then $C_0(X) \rtimes G$ is Morita equivalent to $C_0(X/G)$. But if the G-action is not proper, X/G may be highly non-Hausdorff, while $C_0(X) \rtimes G$ may be a perfectly well-behaved noncommutative algebra. A key case later on will the one where $X = \mathbb{T}$ is the circle group, $G = \mathbb{Z}$, and the generator of G acts by multiplication by $e^{2\pi i\theta}$. When θ is irrational, every orbit is dense, so X/G is an indiscrete space, and $C(\mathbb{T}) \rtimes \mathbb{Z}$ is what's usually denoted A_θ, an irrational rotation algebra or noncommutative 2-torus.

KK^G and crossed products

Now we can explain the relationships between equivariant KK-theory and crossed products. One connection is that if G is discrete and A is a G-C^*-algebra, there is a natural isomorphism $KK^G(A, \mathbb{C}) \cong KK(A \rtimes G, \mathbb{C})$. Dually, if G is compact, there is a natural Green-Julg isomorphism $KK^G(\mathbb{C}, A) \cong KK(\mathbb{C}, A \rtimes G)$.

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$$j, j_r\colon\ KK^G(A, B) \to KK(A \rtimes G, B \rtimes G),\ KK(A \rtimes_r G, B \rtimes_r G)\ \text{(resp.)},$$

sending (when $B = A$) 1_A to $1_{A \rtimes G}$. (In fact, j, j_r can be viewed as functors from the equivariant Kasparov category $\mathbf{KK}^G$ to the non-equivariant Kasparov category $\mathbf{KK}$. Later we will study how close they are to being faithful.) If $B = \mathbb{C}$ and G is discrete, then $j\colon\ KK^G(A, \mathbb{C}) \to KK(A \rtimes G, C^*(G))$ is split injective, and if G is compact, then $j\colon\ KK^G(\mathbb{C}, A) \to KK(C^*(G), A \rtimes G)$ is split injective.

The dual action and Takai duality

When the group G is not just locally compact but also abelian, then it has a Pontrjagin dual group $\widehat{G}$. In this case, given any G-C^*-algebra algebra A, say with α denoting the action of G on A, there is a dual action $\widehat{\alpha}$ of $\widehat{G}$ on the crossed product $A \rtimes G$. When A is unital and G is discrete, so that $A \rtimes G$ is generated by a copy of A and unitaries u_g, $g \in G$, the dual action is given simply by

$$\widehat{\alpha}_\gamma(au_g) = au_g\langle g, \gamma\rangle.$$

The same formula still applies in general, except that the elements a and u_g don't quite live in the crossed product but in a larger algebra. The key fact about the dual action is the Takai duality theorem: $(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \cong A \otimes \mathcal{K}(L^2(G))$, and the double dual action $\hat{\hat{\alpha}}$ of $\tilde{\tilde{G}} \cong G$ on this algebra can be identified with $\alpha \otimes \operatorname{Ad} \lambda$, where λ is the left regular representation of G on $L^2(G)$.

Connes' "Thom isomorphism"

If $\mathbb{C}^n$ (or $\mathbb{R}^{2n}$) acts on X by a trivial action α, then $C_0(X) \rtimes_\alpha \mathbb{C}^n \cong C_0(X) \otimes C^*(\mathbb{C}^n) \cong C_0(X) \otimes C_0(\widehat{\mathbb{C}}^n) \cong C_0(E)$, where E is a trivial rank-n complex vector bundle over X. (We have used Pontrjagin duality and the fact that abelian groups are amenable.) It follows that $K(C_0(X)) \cong K(C_0(X) \rtimes_\alpha \mathbb{C}^n)$. Since any action α of $\mathbb{C}^n$ is homotopic to the trivial action and "K-theory is supposed to be homotopy invariant," that suggests that perhaps $KK(A) \cong KK(A \rtimes_\alpha \mathbb{C}^n)$ for any C^*-algebra A and for any action α of $\mathbb{C}^n$. This is indeed true and the isomorphism is implemented by classes (which are inverse to one another) in $KK(A, A \rtimes_\alpha \mathbb{C}^n)$ and $KK(A \rtimes_\alpha \mathbb{C}^n, A)$. It is clearly enough to prove this in the case $n = 1$, since we can always break a crossed product by $\mathbb{C}^n$ up as an n-fold iterated crossed product.

Connes' Theorem

That A and $A \rtimes_\alpha \mathbb{C}$ are always KK-equivalent or that they at least have the same K-theory, or (this is equivalent since one can always suspend on both sides) that $A \otimes C_0(\mathbb{R})$ and $A \rtimes_\alpha \mathbb{R}$ are always KK-equivalent or that they at least have the same K-theory for any action of $\mathbb{R}$, is called Connes' "Thom isomorphism". Connes' original proof is relatively elementary, but only gives an isomorphism of K-groups, not a KK-equivalence.

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To illustrate Connes' idea, let's suppose A is unital and we have a class in $K_0(A)$ represented by a projection $p \in A$. (One can always reduce to this special case.) If α were to fix p, then $1 \mapsto p$ gives an equivariant map from $\mathbb{C}$ to A and thus would induce a map of crossed products $\mathbb{C} \rtimes \mathbb{R} \cong C_0(\widehat{\mathbb{R}}) \to A \rtimes_\alpha \mathbb{R}$ or $\mathbb{C} \rtimes \mathbb{C} \cong C_0(\widehat{\mathbb{C}}) \to A \rtimes_\alpha \mathbb{C}$ giving a map on K-theory $\beta\colon \mathbb{Z} \to K_0(A \rtimes \mathbb{C})$. The image of $[p]$ under the isomorphism $K_0(A) \to K_0(A \rtimes \mathbb{C})$ will be $\beta(1)$. So the idea is to show that one can modify the action to one fixing p (using a cocycle conjugacy) without changing the isomorphism class of the crossed product.

Proofs of Connes' Theorem

There are now quite a number of proofs of Connes' theorem available, each using somewhat different techniques. We just mention a few of them. A proof using K-theory of Wiener-Hopf extensions was given by Rieffel. There are also fancier proofs using KK-theory. If α is a given action of $\mathbb{R}$ on A and if β is the trivial action, one can try to construct $KK^{\mathbb{R}}$ elements $c \in KK^{\mathbb{R}}((A, \alpha), (A, \beta))$ and $d \in KK^{\mathbb{R}}((A, \beta), (A, \alpha))$ which are inverses of each other in $\mathbf{KK}^{\mathbb{R}}$. Then the morphism j of Section 1 sends these to KK-equivalences $j(c)$ and $j(d)$ between $A \rtimes_{\alpha} \mathbb{R}$ and $A \rtimes_{\beta} \mathbb{R} \cong A \otimes C_0(\mathbb{R})$.

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Fack and Skandalis give another proof using the group $KK^1(A, B)$. This is defined with triples $(\mathcal{H}, \phi, T)$ like those used for $KK(A, B)$, but with two modifications.

The proof of Fack and Skandalis

Conditions for KK^1:

1. $\mathcal{H}$ is no longer graded, and there is no grading condition on ϕ.
2. T is self-adjoint but with no grading condition, and $\phi(a)(T^2 - 1) \in \mathcal{K}(\mathcal{H})$ and $[\phi(a), T] \in \mathcal{K}(\mathcal{H})$ for all $a \in A$.

It turns out that $KK^1(A, B) \cong KK(A \otimes C_0(\mathbb{R}), B)$, and that the Kasparov product can be extended to a graded commutative product on the direct sum of $KK = KK^0$ and KK^1. The product of two classes in KK^1 can by Bott periodicity be taken to land in KK^0.

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We can now explain the proof of Fack and Skandalis as follows. They show that for each separable C^*-algebra A with an action α of $\mathbb{R}$, there is a special element $t_\alpha \in KK^1(A, A \rtimes_\alpha \mathbb{R})$ (constructed using a singular integral operator). Note by the way that doing the construction with the dual action and applying Takai duality gives $t_{\widehat{\alpha}} \in KK^1(A \rtimes_\alpha \mathbb{R}, A)$, since $(A \rtimes_\alpha \mathbb{R}) \rtimes_{\widehat{\alpha}} \mathbb{R} \cong A \otimes \mathcal{K}$, which is Morita equivalent to A.

The elements t_α

These elements have the following properties:

1. (Normalization) If $A = \mathbb{C}$ (so that necessarily $\alpha = 1$ is trivial), then $t_1 \in KK^1(\mathbb{C}, C_0(\mathbb{R}))$ is the usual generator of this group (which is isomorphic to $\mathbb{Z}$).
2. (Naturality) The elements are natural with respect to equivariant homomorphisms $\rho\colon (A, \alpha) \to (C, \gamma)$, in that if $\bar{\rho}$ denotes the induced map on crossed products, then $\bar{\rho}_*(t_\alpha) = \rho^*(t_\gamma) \in KK(A, C \rtimes_\gamma \mathbb{R})$, and similarly, $\bar{\rho}^*(t_{\widehat{\gamma}}) = \rho_*(t_{\widehat{\alpha}}) \in KK(A \rtimes_\alpha \mathbb{R}, C)$.
3. (Compatibility with external products) Given $x \in KK(A, B)$ and $y \in KK(C, D)$,

$$(t_{\widehat{\alpha}} \otimes_A x) \boxtimes y = t_{\widehat{\alpha \otimes 1_C}} \otimes_{A \otimes C} (x \boxtimes y).$$

Similarly, given $x \in KK(B, A)$ and $y \in KK(D, C)$,

$$y \boxtimes (x \otimes_A t_\alpha) = (y \boxtimes x) \otimes_{C \otimes A} t_{1_C \otimes \alpha}. \qquad \square$$

Idea of the proof of Fack-Skandalis

Theorem (Fack-Skandalis)

These properties completely determine t_α, and t_α is a KK-equivalence (of degree 1) between A and $A \rtimes_\alpha \mathbb{R}$.

The Pimsner-Voiculescu Theorem

Now suppose A is a C^*-algebra equipped with an action α of $\mathbb{Z}$ (or equivalently, a single $*$-automorphism θ, the image of $1 \in \mathbb{Z}$ under the action). Then $A \rtimes_\alpha \mathbb{Z}$ is Morita equivalent to $\left(\mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(A, \alpha)\right) \rtimes \mathbb{R}$. The algebra $T_\theta = \mathrm{Ind}_{\mathbb{Z}}^{\mathbb{R}}(A, \alpha)$ is often called the mapping torus of (A, θ); it can be identified with the algebra of continuous functions $f\colon [0,1] \to A$ with $f(1) = \theta(f(0))$. It comes with an obvious short exact sequence

$$0 \to C_0((0,1), A) \to T_\theta \to A \to 0,$$

for which the associated exact sequence in K-theory has the form

$$\cdots \to K_1(A) \xrightarrow{1-\theta_*} K_1(A) \to K_0(T_\theta) \to K_0(A) \xrightarrow{1-\theta_*} K_0(A) \to \cdots .$$

Since $K_0(A \rtimes_\alpha \mathbb{Z}) \cong K_0(T_\theta \rtimes_{\mathrm{Ind}\,\alpha} \mathbb{R}) \cong K_1(T_\theta)$, and similarly for K_0, we obtain the Pimsner-Voiculescu exact sequence

$$\begin{aligned} \cdots \to K_1(A) \xrightarrow{1-\theta_*} K_1(A) \xrightarrow{\iota_*} K_1(A \rtimes_\alpha \mathbb{Z}) \to \\ \to K_0(A) \xrightarrow{1-\theta_*} K_0(A) \xrightarrow{\iota_*} K_0(A \rtimes_\alpha \mathbb{Z}) \to \cdots . \end{aligned} \tag{2}$$

The Baum-Connes Conjecture (without coefficients)

Let G be a locally compact group, and let $\underline{E}G$ be the universal proper G-space. (This is a contractible space on which G acts properly, characterized up to G-homotopy equivalence by two properties: that every compact subgroup of G has a fixed point in $\underline{E}G$, and that the two projections $\underline{E}G \times \underline{E}G \to \underline{E}G$ are G-homotopic. If G has no compact subgroups, then $\underline{E}G$ is the usual universal free G-space EG.)

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Conjecture (Baum-Connes)

Let G be a locally compact group, second-countable for convenience. There is an assembly map

$$\varinjlim_{\substack{X \subseteq \underline{E}G \\ X/G \text{ compact}}} K_*^G(X) \to K_*(C_r^*(G))$$

defined by taking G-indices of G-invariant elliptic operators, and this map is an isomorphism.

The Baum-Connes Conjecture with coefficients

Conjecture (Baum-Connes with coefficients)

With notation as in the previous Conjecture, if A is any separable G-C^*-algebra, the assembly map

$$\varinjlim_{\substack{X \subseteq \underline{E}G \\ X/G \text{ compact}}} KK_*^G(C_0(X), A) \to K_*(A \rtimes_r G)$$

is an isomorphism.

Special cases

If G is compact, $\underline{E}G$ can be taken to be a single point. The conjecture then asserts that the *assembly map* $KK_*^G(\mathrm{pt}, A) \to K_*(A \rtimes G)$ is an isomorphism. This is true by the the Green-Julg theorem.

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If $G = \mathbb{R}$, we can take $\underline{E}G = G = \mathbb{R}$. If A is an $\mathbb{R}$-C^*-algebra, the assembly map is a map $KK_*^{\mathbb{R}}(C_0(\mathbb{R}), A) \to K_*(A \rtimes \mathbb{R})$. This map turns out to be Kasparov's morphism

$$j\colon\ KK_*^{\mathbb{R}}(C_0(\mathbb{R}), A) \to KK_*(C_0(\mathbb{R}) \rtimes \mathbb{R}, A \rtimes \mathbb{R}) = KK_*(\mathcal{K}, A \rtimes \mathbb{R}) \cong K_*(A \rtimes \mathbb{R}),$$

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which is the isomorphism of Connes' Theorem.

Now suppose G is discrete and torsion-free. Then $\underline{E}G = EG$, and the quotient space $\underline{E}G/G$ is the usual classifying space BG. The assembly map $K_*^{\mathrm{cmpct}}(BG) \to K_*(C_r^*(G))$ can be viewed as an index map, since classes in the K-homology group on the left are represented by generalized Dirac operators D over Spin^c manifolds M with a G-covering, and the assembly map takes such an operator to its "Mishchenko-Fomenko index". The conjecture (without coefficients) implies a strong form of the Novikov Conjecture for G.

The approach of Meyer and Nest

Meyer and Nest gave an alternative approach. They observe that the equivariant KK-category, $\mathbf{KK}^G$, is a triangulated category. It has a distinguished class $\mathcal{E}$ of weak equivalences, morphisms $f \in KK^G(A, B)$ which restrict to equivalences in $KK^H(A, B)$ for every compact subgroup H of G. The Baum-Connes Conjecture with coefficients basically amounts to the assertion that if $f \in KK^G(A, B)$ is in $\mathcal{E}$, then $j_r(f) \in KK(A \rtimes_r G, B \rtimes_r G)$ is a KK-equivalence. In particular, suppose G has no nontrivial compact subgroups and satisfies B-C with coefficients. Then if A is a G-C^*-algebra which, forgetting the G-action, is contractible, then the unique morphism in $KK^G(0, A)$ is a weak equivalence, and so (applying j_r), the unique morphism in $KK(0, A \rtimes_r G)$ is a KK-equivalence. Thus $A \rtimes_r G$ is *K-contractible*, i.e., all of its topological K-groups must vanish. When $G = \mathbb{R}$, this follows from Connes' Theorem, and when $G = \mathbb{Z}$, this follows from the Pimsner-Voiculescu exact sequence.

Current status of Baum-Connes

1. There is no known counterexample to Baum-Connes for groups, without coefficients. Counterexamples are now known to Baum-Connes with coefficients (Higson-Lafforgue-Skandalis).

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3. Baum-Connes without coefficients is true for connected reductive Lie groups, connected reductive p-adic groups, for hyperbolic discrete groups, and for cocompact lattice subgroups of $Sp(n, 1)$ or $SL(3, \mathbb{C})$ (Lafforgue).

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4. There is a vast literature; this is just for starters.

Part III

The universal coefficient theorem for KK and some of its applications

Introduction to the UCT

Now that we have discussed KK and KK^G, a natural question arises: how computable are they? In particular, is $KK(A, B)$ determined by $K_*(A)$ and by $K_*(B)$? Is $KK^G(A, B)$ determined by $K_*^G(A)$ and by $K_*^G(B)$?

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A first step was taken by Kasparov: he pointed out that $KK(X, Y)$ is given by an explicit topological formula when X and Y are finite CW complexes.

Let's make a definition — we say the pair of C^*-algebras (A, B) satisfies the Universal Coefficient Theorem for KK (or UCT for short) if there is an exact sequence

$$0 \to \bigoplus_{*\in\mathbb{Z}/2} \operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_{*+1}(B)) \to KK(A, B)$$

$$\xrightarrow{\varphi} \bigoplus_{*\in\mathbb{Z}/2} \operatorname{Hom}_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0.$$

Here φ sends a KK-class to the induced map on K-groups.

The UCT

We need one more definition. Let $\mathcal{B}$ be the bootstrap category, the smallest full subcategory of the separable C^*-algebras containing all separable type I algebras, and closed under extensions, countable C^*-inductive limits, and KK-equivalences. Note that KK-equivalences include Morita equivalences, and type I algebras include commutative algebras.

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Theorem (Rosenberg-Schochet)

The UCT holds for all pairs (A, B) with A an object in $\mathcal{B}$ and B separable.

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The UCT holds for all pairs (A, B) with A an object in $\mathcal{B}$ and B separable.

Unsolved problem: Is every separable nuclear C^*-algebra in $\mathcal{B}$? Skandalis showed that there are non-nuclear algebras not in $\mathcal{B}$.

The proof of Rosenberg and Schochet

First suppose $K_*(B)$ is injective as a $\mathbb{Z}$-module, i.e., divisible as an abelian group. Then $\mathrm{Hom}_{\mathbb{Z}}(\_\_, K_*(B))$ is an exact functor, so $A \mapsto \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_*(B))$ gives a cohomology theory on C^*-algebras. In particular, φ is a natural transformation of homology theories

$$\big(X \mapsto KK_*(C_0(X), B)\big) \rightsquigarrow \big(X \mapsto \mathrm{Hom}_{\mathbb{Z}}(K^*(X), K_*(B))\big).$$

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Since φ is an isomorphism for $X = \mathbb{R}^n$ by Bott periodicity, it is an isomorphism whenever X_+ is a finite CW complex.

The proof of Rosenberg and Schochet

First suppose $K_*(B)$ is injective as a $\mathbb{Z}$-module, i.e., divisible as an abelian group. Then $\mathrm{Hom}_{\mathbb{Z}}(\_\_, K_*(B))$ is an exact functor, so $A \mapsto \mathrm{Hom}_{\mathbb{Z}}(K_*(A), K_*(B))$ gives a cohomology theory on C^*-algebras. In particular, φ is a natural transformation of homology theories

$$\big(X \mapsto KK_*(C_0(X), B)\big) \rightsquigarrow \big(X \mapsto \mathrm{Hom}_{\mathbb{Z}}(K^*(X), K_*(B))\big).$$

Since φ is an isomorphism for $X = \mathbb{R}^n$ by Bott periodicity, it is an isomorphism whenever X_+ is a finite CW complex.
We extend to arbitrary locally compact X by taking limits, and then to the rest of $\mathcal{B}$. (Type I C^*-algebras are colimits of iterated extensions of stably commutative algebras.) So the theorem holds when $K_*(B)$ is injective.

Geometric resolutions

The rest of the proof uses an idea due to Atiyah, of geometric resolutions. The idea is that given arbitrary B, we can change it up to KK-equivalence so that it fits into a short exact sequence

$$0 \to C \to B \to D \to 0$$

for which the induced K-theory sequence is short exact: $K_*(B) \rightarrowtail K_*(D) \twoheadrightarrow K_{*-1}(C)$ and $K_*(D)$, $K_*(C)$ are $\mathbb{Z}$-injective. Then we use the theorem for $KK_*(A, D)$ and $KK_*(A, C)$, along with the long exact sequence in KK in the second variable, to get the UCT for (A, B).

The equivariant case

If one asks about the UCT in the equivariant case, then the homological algebra of the ground ring $R(G)$ becomes relevant. This is not always well behaved, so as noticed by Hodgkin, one needs restrictions on G to get anywhere. But for G a connected compact Lie group with $\pi_1(G)$ torsion-free, $R(G)$ has finite global dimension.

The equivariant case

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Theorem (Rosenberg-Schochet)

If G is a connected compact Lie group with $\pi_1(G)$ torsion-free, and if A, B are separable G-C^*-algebras with A in a suitable bootstrap category containing all commutative G-C^*-algebras, then there is a convergent spectral sequence

$$\mathrm{Ext}^p_{R(G)}(K^G_*(A), K^G_{q+*}(A)) \Rightarrow KK^G_*(A, B).$$

The proof is more complicated than in the non-equivariant case, but in the same spirit.

Categorical aspects

The UCT implies a lot of interesting facts about the bootstrap category $\mathcal{B}$. Here are a few examples.

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Theorem (Rosenberg-Schochet)

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Proof.

$\Rightarrow$ is trivial. So suppose $K_*(A) \cong K_*(B)$. Choose an isomorphism $\psi\colon K_*(A) \to K_*(B)$. Since the map φ in the UCT is surjective, ψ is realized by a class $x \in KK(A, B)$.

The KK-equivalence theorem (cont'd)

Proof (cont'd).

Now consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ext}^1(K_{*+1}(B), K_*(A)) & \longrightarrow & KK_*(B,A) & \xrightarrow{\varphi} & \mathrm{Hom}(K_*(B), K_*(A)) & \longrightarrow & 0 \\ \| & & \cong\downarrow\psi^* & & \downarrow x\otimes_B \_ & & \cong\downarrow\psi^* & & \| \\ 0 & \longrightarrow & \mathrm{Ext}^1(K_{*+1}(A), K_*(A)) & \longrightarrow & KK_*(A,A) & \xrightarrow{\varphi} & \mathrm{Hom}(K_*(A), K_*(A)) & \longrightarrow & 0 \end{array}$$

By the 5-Lemma, Kasparov product with x is an isomorphism $KK_*(B,A) \to KK_*(A,A)$. In particular, there exists $y \in KK(B,A)$ with $x \otimes_B y = 1_A$. Similarly, there exists $z \in KK(B,A)$ with $z \otimes_A x = 1_B$. Then by associativity

$$z = z \otimes_A (x \otimes_B y) = (z \otimes_A x) \otimes_B y = y$$

and we have a KK-inverse to x. □

The KK ring

Recall that $KK(A, A) = \mathrm{End}_{\mathbf{KK}}(A)$ is a ring under Kasparov product.

Theorem (Rosenberg-Schochet)

Suppose A is in $\mathcal{B}$. In the UCT sequence

$$0 \to \bigoplus_{i\in\mathbb{Z}/2} \mathrm{Ext}^1_{\mathbb{Z}}(K_{i+1}(A), K_i(A)) \to KK(A, A) \xrightarrow{\varphi} \bigoplus_{i\in\mathbb{Z}/2} \mathrm{End}(K_i(A)) \to 0,$$

φ is a split surjective homomorphism of rings, and $J = \ker\varphi$ (the Ext term) is an ideal with $J^2 = 0$.

Proof.

Choose A_0 and A_1 commutative with $K_0(A_0) \cong K_0(A)$, $K_1(A_0) = 0$, $K_0(A_1) = 0$, $K_1(A_1) \cong K_1(A)$. Then by the last theorem, $A_0 \oplus A_1$ is KK-equivalent to A, and we may assume $A = A_0 \oplus A_1$. By the UCT, $KK(A_0, A_0) \cong \mathrm{End}\, K_0(A)$ and $KK(A_1, A_1) \cong \mathrm{End}\, K_1(A)$.

The KK-ring (cont'd)

Proof.

So $KK(A_0, A_0) \oplus KK(A_1, A_1)$ is a subring of $KK(A, A)$ mapping isomorphically under φ. This shows φ is split surjective. We also have $J = KK(A_0, A_1) \oplus KK(A_1, A_0)$. If, say, x lies in the first summand and y in the second, then $x \otimes_{A_1} y$ induces the 0-map on $K_0(A)$ and so is 0 in $KK(A_0, A_0)$. Similarly, $y \otimes_{A_0} x$ induces the 0-map on $K_1(A)$ and so is 0 in $KK(A_1, A_1)$. □

The homotopy-theoretic approach

There is a homotopy-theoretic approach to the UCT that topologists might find attractive; it seems to have been discovered independently by several people. Let A and B be C^*-algebras and let $\mathbb{K}(A)$ and $\mathbb{K}(B)$ be their topological K-theory spectra. These are module spectra over $\mathbb{K} = \mathbb{K}(\mathbb{C})$, the usual spectrum of complex K-theory. Then we can define

$$KK^{\text{top}}(A, B) = \pi_0\big(\text{Hom}_{\mathbb{K}}\big(\mathbb{K}(A), \mathbb{K}(B)\big)\big).$$

Theorem

There is a natural map $KK(A, B) \to KK^{top}(A, B)$, and it's an isomorphism if and only if the UCT holds for the pair (A, B).

Observe that $KK^{\text{top}}(A, B)$ even makes sense for Banach algebras, and always comes with a UCT.

An application of KK^{top}

We promised in the first lecture to show that defining $KK(X, Y)$ to be the set of natural transformations

$$(Z \mapsto K(X \times Z)) \rightsquigarrow (Z \mapsto K(Y \times Z))$$

indeed agrees with Kasparov's $KK(C_0(X), C_0(Y))$. Indeed, $Z \mapsto K(X \times Z)$ is basically the cohomology theory defined by $\mathbb{K}(X)$, and $Z \mapsto K(Y \times Z)$ is similarly the cohomology theory defined by $\mathbb{K}(Y)$. So the natural transformations (commuting with Bott periodicity) are basically a model for $KK^{\text{top}}(C_0(X), C_0(Y))$.

Topological applications

The UCT can be used to prove facts about topological K-theory which on their face have nothing to do with C^*-algebras or KK. For example, we have the following purely topological fact:

Theorem

Let X and Y be locally compact spaces such that $K^*(X) \cong K^*(Y)$ just as abelian groups. Then the associated K-theory spectra $\mathbb{K}(X)$ and $\mathbb{K}(Y)$ are homotopy equivalent.

Proof.

We have seen that the hypothesis implies $C_0(X)$ and $C_0(Y)$ are KK-equivalent, which gives the desired conclusion. □

Note that this theorem is quite special to complex K-theory; it fails even for ordinary cohomology (since one needs to consider the action of the Steenrod algebra).

Applications to cohomology operations

Similarly, the UCT implies facts about cohomology operations in complex K-theory and K-theory mod p. For example, one has:

Theorem (Rosenberg-Schochet)

The $\mathbb{Z}/2$-graded ring of homology operations for $K(\_;\mathbb{Z}/n)$ on the category of separable C^*-algebras is the exterior algebra over $\mathbb{Z}/n$ on a single generator, the Bockstein β.

Theorem (Araki-Toda, new proof by Rosenberg-Schochet)

There are exactly n admissible multiplications on K-theory mod n. When n is odd, exactly one is commutative. When $n = 2$, neither is commutative.

Applications to C^*-algebras

Probably the most interesting applications of the UCT for KK are to the classification problem for nuclear C^*-algebras. The Elliott program (to quote M. Rørdam) is to classify "all separable, nuclear C^*-algebras in terms of an invariant that has K-theory as an important ingredient." Kirchberg and Phillips have shown how to do this for Kirchberg algebras, that is simple, purely infinite, separable and nuclear C^*-algebras. The UCT for KK is a key ingredient.

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Theorem (Kirchberg-Phillips)

Two stable Kirchberg algebras A and B are isomorphic if and only if they are KK-equivalent; and moreover every invertible element in $KK(A, B)$ lifts to an isomorphism $A \to B$. Similarly in the unital case if one keeps track of $[1_A] \in K_0(A)$.

More on Kirchberg-Phillips

We will not attempt to explain the proof of Kirchberg-Phillips, but it's based on the idea that a KK-class is given by a quasihomomorphism, which under the specific hypotheses can be lifted to a true homomorphism.

More on Kirchberg-Phillips

We will not attempt to explain the proof of Kirchberg-Phillips, but it's based on the idea that a KK-class is given by a quasihomomorphism, which under the specific hypotheses can be lifted to a true homomorphism. Given the Kirchberg-Phillips result, one is still left with the question of determining when two Kirchberg algebras are KK-equivalent. But those of "Cuntz type" (like $\mathcal{O}_n$) lie in $\mathcal{B}$, and Kirchberg and Phillips show that $\forall$ abelian groups G_0 and G_1 and $\forall g \in G_0$, there is a nonunital Kirchberg algebra $A \in \mathcal{B}$ with these K-groups, and there is a unital Kirchberg algebra $A \in \mathcal{B}$ with these K-groups and with $[1_A] = g$. By the UCT, these algebras are classified by their K-groups.

The opposite extreme: stably finite algebras

The original work on the Elliott program dealt with the opposite extreme: stably finite algebras. Here again, KK can play a useful role. Here is a typical result from the vast literature:

Theorem (Elliott)

If A and B are C^*-algebras of real rank 0 which are inductive limits of certain "basic building blocks", then any $x \in KK(A, B)$ preserving the "graded dimension range" can be lifted to a $*$-homomorphism. If x is a KK-equivalence, it can be lifted to an isomorphism.

This theorem applies for example to the irrational rotation algebras A_θ.