EXAMPLES AND APPLICATIONS OF NONCOMMUTATIVE GEOMETRY AND K-THEORY

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Abstract. These are informal notes from my course at the 3$^{\text{era}}$ Escuela de Invierno Luis Santaló-CIMPA Research School on Topics in Noncommutative Geometry. Feedback, especially from participants at the course, is very welcome.

The course basically is divided into two (related) sections. Lectures 1–3 deal with Kasparov's KK-theory and some of its applications. Lectures 4–5 deal with one of the most fundamental examples in noncommutative geometry, the noncommuative 2-torus.

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Lecture 1. Introduction to Kasparov's KK-theory

1.1. **Why KK?** KK-theory is a bivariant version of topological K-theory, defined for C^*-algebras, with or without a group action. It can be defined for either real or complex algebras, but in these notes we will stick to complex algebras for simplicity. Thus if A and B are complex C^*-algebras, subject to a minor technical requirement (that B be σ-unital, which is certainly the case if it is either unital or separable), an abelian group $KK(A, B)$ is defined, with the property that $KK(\mathbb{C}, B) = K(B) = K_0(B)$ if the first algebra A is just the scalars. (For the basic properties of K_0, I refer you to the courses by Reich and Karoubi.) The theory was defined by Gennadi Kasparov in a remarkable series of papers: [33, 34, 35]. However, the definition at first seems highly technical and unmotivated, so it's worth first seeing where the theory comes from and why one might be interested in it. For purposes of this introduction, we will only be concerned with the case where A and B are commutative. Thus $A = C_0(X)$ and $B = C_0(Y)$, where X and Y are locally compact Hausdorff spaces. We will abbreviate $KK(C_0(X), C_0(Y))$ to $KK(X, Y)$. It is worth pointing out that the study of $KK(X, Y)$ (without considering $KK(A, B)$ more generally) is already highly nontrivial, and encompasses most of the features of the general theory. Note that we expect to have $KK(\mathbb{C}, C_0(Y)) = KK(\text{pt}, Y) = K(Y)$, the K-theory of Y with compact support. Recall that this is the Grothendieck group of complexes of vector bundles that are exact off a compact set. It's actually enough to take complexes of length 2, so an element of $K(Y)$ is represented by a pair of vector bundles V and V' over Y, together with a morphism of vector bundles $V \xrightarrow{\varphi} V'$ that is an isomorphism off a compact set. Alternatively, $K(Y)$ can be identified with the reduced K-theory $\widetilde{K}(Y_+)$ of the one-point compactification Y_+ of Y.

A good place to start in trying to understand KK is Atiyah's paper [4] on the Bott periodicity theorem. Bott periodicity, or more generally, the Thom isomorphism theorem for a complex vector bundle, asserts that if $p\colon E \to X$ is a complex vector bundle (more generally, one could take an even-dimensional real vector bundle with a spin^c structure), then there is a natural isomorphism $\beta_E\colon K(X) \to K(E)$, called the Thom isomorphism in K-theory. In the special case where $X = \text{pt}$, E is just $\mathbb{C}^n$ for some n, and we are asserting that there is a natural isomorphism $\mathbb{Z} = K(\text{pt}) \to K(\mathbb{C}^n) = K(\mathbb{R}^{2n}) = \widetilde{K}(S^{2n})$, the Bott periodicity map. The map β_E can be described by the formula $\beta_E(a) = p^*(a) \cdot \tau_E$. Here $p^*(a)$ is the pull-back of $a \in K(X)$ to E. Since a had compact support, $p^*(a)$ has compact support in the base direction of E, but is constant on fibers of p, so it certainly does not have compact support in the fiber direction. However, we can multiply it by the *Thom class* τ_E, which does have compact support along the fibers, and the product will have compact support in both directions, and will thus give a class in $K(E)$ (remember that since E is necessarily noncompact, assuming $n > 0$, we need to use K-theory with compact support). The Thom class τ_E, in turn, can be described [59, §3] as an explicit complex $\bigwedge^\bullet p^*E$ over E. The vector bundles in this complex are the exterior powers of E pulled back from X to E, and the map at a point $e \in E_x$ from $\bigwedge^j E_x$ to $\bigwedge^{j+1} E_x$ is simply exterior product with e. This complex has compact support in the fiber directions since it is exact off the zero-section of E. (If $e \neq 0$, then the kernel of $e \wedge \_\_$ is spanned by products $e \wedge \omega$.)

So far this is all simple vector bundle theory and KK is not needed. But it comes in at the next step. How do we prove that β_E is an isomorphism? The simplest way would be to construct an inverse map $\alpha_E\colon K(E) \to K(X)$. But there is no easy way to describe such a map using topology alone. As Atiyah recognized, the easiest way to construct α_E uses elliptic operators, in fact the family of Dolbeault operators along the fibers of E. Thus whether we like it or not, some analysis comes in at this stage. In more modern language, what we really want is the class α_E in $KK(E, X)$ corresponding to this family of operators, and the verification of the Thom isomorphism theorem is a Kasparov product calculation, the fact that α_E is a KK inverse to the class $\beta_E \in KK(X, E)$ described (in slightly other terms) before. Atiyah also noticed [4] that it's really just enough (because of certain identities about products) to prove that α_E is a one-way inverse to β_E, or in other words, in the language of Kasparov theory, that $\beta_E \otimes_E \alpha_E = 1_X$. This comes down to an index calculation, which because of naturality comes down to the single calculation $\beta \otimes_{\mathbb{C}} \alpha = 1 \in KK(\mathrm{pt}, \mathrm{pt})$ when X is a point and $E = \mathbb{C}$, which amounts to the Riemann-Roch theorem for $\mathbb{CP}^1$.

What then is $KK(X, Y)$ when X and Y are locally compact spaces? An element of $KK(X, Y)$ defines a map of K-groups $K(X) \to K(Y)$, but is more than this; it is in effect a *natural* family of maps of K-groups $K(X \times Z) \to K(Y \times Z)$ for arbitrary Z. Naturality of course means that one gets a natural transformation of functors, from $Z \mapsto K(X \times Z)$ to $Z \mapsto K(Y \times Z)$. (Nigel Higson has pointed out that one can use this in reverse to *define* $KK(X, Y)$ as a *natural* family of maps of K-groups $K(X \times Z) \to K(Y \times Z)$ for arbitrary Z. The reason why this works will be explained in Lecture 3 in this series.) In particular, since $KK(X \times \mathbb{R}^j)$ can be identified with $K^{-j}(X)$, an element of $KK(X, Y)$ defines a graded map of K-groups $K^j(X) \to K^j(Y)$, at least for $j \leq 0$ (but then for arbitrary j because of Bott periodicity). The example of Atiyah's class $\alpha_E \in KK(E, X)$, based on a family of elliptic operators over E parametrized by X, shows that one gets an element of the bivariant K-group $KK(X, Y)$ from a family of elliptic operators over X parametrized by Y. The element that one gets should be invariant under homotopies of such operators. Hence Kasparov's definition of $KK(A, B)$ is based on a notion of homotopy classes of generalized elliptic operators for the first algebra A, "parametrized" by the second algebra B (and thus commuting with a B-module structure).

1.2. **Kasparov's original definition.** As indicated above in Section 1.1, an element of $KK(A, B)$ is roughly speaking supposed to be a homotopy class of families of elliptic pseudodifferential operators over A parametrized by B. For technical reasons, it's convenient to work with self-adjoint bounded operators[1], but it's well-known that the most interesting elliptic operators send sections of one vector bundle to sections of another. The way to get around this is to take our operators to be self-adjoint, but odd with respect to a grading, i.e., of the form

$$T = T^* = \begin{pmatrix} 0 & F^* \\ F & 0 \end{pmatrix}. \tag{1.1}$$

[1]Often we want to apply the theory to self-adjoint differential operators D, which are never bounded on L^2 spaces. The trick is to replace D by $D(1 + D^2)^{-\frac{1}{2}}$, which has the same index theory as D and is bounded.

The operator F here really does act between different spaces, but T, built from F and F^*, is self-adjoint, making it easier to work with. Then we need various conditions on T that correspond to the terms "elliptic," "pseudodifferential," and "parametrized by B." So this boils down to the following. A class in $KK(A,B)$ is represented by a *Kasparov A-B-bimodule*, that is, a $\mathbb{Z}/2$-graded (right) Hilbert B-module $\mathcal{H}=\mathcal{H}_0\oplus\mathcal{H}_1$, together with a B-linear operator $T\in\mathcal{L}(\mathcal{H})$ of the form (1.1), and a (grading-preserving) $*$-representation ϕ of A on $\mathcal{H}$, subject to the conditions that $\phi(a)(T^2-1)\in\mathcal{K}(\mathcal{H})$ and $[\phi(a),T]\in\mathcal{K}(\mathcal{H})$ for all $a\in A$. These conditions require a few comments. The condition that $\phi(a)(T^2-1)\in\mathcal{K}(\mathcal{H})$ is "ellipticity" and the condition that $[\phi(a),T]\in\mathcal{K}(\mathcal{H})$ is "pseudolocality." If $B=\mathbb{C}$, a Hilbert B-module is just a Hilbert space, $\mathcal{L}(\mathcal{H})$ is the set of bounded linear operators on $\mathcal{H}$, and $\mathcal{K}$ is the set of compact operators on $\mathcal{H}$. If $B=C_0(Y)$, a Hilbert B-module is equivalent to a continuous field of Hilbert spaces over Y. In this case, $\mathcal{K}(\mathcal{H})$ is the associated set of continuous fields of compact operators, while $\mathcal{L}(\mathcal{H})$ consists of continuous fields (continuity taken in the strong-$*$ operator topology) of bounded Hilbert space operators. If X is another locally compact space, then it is easy to see that Kasparov's conditions are an abstraction of a continuous family of elliptic pseudolocal Hilbert space operators over X, parametrized by Y. Finally, if B is arbitrary, a Hilbert B-module means a right B-module equipped with a B-valued inner product $\langle \_\_, \_\_\rangle_B$, right B-linear in the second variable, satisfying $\langle\xi,\eta\rangle_B=\langle\eta,\xi\rangle_B^*$ and $\langle\xi,\xi\rangle_B\geq 0$ (in the sense of self-adjoint elements of B), with equality only if $\xi=0$. Such an inner product gives rise to a norm on $\mathcal{H}$: $\|\xi\|=\|\langle\xi,\xi\rangle_B\|_B^{1/2}$, and we require $\mathcal{H}$ to be complete with respect to this norm. Given a Hilbert B-module $\mathcal{H}$, there are two special C^*-algebras associated to it. The first, called $\mathcal{L}(\mathcal{H})$, consists of bounded B-linear operators a on $\mathcal{H}$, admitting an adjoint a^* with the usual property that $\langle a\xi,\eta\rangle_B=\langle\xi,a^*\eta\rangle_B$ for all $\xi,\eta\in\mathcal{H}$. Unlike the case where $B=\mathbb{C}$, existence of an adjoint is not automatic, so it must be explicitly assumed. Then inside $\mathcal{L}(\mathcal{H})$ is the ideal of *B-compact operators*. This is the closed linear span of the "rank-one operators" $T_{\xi,\eta}$ defined by $T_{\xi,\eta}(\nu)=\xi\langle\eta,\nu\rangle_B$. Note that

$$\begin{aligned}
\langle T_{\xi,\eta}(\nu),\omega\rangle_B &= \langle\xi\langle\eta,\nu\rangle_B,\omega\rangle_B=\langle\omega,\xi\langle\eta,\nu\rangle_B\rangle_B^*\\
&=\left(\langle\omega,\xi\rangle_B\langle\eta,\nu\rangle_B\right)^*=\langle\eta,\nu\rangle_B^*\langle\omega,\xi\rangle_B^*\\
&=\langle\nu,\eta\rangle_B\langle\xi,\omega\rangle_B=\langle\nu,\eta\langle\xi,\omega\rangle_B\rangle_B\\
&=\langle\nu,T_{\eta,\xi}(\omega)\rangle_B,
\end{aligned}$$

so that $T_{\xi,\eta}^*=T_{\eta,\xi}$. It is also obvious that if $a\in\mathcal{L}(\mathcal{H})$, then $aT_{\xi,\eta}=T_{a\xi,\eta}$, while $T_{\xi,\eta}a=T_{\eta,\xi}^*\big(a^*\big)^*=\big(a^*T_{\eta,\xi}\big)^*=T_{a^*\eta,\xi}^*=T_{\xi,a^*\eta}$, so these rank-one operators generate an ideal in $\mathcal{L}(\mathcal{H})$, which is just the usual ideal of compact operators in case $B=\mathbb{C}$. For more on Hilbert C^*-modules and the C^*-algebras acting on them, see [40] or [46, Ch. 2].

The simplest kind of Kasparov bimodule is associated to a homomorphism $\phi\colon A\to B$. In this case, we simply take $\mathcal{H}=\mathcal{H}_0=B$, viewed as a right B-module, with the B-valued inner product $\langle b_1,b_2\rangle_B=b_1^*b_2$, and take $\mathcal{H}_1=0$ and $T=0$. In this case, $\mathcal{L}(\mathcal{H})=M(B)$ (the multiplier algebra of B, the largest C^*-algebra containing B as an essential ideal), and $\mathcal{K}(\mathcal{H})=B$. So ϕ maps A into $\mathcal{K}(\mathcal{H})$, and even though $T=0$, the condition that $\phi(a)(T^2-1)\in\mathcal{K}(\mathcal{H})$ is satisfied for any $a\in A$.

One special case which is especially important is the case where $A = B$ and ϕ is the identity map. The above construction then yields a distinguished element $1_A \in KK(A, A)$, which will play an important role later.

In applications to index theory, Kasparov A-B-bimodules typically arise from elliptic (or hypoelliptic) pseudodifferential operators. However, there are other ways to generate Kasparov bimodules which we will discuss in Section 1.4 below.

So far we have explained what the *cycles* are for KK-theory, but not the equivalence relation that determines when two such cycles give the same KK-element. First of all, there is a natural associative addition on Kasparov bimodules, obtained by taking the direct sum of Hilbert B-modules and the block direct sum of homomorphisms and operators. Then we divide out by the equivalence relation generated by addition of *degenerate* Kasparov bimodules (those for which for all $a \in A$, $\phi(a)(T^2 - 1) = 0$ and $[\phi(a), T] = 0$) and by *homotopy.* (A homotopy of Kasparov A-B-bimodules is just a Kasparov A-$C([0, 1], B)$-bimodule.) Then it turns out that the resulting semigroup $KK(A, B)$ is actually an abelian group, with inversion given by reversing the grading, i.e., reversing the roles of $\mathcal{H}_0$ and $\mathcal{H}_1$, and interchanging F and F^*. Actually, it was not really necessary to divide out by degenerate bimodules, since if $(\mathcal{H}, \phi, T)$ is degenerate, then $(C_0((0, 1], \mathcal{H})$ (along with the action of A and the operator which are given by ϕ and T at each point of $(0, 1])$ is a homotopy from $(\mathcal{H}, \phi, T)$ to the 0-module.

An interesting exercise is to consider what happens when $A = \mathbb{C}$ and B is a unital C^*-algebra. Then if $\mathcal{H}_0$ and $\mathcal{H}_1$ are finitely generated projective (right) B-modules and we take $T = 0$ and ϕ to be the usual action of $\mathbb{C}$ by scalar multiplication, we get a Kasparov $\mathbb{C}$-B-bimodule corresponding to the element $[\mathcal{H}_0] - [\mathcal{H}_1]$ of $K_0(B)$. With some work one can show that this gives an isomorphism between the Grothendieck group $K_0(B)$ of usual K-theory and $KK(\mathbb{C}, B)$. By considering what happens when one adjoins a unit, one can then show that there is still a natural isomorphism between $K_0(B)$ and $KK(\mathbb{C}, B)$, even if B is nonunital.

Another important special case is when A and B are Morita equivalent in the sense of Rieffel [47, 50] — see [46] for a very good textbook treatment. That means we have an A-B-bimodule X with the following special properties:

(1) X is a right Hilbert B-module and a left Hilbert A-module.
(2) The left action of A is by bounded adjointable operators for the B-valued inner product, and the right action of B is by bounded adjointable operators for the A-valued inner product.
(3) The A- and B-valued inner products on X are compatible in the sense that if $\xi, \eta, \nu \in X$, then ${}_A\langle \xi, \eta\rangle \nu = \xi \langle \eta, \nu\rangle_B$.
(4) The inner products are "full," in the sense that the image of ${}_A\langle \_\_, \_\_\rangle$ is dense in A, and the image of $\langle \_\_, \_\_\rangle_B$ is dense in B.

Under these circumstances, X defines classes in $[X] \in KK(A, B)$ and $[\widetilde{X}] \in KK(B, A)$ which are inverses to each other (with respect to the product discussed below in Section 1.3). Thus as far as KK-theory is concerned, A and B are essentially equivalent. The construction of $[X]$ and of $[\widetilde{X}]$ is fairly straightforward; for example, to construct $[X]$, take $\mathcal{H}_0 = X$ (viewed as a right Hilbert B-module), $\mathcal{H}_1 = 0$, and $T = 0$, and let $\phi\colon A \to \mathcal{L}(\mathcal{H})$ be the left action of A (which factors through $\mathcal{L}(\mathcal{H})$ by axiom (2)). By axiom (4) (which is really the key property), any element of A can be approximated by linear combinations of inner products ${}_A\langle \xi, \eta\rangle$.

For such an inner product, we have

$$\phi({}_A\langle \xi, \eta\rangle)\nu = \xi\langle \eta, \nu\rangle_B = T_{\xi,\eta}(\nu),$$

so the action of A on $\mathcal{H}$ is by operators in $\mathcal{K}(\mathcal{H})$, which is what is needed for the conditions for a Kasparov bimodule.

The prototype example of a Morita equivalence has $A = \mathbb{C}$, $B = \mathcal{K}(\mathcal{H})$ (we usually drop the $\mathcal{H}$ and just write $\mathcal{K}$ if the Hilbert space is infinite-dimensional and separable), and $X = \mathcal{H}$, with the B-valued inner product taking a pair of vectors in $\mathcal{H}$ to the corresponding rank-one operator. Thus from the point of KK-theory, $\mathbb{C}$ and $\mathcal{K}$ are essentially indistinguishable, and so are B and $B \otimes \mathcal{K}$ for any B. There is a converse [10]; separable C^*-algebras A and B are Morita equivalent if and only if $A \otimes \mathcal{K}$ and $B \otimes \mathcal{K}$ are isomorphic. (This condition, called *stable isomorphism*, is obviously satisfied by B and $B \otimes \mathcal{K}$, since $(B \otimes \mathcal{K}) \otimes \mathcal{K} \cong B \otimes (\mathcal{K} \otimes \mathcal{K}) \cong B \otimes \mathcal{K}$.) However, a Morita equivalence between A and B leads directly to a KK-equivalence, but not directly to an isomorphism $A \otimes \mathcal{K} \cong B \otimes \mathcal{K}$ (which requires some arbitrary choices).

The most readable references for the material of this section are the book by Blackadar [6], Chapter VIII, and the "primer" of Higson [29].

1.3. **Connections and the product.** The hardest aspect of Kasparov's approach to KK is to prove that there is a well-defined, functorial, bilinear, and associative product $\otimes_B \colon KK(A, B) \times KK(B, C) \to KK(A, C)$. There is also an *external* product $\boxtimes\colon KK(A, B) \times KK(C, D) \to KK(A \otimes C, B \otimes D)$, where $\otimes$ denotes the completed tensor product. (For our purposes, the *minimal* or *spatial* C^*-tensor product will suffice.) The external product is actually built from the usual product using an operation called *dilation* (external product with 1). We can dilate a class $a \in KK(A, B)$ to a class $a \boxtimes 1_C \in KK(A \otimes C, B \otimes C)$, by taking a representative $(\mathcal{H}, \phi, T)$ for a to the bimodule $(\mathcal{H} \otimes C, \phi \otimes 1_C, T \otimes 1)$. Similarly, we can dilate a class $b \in KK(C, D)$ (on the other side) to a class $1_B \boxtimes b \in KK(B \otimes C, B \otimes D)$. Then

$$a \boxtimes b = (a \boxtimes 1_C) \otimes_{B\otimes C} (1_B \boxtimes b) \in KK(A \otimes C, B \otimes D),$$

and one can check that this is the same as what one gets by computing in the other order as $(1_A \boxtimes b) \otimes_{A\otimes D} (a \boxtimes 1_D)$.

The *Kasparov products*, as they are called, encompass the usual cup and cap products relating K-theory and K-homology. For example, the cup product in ordinary topological K-theory for a compact space X, $\cup\colon K(X) \times K(X) \to K(X)$, is a composite of two products. Given $a \in K(X) = KK(\mathbb{C}, C(X))$ and $b \in K(X) = KK(\mathbb{C}, C(X))$, we first form the external product $a \boxtimes b \in KK(\mathbb{C}, C(X) \otimes C(X)) = KK(\mathbb{C}, C(X \times X))$. Then we have

$$a \cup b = (a \boxtimes b) \otimes_{C(X\times X)} \Delta,$$

where $\Delta \in KK(C(X \times X), C(X))$ is the class of the homomorphism defined by restriction of functions on $X \times X$ to the diagonal copy of X.

In any event, it still remains to construct the product $KK(A, B) \times KK(B, C) \to KK(A, C)$. Suppose we have classes represented by $(\mathcal{E}_1, \phi_1, T_1)$ and $(\mathcal{E}_2, \phi_2, T_2)$, where $\mathcal{E}_1$ is a right Hilbert B-module, $\mathcal{E}_2$ is a right Hilbert C-module, $\phi_1\colon A \to \mathcal{L}(\mathcal{E}_1)$, $\phi_2\colon B \to \mathcal{L}(\mathcal{E}_2)$, T_1 essentially commutes with the image of ϕ_1, and T_2 essentially commutes with the image of ϕ_2. It is clear that we want to construct the product using $\mathcal{H} = \mathcal{E}_1 \otimes_{B,\phi_2} \mathcal{E}_2$ and $\phi = \phi_1 \otimes 1\colon A \to \mathcal{L}(\mathcal{H})$. The main difficulty is getting the correct operator T. In fact there is no canonical choice; the choice is only unique up to homotopy. The most convenient method of doing the construction seems to be using the notion of a *connection* due to Connes and Skandalis [13], nicely explained in [6, §18] or [61].

To motivate this, let's just consider a simple example that comes up in index theory, the construction of an "elliptic operator with coefficients in a vector bundle." Let T be an elliptic operator on a compact manifold M, which we take to be a bounded operator of the form (1.1) (acting on a $\mathbb{Z}/2$-graded Hilbert space $\mathcal{H}$), and let E be a complex vector bundle over M. Often we want to form T_E, the same operator with coefficients in the vector bundle E. This is actually a special case of the Kasparov product. The sections $\Gamma(M, E)$ are a finitely generated projective $C(M)$-module $\mathcal{E}$; since $C(M)$ is commutative, we can regard this as a $C(M)$-$C(M)$-bimodule, with the same action on the left and on the right. Then $\mathcal{E}$ (concentrated entirely in degree 0, together with the 0-operator), defines a KK-class $[[E]] \in KK(M, M)$, while T defines a class $[T]$ in $KK(M, \mathrm{pt})$. Note that forgetting the left $C(M)$-action on $\mathcal{E}$ is the same as composing with inclusion of the scalars $\mathbb{C} \hookrightarrow C(M)$ to get from $[[E]]$ a class $[E] \in KK(\mathrm{pt}, M) = K(M)$, which is the usual K-theory class of E. The class of the operator T_E will be the Kasparov product $[[E]] \otimes_M [T] \in KK(M, \mathrm{pt})$. Defining the operator, however, requires a choice of connection on the bundle E. One way to get this is to embed E as a direct summand in a trivial bundle $M \times \mathbb{C}^n$. Then orthogonal projection onto E is given by a self-adjoint projection $p \in C(M, M_n(\mathbb{C}))$. We can certainly form $T \otimes 1$ acting on $\mathcal{H} \otimes \mathbb{C}^n$, on which we have an obvious action of $C(M) \otimes M_n(\mathbb{C})$, but there is no reason why $T \otimes 1$ and p should commute, so there is no "natural" cut-down of T to E. Thus we simply take the compression $T' = p(T \otimes 1)p$ acting on $\mathcal{H}' = p\mathcal{H}$ with the obvious action ϕ' of $C(M)$. Since T commutes with the action of $C(M)$ up to compact operators, the commutator $[p, T \otimes 1]$ is also compact, so T' satisfies the requirements that $(T')^2 - 1 \in \mathcal{K}(\mathcal{H}')$ and $[\phi'(f), T'] \in \mathcal{K}(\mathcal{H}')$. Its Kasparov class is well-defined, even though there is great freedom in choosing the operator (corresponding to the freedom to embed E in a trivial bundle in many different ways). (When n is large enough, all vector bundle embeddings of E into $M \times \mathbb{C}^n$ are isotopic, and thus the operators obtained by the above construction will be homotopic in a way preserving the Kasparov requirements.)

1.4. **Cuntz's approach.** Joachim Cuntz noticed in [17] that all Kasparov bimodules can be taken to come from the basic notion of a *quasihomomorphism* between C^*-algebras A and B. A quasihomomorphism $A \rightrightarrows D \trianglerighteq B$ is roughly speaking a formal difference of two homomorphisms $f_\pm \colon A \to D$, neither of which maps into B itself, but which agree modulo an ideal isomorphic to B. Thus $a \mapsto f_+(a) - f_-(a)$ is a linear map $A \to B$. Suppose for simplicity (one can always reduce to this case) that $D/B \cong A$, so that $f_\pm$ are two splittings for an extension

$$0 \to B \to D \to A \to 0.$$

Then for any *split-exact* functor F from C^*-algebras to abelian groups (meaning it sends split extensions to short exact sequences — an example would be $F(A) = K(A \otimes C)$ for some coefficient algebra C), we get an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & F(B) \arrow[r] & F(D) \arrow[r] & F(A) \arrow[l, bend right, "(f_+)_*"'] \arrow[l, bend left, "(f_-)_*"] \arrow[r] & 0. \end{tikzcd}$$

Thus $(f_+)_* - (f_-)_*$ gives a well-defined homomorphism $F(A) \to F(B)$, which we might well imagine should come from a class in $KK(A, B)$. (Think about Section 1.1, where we mentioned Higson's idea of *defining* $KK(X, Y)$ in terms of natural transformations of functors, from $Z \mapsto K(X \times Z)$ to $Z \mapsto K(Y \times Z)$. We will certainly get such a natural transformation from a quasihomomorphism $C_0(X) \rightrightarrows D \trianglerighteq C_0(Y) \otimes \mathcal{K}$, since $C_0(Y) \otimes \mathcal{K}$ and Y have the same K-theory.) And indeed, given a quasihomomorphism as above, we get a Kasparov A-B-bimodule, with $B \oplus B$ as the Hilbert B-module (with the obvious grading), with $\phi\colon\ A \to \mathcal{L}(B \oplus B)$ defined by

$$\begin{pmatrix} f_+ & 0 \\ 0 & f_- \end{pmatrix}, \quad \text{and} \quad T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The "almost commutation" relation is

$$\left[\begin{pmatrix} f_+(a) & 0 \\ 0 & f_-(a) \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & f_+(a) - f_-(a) \\ f_-(a) - f_+(a) & 0 \end{pmatrix} \in \mathcal{K}(B \oplus B),$$

since $\mathcal{K}(B \oplus B) = M_2(B)$. In the other direction, given a Kasparov A-B-bimodule, one can add on a degenerate bimodule and do a homotopy to reduce it to something roughly of this form, showing that all of $KK(A, B)$ comes from quasihomomorphisms (see [6, §17.6]).

The quasihomomorphism approach to KK makes it possible to define $KK(A, B)$ in a seemingly simpler way [18]. To do this, Cuntz observed that a quasihomomorphism $A \rightrightarrows D \trianglerighteq B$ factors through a *universal* algebra qA constructed as follows. Start with the *free product* C^*-algebra $QA = A * A$, the completion of linear combinations of words in two copies of A. There is an obvious surjective homomorphism $QA \twoheadrightarrow A$ obtained by identifying the two copies of A. The kernel of $QA \twoheadrightarrow A$ is called qA, and if

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r] & D \arrow[r, shift left] \arrow[r, shift right] & A \arrow[l, bend right, "f_+"'] \arrow[l, bend left, "f_-"] \arrow[r] & 0 \end{tikzcd}$$

is a quasihomomorphism, we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & qA \arrow[r] \arrow[d] & QA \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r] & D \arrow[r] & A \arrow[r] & 0, \end{tikzcd}$$

with the first copy of A in QA mapping to D via f_+, and the second copy of A in QA mapping to D via f_-. Thus homotopy classes of (strict) quasihomomorphisms from A to B can be identified with homotopy classes of $*$-homomorphisms from qA to B, and $KK(A, B)$ turns out to be simply the set of homotopy classes of $*$-homomorphisms from qA to $B \otimes \mathcal{K}$.

1.5. **Higson's approach.** There is still another very elegant approach to KK-theory due to Nigel Higson [28]. Namely, one can construct an additive category **KK** whose objects are the separable C^*-algebras, and where the morphisms from A to B are given by $KK(A, B)$. Associativity and bilinearity of the Kasparov product, along with properties of the special elements $1_A \in KK(A, A)$, ensure that this is indeed an additive category. What Higson did is to give an alternative construction of this category. Namely, start with the *homotopy category* of separable C^*-algebras, where the morphisms from A to B are the homotopy classes of $*$-homomorphisms $A \to B$. Then **KK** is the smallest additive category with the same objects, these morphisms, plus enough additional morphisms so that two basic properties are satisfied:

(1) Matrix stability. If A is an object in **KK** (that is, a separable C^*-algebra) and if e is a rank-one projection in $\mathcal{K} = \mathcal{K}(\mathcal{H})$, $\mathcal{H}$ a separable Hilbert space, then the homomorphism $a \mapsto a \otimes e$, viewed as an element of $\mathrm{Hom}(A, A \otimes \mathcal{K})$, is an equivalence in **KK**, i.e., has an inverse in $KK(A \otimes \mathcal{K}, A)$.

(2) Split exactness. If $0 \longrightarrow A \longrightarrow B \underset{\xleftarrow{s}}{\longrightarrow} C \longrightarrow 0$ is a split short exact sequence of separable C^*-algebras, then for any separable C^*-algebra D,

$$\begin{tikzcd} 0 \arrow[r] & KK(D,A) \arrow[r] & KK(D,B) \arrow[r] & KK(D,C) \arrow[l, bend right, "s_*"'] \arrow[r] & 0 \end{tikzcd}$$

and

$$\begin{tikzcd} 0 \arrow[r] & KK(C,D) \arrow[r] & KK(B,D) \arrow[l, bend right, "s^*"'] \arrow[r] & KK(A,D) \arrow[r] & 0 \end{tikzcd}$$

are split exact.

Incidentally, if one just starts with the homotopy category and requires (1), matrix stability, that is already enough to guarantee that the resulting category has Hom-sets which are commutative monoids and that composition is bilinear [54, Theorem 3.1]. So it's not asking much additional to require that one have an additive category.

The proof of Higson's theorem very much depends on the Cuntz construction in Section 1.4 above.

Lecture 2. K-THEORY AND KK-THEORY OF CROSSED PRODUCTS

2.1. **Equivariant Kasparov theory.** Many of the interesting applications of KK-theory involve actions of groups in some way. For this, Kasparov also invented an equivariant version of the theory. In what follows, G will always be a second-countable locally compact group. A G-C^*-algebra will mean a C^*-algebra A, along with an action of G on A by $*$-automorphisms, continuous in the sense that the map $G \times A \to A$ is jointly continuous. (Another way to say this is that if we give $\operatorname{Aut} A$ the topology of pointwise convergence, then $G \to \operatorname{Aut} A$ is a continuous group homomorphism.) If G is compact, making the theory equivariant is rather straightforward. We just require all algebras and Hilbert modules to be equipped with G-actions, we require $\phi\colon A \to \mathcal{L}(\mathcal{H})$ to be G-equivariant, and we require the operator $T \in \mathcal{L}(\mathcal{H})$ to be G-invariant. This produces groups $KK^G(A, B)$ for (separable, say) G-C^*-algebras A and B, and the same argument as before shows that $KK^G(\mathbb{C}, B) \cong K_0^G(B)$, equivariant K-theory. (In the commutative case, this is described in [59]. A general description may be found in [6, §11].) In particular, $KK^G(\mathbb{C}, \mathbb{C}) \cong R(G)$, the *representation ring* of G, in other words, the Grothendieck group of the category of finite-dimensional representations of G, with product coming from the tensor product of representations. The rings $R(G)$ are commutative, Noetherian if G is a compact Lie group, and often easily computable; for example, if G is compact and abelian, $R(G) \cong \mathbb{Z}[\widehat{G}]$, the group ring of the Pontrjagin dual. If G is a compact connected Lie group with maximal torus T and Weyl group $W = N_G(T)/T$, then $R(G) \cong R(T)^W \cong \mathbb{Z}[\widehat{T}]^W$. The properties of the Kasparov product all go through without change, since it is easy to "average" things with respect to a compact group action. Then Kasparov product with $KK^G(\mathbb{C}, \mathbb{C})$ makes all KK^G-groups into modules over the ground ring $R(G)$, so that homological algebra of the ring $R(G)$ comes into play in understanding the equivariant KK-category $\mathbf{KK}^G$.

When G is noncompact, the definition and properties of KK^G are considerably more subtle, and were worked out in [35]. A shorter exposition may be found in [36]. The problem is that in this case, topological vector spaces with a continuous G-action are very rarely completely decomposable, and there are rarely enough G-equivariant operators to give anything useful. Kasparov's solution was to work with G-continuous rather than G-equivariant Hilbert modules and operators; rather remarkably, these still give a useful theory with all the same formal properties as before. The KK^G-groups are again modules over the commutative ring $R(G) = KK^G(\mathbb{C}, \mathbb{C})$, though this ring no longer has such a simple interpretation as before, and in fact, is not known for most connected semisimple Lie groups.

A few functorial properties of the KK^G-groups will be needed below, so we just mention a few of them. First of all, if H is a closed subgroup of G, then any G-C^*-algebra is by restriction also an H-C^*-algebra, and we have restriction maps $KK^G(A, B) \to KK^H(A, B)$. To go the other way, we can "induce" an H-C^*-algebra A to get a G-C^*-algebra $\operatorname{Ind}_H^G(A)$, defined by

$$\begin{aligned}\operatorname{Ind}_H^G(A) = \{ f \in C(G, A) \mid f(gh) = h \cdot f(g) \quad & \forall g \in G, h \in H, \\ & \|f(g)\| \to 0 \text{ as } g \to \infty \mod H \}.\end{aligned}$$

The induced action of G on $\operatorname{Ind}_H^G(A)$ is just left translation. An "imprimitivity theorem" due to Green shows that $\operatorname{Ind}_H^G(A) \rtimes G$ and $A \rtimes H$ are Morita equivalent.

If A and B are H-C^*-algebras, we then have an induction homomorphism

$$KK^H(A,B) \to KK^G(\mathrm{Ind}_H^G(A), \mathrm{Ind}_H^G(B)).$$

The last basic operation on the KK^G-groups depends on *crossed products*, so we consider these next.

2.2. **Basic properties of crossed products.** Suppose A is a G-C^*-algebra. Then one can define two new C^*-algebras, called the full and reduced *crossed products* of A by G, which capture the essence of the group action. These are easiest to define when G is discrete and A is unital. Then the full crossed product $A \rtimes_\alpha G$ (we often omit the α if there is no possibility of confusion) is the universal C^*-algebra generated by a copy of A and unitaries u_g, $g \in G$, subject to the commutation condition $u_g a u_g^* = \alpha_g(a)$, where α denotes the action of G on A. The reduced crossed product $A \rtimes_{\alpha,r} G$ is the image of $A \rtimes_\alpha G$ in its "regular representation" π on $L^2(G, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space on which A acts faithfully, say by a representation ρ. Here A acts by $(\pi(a)f)(g) = \rho(\alpha_{g^{-1}}(a))f(g)$ and G acts by left translation. The compatibility condition is satisfied since

$$\begin{aligned}
\pi(u_g)\pi(a)\pi(u_g^*)f(g') &= (\pi(a)\pi(u_g^*)f)(g^{-1}g') \\
&= \rho(\alpha_{g'^{-1}g}(a))(\pi(u_g^*)f)(g^{-1}g') \\
&= \rho(\alpha_{g'^{-1}g}(a))(f(g')) \\
&= \rho(\alpha_{g'^{-1}}(\alpha_g(a))(f(g')) = \pi(\alpha_g(a))f(g').
\end{aligned}$$

In the general case (where A is not necessarily unital and G is not necessarily discrete), the full crossed product is still defined as the universal C^*-algebra for *covariant pairs* of a $*$-representation ρ of A and a unitary representation π of G, satisfying the compatibility condition $\pi(g)\rho(a)\pi(g^{-1}) = \rho(\alpha_g(a))$. It may be constructed by defining a convolution multiplication on $C_c(G, A)$ and then completing in the greatest C^*-algebra norm. The reduced crossed product $A \rtimes_{\alpha,r} G$ is again the image of $A \rtimes_\alpha G$ in its "regular representation" on $L^2(G, \mathcal{H})$. For details of the construction, see [44, §7.6] and [62, Ch. 2].

If $A = \mathbb{C}$, the crossed product $A \rtimes G$ is simply the universal C^*-algebra for unitary representations of G, or the group C^*-algebra $C^*(G)$, and $A \rtimes_r G$ is $C^*_r(G)$, the image of $C^*(G)$ in the left regular representation on $L^2(G)$. The natural map $C^*(G) \twoheadrightarrow C^*_r(G)$ is an isomorphism if and only if G is amenable. When the action α is trivial (factors through the trivial group $\{1\}$), then $A \rtimes G$ is the maximal tensor product $A \otimes_{\max} C^*(G)$ while $A \rtimes_r G$ is the minimal tensor product $A \otimes C^*_r(G)$. Again, the natural map from $A \otimes_{\max} C^*(G)$ to $A \otimes C^*_r(G)$ is an isomorphism if and only if G is amenable.

When A and the action α are arbitrary, the natural map $A \rtimes_\alpha G \twoheadrightarrow A \rtimes_{\alpha,r} G$ is an isomorphism if G is amenable, but also more generally if the action α is amenable in a certain sense. For example, if X is a locally compact G-space, the action is automatically amenable if it is proper, whether or not G is amenable. A good short survey of amenability for group actions may be found in [1].

When X is a locally compact G-space, the crossed product $C_0(G) \rtimes G$ often serves as a good substitute for the "quotient space" X/G in cases where the latter is badly behaved. Indeed, if G acts freely and properly on X, then $C_0(X) \rtimes G$ is Morita equivalent to $C_0(X/G)$. If G acts locally freely and properly on X, then $C_0(X) \rtimes G$ is Morita equivalent to an "orbifold algebra" that encompasses not only the topology of X/G but also the finite isotropy groups. But if the G-action is not proper, X/G may be highly non-Hausdorff, while $C_0(X) \rtimes G$ may be a perfectly well-behaved noncommutative algebra. A key case later on will the one where $X = \mathbb{T}$ is the circle group, $G = \mathbb{Z}$, and the generator of G acts by multiplication by $e^{2\pi i\theta}$. When θ is irrational, every orbit is dense, so X/G is an indiscrete space, and $C(\mathbb{T}) \rtimes \mathbb{Z}$ is what's usually denoted A_θ, an *irrational rotation algebra* or *noncommutative* 2*-torus*.

Now we can explain the relationships between equivariant KK-theory and crossed products. One connection is that if G is discrete and A is a G-C^*-algebra, there is a natural isomorphism $KK^G(A, \mathbb{C}) \cong KK(A \rtimes G, \mathbb{C})$. Dually, if G is compact, there is a natural *Green-Julg isomorphism* [6, §11.7] $KK^G(\mathbb{C}, A) \cong KK(\mathbb{C}, A \rtimes G)$. Still another connection is that there is (for arbitrary G) a functorial homomorphism

$$j\colon\ KK^G(A, B) \to KK(A \rtimes G, B \rtimes G)$$

sending (when $B = A$) 1_A to $1_{A\rtimes G}$. (In fact, j can be viewed as a functor from the equivariant Kasparov category $\mathbf{KK}^G$ to the non-equivariant Kasparov category $\mathbf{KK}$. Later we will study how close it is to being faithful.) There is also a variant of j using reduced crossed products, denoted j_r [35, §3.11]. If $B = \mathbb{C}$ and G is discrete, then j can be identified with the map $KK(A\rtimes G, \mathbb{C}) \to KK(A\rtimes G, C^*(G))$ induced by the inclusion of scalars $\mathbb{C} \hookrightarrow C^*(G)$. (The fact that G is discrete means that $C^*(G)$ is unital.) The map j is split injective in this case since it is split by the map induced by $C^*(G) \to \mathbb{C}$, corresponding to the trivial representation of G. Similarly, if G is compact, then via Green-Julg, j can be identified with the map $KK(\mathbb{C}, A \rtimes G) \to KK(C^*(G), A \rtimes G)$ induced by the map $C^*(G) \to \mathbb{C}$ corresponding to the trivial representation of G. This is again a split injection since $C^*(G)$ splits as the direct sum of $\mathbb{C}$ and summands associated to other representations.

2.3. **The dual action and Takai duality.** When the group G is not just locally compact but also abelian, then it has a Pontrjagin dual group $\widehat{G}$. In this case, given any G-C^*-algebra algebra A, say with α denoting the action of G on A, there is a *dual action* $\widehat{\alpha}$ of $\widehat{G}$ on the crossed product $A \rtimes G$. When A is unital and G is discrete, so that $A \rtimes G$ is generated by a copy of A and unitaries u_g, $g \in G$, the dual action is given simply by

$$\widehat{\alpha}_\gamma(au_g) = au_g\langle g, \gamma\rangle.$$

The same formula still applies in general, except that the elements a and u_g don't quite live in the crossed product but in a larger algebra. The key fact about the dual action is the *Takai duality theorem*: $(A \rtimes_\alpha G) \rtimes_{\widehat{\alpha}} \widehat{G} \cong A \otimes \mathcal{K}(L^2(G))$, and the double dual action $\hat{\hat{\alpha}}$ of $\tilde{\hat{G}} \cong G$ on this algebra can be identified with $\alpha \otimes \operatorname{Ad}\lambda$, where λ is the left regular representation of G on $L^2(G)$. Good expositions may be found in [44, §7.9] and in [62, Ch. 7].

2.4. **Connes' "Thom isomorphism".** Recall that the Thom isomorphism theorem in K-theory (see Section 1.1) asserts that if E is a complex vector bundle over X, there is an isomorphism of K-groups $K(X) \to K(E)$, implemented by a KK-class in $KK(X, E)$. Now if $\mathbb{C}^n$ (or $\mathbb{R}^{2n}$ — there is no difference since we are just considering the additive group structure) acts on X by a trivial action α, then $C_0(X) \rtimes_\alpha \mathbb{C}^n \cong C_0(X) \otimes C^*(\mathbb{C}^n) \cong C_0(X) \otimes C_0(\widehat{\mathbb{C}}^n) \cong C_0(E)$, where E is a trivial rank-n complex vector bundle over X. (We have used Pontrjagin duality and the fact that abelian groups are amenable.) It follows that $K(C_0(X)) \cong K(C_0(X) \rtimes_\alpha \mathbb{C}^n)$. Since any action α of $\mathbb{C}^n$ is homotopic to the trivial action and "K-theory is supposed to be homotopy invariant," that suggests that perhaps $KK(A) \cong KK(A \rtimes_\alpha \mathbb{C}^n)$ for any C^*-algebra A and for any action α of $\mathbb{C}^n$. This is indeed true and the isomorphism is implemented by classes (which are inverse to one another) in $KK(A, A \rtimes_\alpha \mathbb{C}^n)$ and $KK(A \rtimes_\alpha \mathbb{C}^n, A)$. It is clearly enough to prove this in the case $n = 1$, since we can always break a crossed product by $\mathbb{C}^n$ up as an n-fold iterated crossed product.

That A and $A \rtimes_\alpha \mathbb{C}$ are always KK-equivalent or that they at least have the same K-theory, or (this is equivalent since one can always suspend on both sides) that $A \otimes C_0(\mathbb{R})$ and $A \rtimes_\alpha \mathbb{R}$ are always KK-equivalent or that they at least have the same K-theory for any action of $\mathbb{R}$, is called *Connes' "Thom isomorphism"* (with the name "Thom" in quotes since the only connection with the classical Thom isomorphism is the one we have already explained). Connes' original proof is relatively elementary, but only gives an isomorphism of K-groups, not a KK-equivalence, and can be found in [12] or in [20, §10.2].

To illustrate Connes' idea, let's suppose A is unital and we have a class in $K_0(A)$ represented by a projection $p \in A$. (One can always reduce to this special case.) If α were to fix p, then $1 \mapsto p$ gives an equivariant map from $\mathbb{C}$ to A and thus would induce a map of crossed products $\mathbb{C} \rtimes \mathbb{R} \cong C_0(\widehat{\mathbb{R}}) \to A \rtimes_\alpha \mathbb{R}$ or $\mathbb{C} \rtimes \mathbb{C} \cong C_0(\widehat{\mathbb{C}}) \to A \rtimes_\alpha \mathbb{C}$ giving a map on K-theory $\beta\colon \mathbb{Z} \to K_0(A \rtimes \mathbb{C})$. The image of $[p]$ under the isomorphism $K_0(A) \to K_0(A \rtimes \mathbb{C})$ will be $\beta(1)$. So the idea is to show that one can modify the action to one fixing p (using a cocycle conjugacy) without changing the isomorphism class of the crossed product.

There are now quite a number of proofs of Connes' theorem available, each using somewhat different techniques. We just mention a few of them. A proof using K-theory of Wiener-Hopf extensions is given in [49]. There are also fancier proofs using KK-theory. If α is a given action of $\mathbb{R}$ on A and if β is the trivial action, one can try to construct $KK^{\mathbb{R}}$ elements $c \in KK^{\mathbb{R}}((A, \alpha), (A, \beta))$ and $d \in KK^{\mathbb{R}}((A, \beta), (A, \alpha))$ which are inverses of each other in $\mathbf{KK}^{\mathbb{R}}$. Then the morphism j of Section 2.1 sends these to KK-equivalences $j(c)$ and $j(d)$ between $A \rtimes_\alpha \mathbb{R}$ and $A \rtimes_\beta \mathbb{R} \cong A \otimes C_0(\mathbb{R})$.

Another rather elegant approach, using KK-theory but not the equivariant groups, may be found in [26]. Fack and Skandalis use the group $KK^1(A, B)$, which we have avoided so far in order to simplify the theory, but it can be defined with triples $(\mathcal{H}, \phi, T)$ like those used for $KK(A, B)$, but with two modifications:

(1) $\mathcal{H}$ is no longer graded, and there is no grading condition on ϕ.
(2) T is self-adjoint but with no grading condition, and $\phi(a)(T^2 - 1) \in \mathcal{K}(\mathcal{H})$ and $[\phi(a), T] \in \mathcal{K}(\mathcal{H})$ for all $a \in A$.

It turns out that $KK^1(A, B) \cong KK(A \otimes C_0(\mathbb{R}), B)$, and that the Kasparov product can be extended to a graded commutative product on the direct sum of $KK = KK^0$ and KK^1. The product of two classes in KK^1 can by Bott periodicity be taken to land in KK^0.

We can now explain the proof of Fack and Skandalis as follows. They show that for each separable C^*-algebra A with an action α of $\mathbb{R}$, there is a special element $t_\alpha \in KK^1(A, A \rtimes_\alpha \mathbb{R})$ (constructed using a singular integral operator). Note by the way that doing the construction with the dual action and applying Takai duality gives $t_{\widehat{\alpha}} \in KK^1(A\rtimes_\alpha\mathbb{R}, A)$, since $(A\rtimes_\alpha\mathbb{R})\rtimes_{\widehat{\alpha}}\mathbb{R} \cong A\otimes\mathcal{K}$, which is Morita equivalent to A. These elements have the following properties:

(1) (Normalization) If $A = \mathbb{C}$ (so that necessarily $\alpha = 1$ is trivial), then $t_1 \in KK^1(\mathbb{C}, C_0(\mathbb{R}))$ is the usual generator of this group (which is isomorphic to $\mathbb{Z}$).

(2) (Naturality) The elements are natural with respect to equivariant homomorphisms $\rho\colon (A,\alpha) \to (C,\gamma)$, in that if $\bar\rho$ denotes the induced map on crossed products, then $\bar\rho_*(t_\alpha) = \rho^*(t_\gamma) \in KK(A, C\rtimes_\gamma \mathbb{R})$, and similarly, $\bar\rho^*(t_{\widehat{\gamma}}) = \rho_*(t_{\widehat{\alpha}}) \in KK(A\rtimes_\alpha \mathbb{R}, C)$.

(3) (Compatibility with external products) Given $x \in KK(A,B)$ and $y \in KK(C,D)$,

$$(t_{\widehat{\alpha}} \otimes_A x) \boxtimes y = t_{\widehat{\alpha\otimes 1_C}} \otimes_{A\otimes C} (x \boxtimes y).$$

Similarly, given $x \in KK(B,A)$ and $y \in KK(D,C)$,

$$y \boxtimes (x \otimes_A t_\alpha) = (y \boxtimes x) \otimes_{C\otimes A} t_{1_C\otimes\alpha}. \qquad \square$$

Theorem 2.1 (Fack-Skandalis [26])**.** *These properties completely determine t_α, and t_α is a KK-equivalence* (*of degree* 1) *between A and $A\rtimes_\alpha \mathbb{R}$.*

Proof. Suppose we have elements t_α satisfying the properties above. Let us first show that $t_\alpha\otimes_{A\rtimes_\alpha\mathbb{R}} t_{\widehat{\alpha}} = 1_A$. For $s \in \mathbb{R}$, let α^s be the rescaled action $\alpha^s_t = \alpha_{st}$. Then define an action β of $\mathbb{R}$ on $B = C([0,1], A)$ by $(\beta_t f)(s) = \alpha^s_t(f(s))$. Let $g_s\colon B \to A$ be evaluation at s, which is clearly an equivariant map $(B,\beta) \twoheadrightarrow (A,\alpha^s)$. We also get maps $\widehat{g}_s\colon B\rtimes_\beta\mathbb{R} \to A\rtimes_{\alpha^s}\mathbb{R}$, and the double dual map $\hat{\hat{g}}_s$ can be identified with $g_s\otimes 1\colon B\otimes\mathcal{K} \to A\otimes\mathcal{K}$. By Axiom (2), $(\bar g_s)_*(t_\beta) = g_s^*(t_{\alpha^s})$ and $(g_s)_*(t_{\widehat{\beta}}) = \bar g_s^*(t_{\widehat{\alpha}^s})$. Let $\sigma_s = t_{\alpha^s} \otimes_{A\rtimes_{\alpha^s}\mathbb{R}} t_{\widehat{\alpha}^s} \in KK(A,A)$. By associativity of Kasparov products,

$$\begin{aligned}
(g_s)_*\big(t_\beta \otimes_{B\rtimes_\beta\mathbb{R}} t_{\widehat{\beta}}\big) &= t_\beta \otimes_{B\rtimes_\beta\mathbb{R}} \big(t_{\widehat{\beta}} \otimes_B [g_s]\big)\\
&= t_\beta \otimes_{B\rtimes_\beta\mathbb{R}} \big([\bar g_s] \otimes_{A\rtimes_{\alpha^s}\mathbb{R}} t_{\widehat{\alpha}^s}\big)\\
&= \big(t_\beta \otimes_{B\rtimes_\beta\mathbb{R}} [\bar g_s]\big) \otimes_{A\rtimes_{\alpha^s}\mathbb{R}} t_{\widehat{\alpha}^s}\\
&= \big([g_s] \otimes_A t_{\alpha^s}\big) \otimes_{A\rtimes_{\alpha^s}\mathbb{R}} t_{\widehat{\alpha}^s}\\
&= [g_s] \otimes_A \sigma_s.
\end{aligned}$$

Since g_s is a homotopy of maps $B \to A$ and KK is homotopy-invariant, $[g_s] = [g_0]$. But g_0 is a homotopy equivalence with homotopy inverse $f\colon a \mapsto a\otimes 1$, so we see that

$$\sigma_s = [f] \otimes_B \big(t_\beta \otimes_{B\rtimes_\beta\mathbb{R}} t_{\widehat{\beta}}\big) \otimes_B [g_0]$$

is independent of s. In particular, $\sigma_1 = t_\alpha \otimes_{A\rtimes_\alpha\mathbb{R}} t_{\widehat{\alpha}}$ agrees with σ_0, which can be computed to be 1_A by Axioms (1) and (3) since the action of $\mathbb{R}$ is trivial in this case. So $t_\alpha \otimes_{A\rtimes_\alpha\mathbb{R}} t_{\widehat{\alpha}} = 1_A$. Replacing α by $\widehat{\alpha}$ and using Takai duality, this also implies that $t_{\widehat{\alpha}} \otimes_A t_\alpha = 1_{A\rtimes_\alpha\mathbb{R}}$. So t_α and $t_{\widehat{\alpha}}$ give KK-equivalences.

The uniqueness falls out at the same time, since we see from the above that $[g_s] \otimes_A t_{\alpha^s} = t_\beta \otimes_{B\rtimes_\beta\mathbb{R}} [\bar g_s] \in KK(B, A\rtimes_{\alpha^s}\mathbb{R})$, and that all the KK-elements involved are KK-equivalences. Furthermore, we know by Axioms (1) and (3) that $t_{\alpha^0} = 1_A \boxtimes t_1$, where t_1 is the special element of $KK^1(\mathbb{C}, C_0(\mathbb{R}))$ mentioned in Axiom (1). This determines t_β (from the identity $[g_0]\otimes_A t_{\alpha^0} = t_\beta \otimes_{B\rtimes_\beta\mathbb{R}} [\bar g_0]$), and then t_α is determined from the identity $[g_0] \otimes_A t_\alpha = t_\beta \otimes_{B\rtimes_\beta\mathbb{R}} [\bar g_1]$. $\square$

2.5. **The Pimsner-Voiculescu Theorem.** Connes' Theorem from Section 2.4 computes K-theory or KK-theory for crossed products by $\mathbb{R}$. This can be used to compute K-theory or KK-theory for crossed products by $\mathbb{Z}$, using the fact from Section 2.2 that if A is a C^*-algebra equipped with an action α of $\mathbb{Z}$ (or equivalently, a single $*$-automorphism θ, the image of $1 \in \mathbb{Z}$ under the action), then $A \rtimes_\alpha \mathbb{Z}$ is Morita equivalent to $\left(\operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}}(A, \alpha)\right) \rtimes \mathbb{R}$. The algebra $T_\theta = \operatorname{Ind}_{\mathbb{Z}}^{\mathbb{R}}(A, \alpha)$ is often called the *mapping torus* of (A, θ); it can be identified with the algebra of continuous functions $f\colon [0,1] \to A$ with $f(1) = \theta(f(0))$. It comes with an obvious short exact sequence

$$0 \to C_0((0,1), A) \to T_\theta \to A \to 0,$$

for which the associated exact sequence in K-theory has the form

$$\cdots \to K_1(A) \xrightarrow{1-\theta_*} K_1(A) \to K_0(T_\theta) \to K_0(A) \xrightarrow{1-\theta_*} K_0(A) \to \cdots .$$

Since

$$K_0(A \rtimes_\alpha \mathbb{Z}) \cong K_0(T_\theta \rtimes_{\operatorname{Ind}\alpha} \mathbb{R}) \cong K_1(T_\theta),$$

and similarly for K_0, we obtain the *Pimsner-Voiculescu exact sequence*

$$\begin{aligned} &\cdots \to K_1(A) \xrightarrow{1-\theta_*} K_1(A) \to K_1(A \rtimes_\alpha \mathbb{Z}) \to \\ &\to K_0(A) \xrightarrow{1-\theta_*} K_0(A) \to K_0(A \rtimes_\alpha \mathbb{Z}) \to \cdots . \end{aligned} \tag{2.1}$$

Here one can check that the maps $K_j(A) \to K_j(A \rtimes_\alpha \mathbb{Z})$ are induced by the inclusion of A into the crossed product. For another proof, closer to the original argument of Pimsner and Voiculescu, see [20, Ch. 5].

2.6. **The Baum-Connes Conjecture.** The theorems of Connes and Pimsner-Voiculescu on K-theory of crossed products by $\mathbb{R}$ and $\mathbb{Z}$ suggest the question of whether there are similar results for other groups G. In particular, one would like to know if the K-theory of $C^*_r(G)$, or better still, the K-theory of reduced crossed products $A \rtimes G$, can be computed in a "topological" way. The answer in many cases seems to be "yes," and the conjectured answer is what is usually called the *Baum-Connes Conjecture*, with or without coefficients. The special case of the Baum-Connes Conjecture (without coefficients) for connected Lie groups is also known as the *Connes-Kasparov Conjecture*, and is now a known theorem.

The Baum-Connes conjecture also has other origins, such as the Novikov Conjecture on higher signatures and conjectures about algebraic K-theory of group rings, which will be touched on in Reich's lectures. These other motivations for the conjecture mostly concern the case where G is discrete, which is actually the most interesting case of the conjecture, though there are good reasons for not restricting only to this case. (For example, as we already saw in the case of $\mathbb{Z}$, information about discrete groups can often be obtained by embedding them in a Lie group.)

Here is the formal statement of the conjecture.

Conjecture 2.2 (Baum-Connes)**.** *Let G be a locally compact group, second-countable for convenience. Let $\underline{E}G$ be the universal proper G-space.* (*This is a contractible space on which G acts properly, characterized* [5] *up to G-homotopy equivalence by two properties: that every compact subgroup of G has a fixed point in $\underline{E}G$, and that the two projections $\underline{E}G \times \underline{E}G \to \underline{E}G$ are G-homotopic. Here the product* *space is given the diagonal G-action. If G has no compact subgroups, then $\underline{E}G$ is the usual universal free G-space EG.) There is an* assembly map

$$\varinjlim_{\substack{X\subseteq \underline{E}G\\ X/G \text{ compact}}} K_*^G(X) \to K_*(C_r^*(G))$$

defined by taking G-indices of G-invariant elliptic operators, and this map is an isomorphism.

Conjecture 2.3 (Baum-Connes with coefficients)**.** *With notation as in Conjecture* 2.2*, if A is any separable G-C^*-algebra, the assembly map*

$$\varinjlim_{\substack{X\subseteq \underline{E}G\\ X/G \text{ compact}}} KK_*^G(C_0(X), A) \to K_*(A \rtimes_r G)$$

is an isomorphism.

Let's see what the conjecture amounts to in some special cases. If G is compact, $\underline{E}G$ can be taken to be a single point. The conjecture then asserts that the *assembly map* $KK_*^G(\mathrm{pt}, \mathrm{pt}) \to K_*(C^*(G))$ is an isomorphism. For G compact, $C^*(G)$ is by the Peter-Weyl Theorem the completed direct sum of matrix algebras $\bigoplus_V \mathrm{End}(V)$, where V runs over a set of representatives for the irreducible representations of G. Thus $K_1(C^*(G))$ (remember this is topological K_1) vanishes and $K_0(C^*(G)) \cong R(G)$. The assembly map in this case is the Green-Julg isomorphism of Section 2.2. In fact, the same holds with coefficients; the assembly map $KK_*^G(\mathbb{C}, A) = K_*^G(A) \to K_*(A \rtimes G)$ is the Green-Julg isomorphism, and Conjecture 2.3 is true.

Next, suppose $G = \mathbb{R}$. Since G has no compact subgroups and is contractible, we can take $\underline{E}G = \mathbb{R}$ with $\mathbb{R}$ acting on itself by translations. If A is an $\mathbb{R}$-C^*-algebra, the assembly map is a map $KK_*^{\mathbb{R}}(C_0(\mathbb{R}), A) \to K_*(A \rtimes \mathbb{R})$. This map turns out to be Kasparov's morphism

$$j\colon\ KK_*^{\mathbb{R}}(C_0(\mathbb{R}), A) \to KK_*(C_0(\mathbb{R}) \rtimes \mathbb{R}, A \rtimes \mathbb{R}) = KK_*(\mathcal{K}, A \rtimes \mathbb{R}) \cong K_*(A \rtimes \mathbb{R}),$$

which is the isomorphism of Connes' Theorem (Section 2.4). (The isomorphism $C_0(\mathbb{R}) \rtimes \mathbb{R} \cong \mathcal{K}$ is a special case of the Imprimitivity Theorem giving a Morita equivalence between $\left(\mathrm{Ind}_{\{1\}}^G A\right) \rtimes G$ and A, or if you prefer, of Takai duality from Section 2.3.) So again the conjecture is true.

Another good test case is $G = \mathbb{Z}$. Then $\underline{E}G = EG = \mathbb{R}$, with $\mathbb{Z}$ acting by translations and quotient space $\mathbb{T}$. The left-hand side of the conjecture is thus $KK^{\mathbb{Z}}(C_0(\mathbb{R}), A)$, while the right-hand side is $K(A \rtimes \mathbb{Z})$, which is computed by the Pimsner-Voiculescu sequence.

More generally, suppose G is discrete and torsion-free. Then $\underline{E}G = EG$, and the quotient space $\underline{E}G/G$ is the usual classifying space BG. The assembly map (for the conjecture without coefficients) maps $K_*^{\mathrm{cmpct}}(BG) \to K_*(C_r^*(G))$. (The left-hand side is K-homology with compact supports.) This map can be viewed as an index map, since classes in the K-homology group on the left are represented by generalized Dirac operators D over Spin^c manifolds M with a G-covering, and the assembly map takes such an operator to its "Mishchenko-Fomenko index" with values in the K-theory of the (reduced) group C^*-algebra. The connection between this assembly map and the usual sort of assembly map studied by topologists is discussed in [55]. In particular, Conjecture 2.2 implies a strong form of the Novikov Conjecture for G.

2.7. **The approach of Meyer and Nest.** An interesting alternative approach to the Baum-Connes Conjecture has been proposed by Meyer and Nest [42, 43]. This approach is also briefly sketched (in somewhat simplified form) in [20, §5.3] and in [56, Ch. 5]. Meyer and Nest begin by observing that the equivariant KK-category, $\mathbf{KK}^G$, naturally has the structure of a triangulated category. It has a distinguished class $\mathcal{E}$ of *weak equivalences*, morphisms $f \in KK^G(A, B)$ which restrict to equivalences in $KK^H(A, B)$ for every compact subgroup H of G. (Note that if G has no nontrivial compact subgroups, for example if G is discrete and torsion-free, then this condition just says that f is a KK-equivalence after forgetting the G-equivariant structure.) The Baum-Connes Conjecture with coefficients, Conjecture 2.3, basically amounts to the assertion that if $f \in KK^G(A, B)$ is in $\mathcal{E}$, then $j_r(f) \in KK(A \rtimes_r G, B \rtimes_r G)$ is a KK-equivalence.[2] In particular, suppose G has no nontrivial compact subgroups and satisfies Conjecture 2.3. Then if A is a G-C^*-algebra which, forgetting the G-action, is contractible, then the unique morphism in $KK^G(0, A)$ is a weak equivalence, and so (applying j_r), the unique morphism in $KK(0, A \rtimes_r G)$ is a KK-equivalence. Thus $A \rtimes_r G$ is K-*contractible*, i.e., all of its topological K-groups must vanish. When $G = \mathbb{R}$, this follows from Connes' Theorem, and when $G = \mathbb{Z}$, this follows from the Pimsner-Voiculescu exact sequence, (2.1).

Now that we have several different formulations of the Baum-Connes Conjecture, it is natural to ask how widely the conjecture is valid. Here are some of the things that are known:

(1) There is no known counterexample to Conjecture 2.2 (Baum-Connes for groups, without coefficients). Counterexamples are now known [27] to Conjecture 2.3 with G discrete and A even commutative.
(2) Conjecture 2.3 is true if G is amenable, or more generally, if it is *a-T-menable*, that is, if it has an affine, isometric and metrically proper action on a Hilbert space [30]. Such groups include all Lie groups whose noncompact semisimple factors are all locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$ for some n.
(3) Conjecture 2.2 is true for connected reductive Lie groups, connected reductive p-adic groups, for hyperbolic discrete groups, and for cocompact lattice subgroups of $Sp(n, 1)$ or $SL(3, \mathbb{C})$ [39].

There is now a vast literature on this subject, but our intention here is not to be exhaustive, but just to give the reader some flavor of what's going on.

[2] The reason for using j_r in place of j for can be seen from the case of G nonamenable with property T. In this case, $C^*(G)$ has a projection corresponding to the trivial representation of G which is "isolated," and thus maps to 0 in $C^*_r(G)$. So these two algebras do not have the same K-theory. It turns out at least in many examples that $K_0(C^*_r(G))$ can be described in purely topological terms, but $K_0(C^*(G))$ cannot.

Lecture 3. The universal coefficient theorem for KK and some of its applications

3.1. **Introduction to the UCT.** Now that we have discussed KK and KK^G, a natural question arises: *how computable are they?* In particular, is $KK(A, B)$ determined by $K_*(A)$ and by $K_*(B)$? Is $KK^G(A, B)$ determined by $K_*^G(A)$ and by $K_*^G(B)$?

A first step was taken by Kasparov [34]: he pointed out that $KK(X, Y)$ is given by an explicit topological formula when the one-point compactifications X_+ and Y_+ are finite CW complexes: $KK(X, Y) \cong \widetilde{K}(Y_+ \wedge D(X_+))$, where $D(X_+)$ denotes the Spanier-Whitehead dual of X_+.[3]

Let's make a definition — we say the pair of C^*-algebras (A, B) *satisfies the Universal Coefficient Theorem for* KK (or *UCT* for short) if there is an exact sequence

$$0 \to \bigoplus_{*\in\mathbb{Z}/2} \operatorname{Ext}^1_{\mathbb{Z}}(K_*(A), K_{*+1}(B)) \to KK(A, B) \xrightarrow{\varphi} \bigoplus_{*\in\mathbb{Z}/2} \operatorname{Hom}_{\mathbb{Z}}(K_*(A), K_*(B)) \to 0. \tag{3.1}$$

Here φ sends a KK-class to the induced map on K-groups.

We need one more definition. Let $\mathcal{B}$ be the *bootstrap category*, the smallest full subcategory of the separable C^*-algebras (with the $*$-homomorphisms as morphisms) containing all separable type I algebras, and closed under extensions, countable C^*-inductive limits, and KK-equivalences. Note that KK-equivalences include Morita equivalences, and type I algebras include commutative algebras. Recall from Section 1.2 that stably isomorphic separable C^*-algebras are Morita equivalent, hence KK-equivalent. Furthermore, separable type I C^*-algebras are inductive limits of finite iterated extensions of stably commutative C^*-algebras [44, Ch. 6]. Thus we could just as well replace the words "type I" by "commutative" in the definition of $\mathcal{B}$. Furthermore, any compact metric space is a countable limit of finite CW complexes. Dualizing, that means that any unital separable commutative C^*-algebra is a countable inductive limit (i.e., categorical colimit) of algebras of the form $C(X)$, X a finite CW complex, and any separable commutative C^*-algebra is a countable inductive limit (i.e., colimit) of algebras of the form $C_0(X)$, X_+ a finite CW complex. We will use this fact shortly.

Theorem 3.1 (Rosenberg-Schochet [58])**.** *The UCT holds for all pairs (A, B) with A an object in $\mathcal{B}$ and B separable.*

Unsolved problem: Is every separable nuclear C^*-algebra in $\mathcal{B}$? Skandalis [60] showed that there are *non-nuclear algebras* not in $\mathcal{B}$, for which the UCT fails.

3.2. **The proof of Rosenberg and Schochet.** First suppose $K_*(B)$ is injective as a $\mathbb{Z}$-module, i.e., divisible as an abelian group. Then $\operatorname{Hom}_{\mathbb{Z}}(\underline{\quad}, K_*(B))$ is an exact functor, so $A \mapsto \operatorname{Hom}_{\mathbb{Z}}(K_*(A), K_*(B))$ gives a cohomology theory on C^*-algebras. In particular, φ is a natural transformation of homology theories for spaces

$$\big(X \mapsto KK_*(C_0(X), B)\big) \rightsquigarrow \big(X \mapsto \operatorname{Hom}_{\mathbb{Z}}(K^*(X), K_*(B))\big).$$

[3] Spanier-Whitehead duality basically interchanged homology and cohomology. $\wedge$ denotes the smash product, the product in the category of spaces with distinguished basepoint.

Since φ is an isomorphism for $X = \mathbb{R}^n$ by Bott periodicity, it is an isomorphism whenever X_+ is a sphere, and thus (by the analogue of the Eilenberg-Steenrod uniqueness theorem for generalized homology theories) whenever X_+ is a finite CW complex.

We extend to arbitrary locally compact X by taking limits, and then to the rest of $\mathcal{B}$, using the observations we made before the proof of the theorem. In order to know we can pass to countable inductive limits, we need one additional fact about KK, namely that it is "countably additive" (sends countable C^*-algebra direct sums in the first variable to products of abelian groups). This fact is not hard to check from Kasparov's original definition. So the theorem holds when $K_*(B)$ is injective.

The rest of the proof uses an idea due to Atiyah [3], of *geometric resolutions*. The idea is that given arbitrary B, we can change it up to KK-equivalence so that it fits into a short exact sequence

$$0 \to C \to B \to D \to 0$$

for which the induced K-theory sequence is short exact:

$$K_*(B) \rightarrowtail K_*(D) \twoheadrightarrow K_{*-1}(C)$$

and $K_*(D)$, $K_*(C)$ are $\mathbb{Z}$-injective. Then we use the theorem for $KK_*(A, D)$ and $KK_*(A, C)$, along with the long exact sequence in KK in the second variable, to get the UCT for (A, B). $\square$

3.3. **The equivariant case.** If one asks about the UCT in the equivariant case, then the homological algebra of the ground ring $R(G)$ becomes relevant. This is not always well behaved, so as noticed by Hodgkin [31], one needs restrictions on G to get anywhere. But for G a connected compact Lie group with $\pi_1(G)$ torsion-free, $R(G)$ has finite global dimension, and the spectral sequence one ends up with does converge to the right limit.

Theorem 3.2 (Rosenberg-Schochet [57])**.** *If G is a connected compact Lie group with $\pi_1(G)$ torsion-free, and if A, B are separable G-C^*-algebras with A in a suitable bootstrap category containing all commutative G-C^*-algebras, then there is a convergent spectral sequence*

$$\operatorname{Ext}^p_{R(G)}(K^G_*(A), K^G_{q+*}(A)) \Rightarrow KK^G_*(A, B).$$

The proof is more complicated than in the non-equivariant case, but in the same spirit.

Also along the same lines, there is a UCT for KK of real C^*-algebras, due to Boersema [8]. The homological algebra involved in this case is appreciably more complicated than in the complex C^*-algebra case, and is based on ideas on Bousfield [9] on the classification of K-local spectra.

3.4. **The categorical approach.** The UCT implies a lot of interesting facts about the bootstrap category $\mathcal{B}$. Here are a few examples.

Theorem 3.3 (Rosenberg-Schochet [58])**.** *Let A, B be C^*-algebras in $\mathcal{B}$. Then A and B are KK-equivalent if and only if they have the isomorphic topological K-groups.*

Proof. $\Rightarrow$ is trivial. So suppose $K_*(A) \cong K_*(B)$. Choose an isomorphism

$$\psi\colon K_*(A) \to K_*(B).$$

Since the map φ in the UCT (3.1) is surjective, ψ is realized by a class $x \in KK(A, B)$ (not necessarily unique, but just pick one).

Now consider the commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, equal] & \mathrm{Ext}^1_{\mathbb{Z}}(K_{*+1}(B), K_*(A)) \arrow[r] \arrow[d, "\psi^*", "\cong"'] & KK_*(B, A) \arrow[r, "\varphi"] \arrow[d, "x\otimes_B _"] & \mathrm{Hom}(K_*(B), K_*(A)) \arrow[r] \arrow[d, "\psi^*", "\cong"'] & 0 \arrow[d, equal] \\
0 \arrow[r] & \mathrm{Ext}^1_{\mathbb{Z}}(K_{*+1}(A), K_*(A)) \arrow[r] & KK_*(A, A) \arrow[r, "\varphi"] & \mathrm{Hom}(K_*(A), K_*(A)) \arrow[r] & 0
\end{tikzcd}$$

By the 5-Lemma, Kasparov product with x is an isomorphism $KK_*(B, A) \to KK_*(A, A)$. In particular, there exists $y \in KK(B, A)$ with $x \otimes_B y = 1_A$. Similarly, there exists $z \in KK(B, A)$ with $z \otimes_A x = 1_B$. Then by associativity

$$z = z \otimes_A (x \otimes_B y) = (z \otimes_A x) \otimes_B y = y$$

and we have a KK-inverse to x. □

Corollary 3.4. *We can also describe $\mathcal{B}$ as the smallest full subcategory of the separable C^*-algebras closed under KK-equivalence and containing the separable commutative C^*-algebras. A separable C^*-algebra A has the property that (A, B) satisfies the UCT for all separable C^*-algebras if and only if it lies in $\mathcal{B}$.*

Proof. Let $\mathcal{B}'$ be the smallest full subcategory of the separable C^*-algebras closed under KK-equivalence and containing the separable commutative C^*-algebras. By definition of $\mathcal{B}$, $\mathcal{B}'$ is a subcategory of $\mathcal{B}$. But if A is in $\mathcal{B}$, its K-groups are countable. For any countable groups G_0 and G_1, it is easy to construct a second-countable locally compact space with these K-groups. So there is a separable commutative C^*-algebra $C_0(Y)$ with $K_*(C_0(Y)) \cong K_*(A)$ (just as abelian groups). By Theorem 3.3, there is a KK-equivalence between A and $C_0(Y)$, so A lies in $\mathcal{B}'$.

As far as the last statement is concerned, one direction is the UCT itself. For the other direction, suppose that (A, B) satisfies the UCT for all separable C^*-algebras B. In particular, it holds for a commutative B with the same K-groups as A, and by the argument above, A is KK-equivalent to B, hence lies in $\mathcal{B}$. □

Recall that $KK(A, A) = \mathrm{End}_{\mathbf{KK}}(A)$ is a ring under Kasparov product. We can now compute the ring structure.

Theorem 3.5 (Rosenberg-Schochet)**.** *Suppose A is in $\mathcal{B}$. In the UCT sequence*

$$0 \to \bigoplus_{i\in\mathbb{Z}/2} \mathrm{Ext}^1_{\mathbb{Z}}(K_{i+1}(A), K_i(A)) \to KK(A, A) \xrightarrow{\varphi} \bigoplus_{i\in\mathbb{Z}/2} \mathrm{End}(K_i(A)) \to 0,$$

φ is a split surjective homomorphism of rings, and $J = \ker\varphi$ (the Ext *term) is an ideal with $J^2 = 0$.*

Proof. Choose A_0 and A_1 commutative with $K_0(A_0) \cong K_0(A)$, $K_1(A_0) = 0$, $K_0(A_1) = 0$, $K_1(A_1) \cong K_1(A)$. Then by Theorem 3.3, $A_0 \oplus A_1$ is KK-equivalent to A, and without loss of generality, we may assume we have an actual splitting $A = A_0 \oplus A_1$. By the UCT, $KK(A_0, A_0) \cong \mathrm{End}\, K_0(A)$ and $KK(A_1, A_1) \cong \mathrm{End}\, K_1(A)$.

So $KK(A_0, A_0) \oplus KK(A_1, A_1)$ is a subring of $KK(A, A)$ mapping isomorphically under φ. This shows φ is split surjective. We also have $J = KK(A_0, A_1) \oplus$

$KK(A_1, A_0)$. If, say, x lies in the first summand and y in the second, then $x \otimes_{A_1} y$ induces the 0-map on $K_0(A)$ and so is 0 in $KK(A_0, A_0) \cong \mathrm{End}(K_0(A))$. Similarly, $y \otimes_{A_0} x$ induces the 0-map on $K_1(A)$ and so is 0 in $KK(A_1, A_1) \cong \mathrm{End}(K_1(A))$. $\square$

3.5. **The homotopy-theoretic approach.** There is a homotopy-theoretic approach to the UCT that topologists might find attractive; it seems to have been discovered independently by several people (e.g., [11, 32] — see also the review of [11] in MathSciNet). Let A and B be C^*-algebras and let $\mathbb{K}(A)$ and $\mathbb{K}(B)$ be their topological K-theory spectra. These are module spectra over $\mathbb{K} = \mathbb{K}(\mathbb{C})$, the usual spectrum of complex K-theory. Then we can define

$$KK^{\mathrm{top}}(A, B) = \pi_0\big(\mathrm{Hom}_{\mathbb{K}}\big(\mathbb{K}(A), \mathbb{K}(B)\big)\big).$$

This is again using ideas of Bousfield [9].

Theorem 3.6. *There is a natural map $KK(A, B) \to KK^{top}(A, B)$, and it's an isomorphism if and only if the UCT holds for the pair (A, B).*

Observe that $KK^{\mathrm{top}}(A, B)$ even makes sense for Banach algebras, and always comes with a UCT.

We promised in the first lecture to show that defining $KK(X, Y)$ to be the set of natural transformations

$$(Z \mapsto K(X \times Z)) \rightsquigarrow (Z \mapsto K(Y \times Z))$$

indeed agrees with Kasparov's $KK(C_0(X), C_0(Y))$. Indeed, $Z \mapsto K(X \times Z)$ is basically the cohomology theory defined by $\mathbb{K}(X)$, and $Z \mapsto K(Y \times Z)$ is similarly the cohomology theory defined by $\mathbb{K}(Y)$. So the natural transformations (commuting with Bott periodicity) are basically a model for $KK^{\mathrm{top}}(C_0(X), C_0(Y))$.

3.6. **Topological applications.** The UCT can be used to prove facts about topological K-theory which on their face have nothing to do with C^*-algebras or KK. For example, we have the following purely topological fact:

Theorem 3.7. *Let X and Y be locally compact spaces such that $K^*(X) \cong K^*(Y)$ just as abelian groups. Then the associated K-theory spectra $\mathbb{K}(X)$ and $\mathbb{K}(Y)$ are homotopy equivalent.*

Proof. We have seen (Theorem 3.3) that the hypothesis implies $C_0(X)$ and $C_0(Y)$ are KK-equivalent, which gives the desired conclusion. $\square$

Note that this theorem is quite special to complex K-theory; it fails even for ordinary cohomology (since one needs to consider the action of the Steenrod algebra).

Similarly, the UCT implies facts about cohomology operations in complex K-theory and K-theory mod p. For example, one has:

Theorem 3.8 (Rosenberg-Schochet [58])**.** *The $\mathbb{Z}/2$-graded ring of homology operations for $K(\_\_; \mathbb{Z}/n)$ on the category of separable C^*-algebras is the exterior algebra over $\mathbb{Z}/n$ on a single generator, the Bockstein β.*

Theorem 3.9 (Araki-Toda [2], new proof by Rosenberg-Schochet in [58])**.** *There are exactly n admissible multiplications on K-theory mod n. When n is odd, exactly one is commutative. When $n = 2$, neither is commutative.*

3.7. **Applications to C^*-algebras.** Probably the most interesting applications of the UCT for KK are to the classification problem for nuclear C^*-algebras. The *Elliott program* (to quote M. Rørdam from his review of the Kirchberg-Phillips paper [37]) is to classify "all separable, nuclear C^*-algebras in terms of an invariant that has K-theory as an important ingredient." Kirchberg and Phillips have shown how to do this for *Kirchberg algebras*, that is simple, purely infinite, separable and nuclear C^*-algebras. The UCT for KK is a key ingredient.

Theorem 3.10 (Kirchberg-Phillips [37, 45])**.** *Two stable Kirchberg algebras A and B are isomorphic if and only if they are KK-equivalent; and moreover every invertible element in $KK(A, B)$ lifts to an isomorphism $A \to B$. Similarly in the unital case if one keeps track of $[1_A] \in K_0(A)$.*

We will not attempt to explain the proof of Kirchberg-Phillips, but it's based on the idea that a KK-class is given by a quasihomomorphism, which under the specific hypotheses can be lifted to a true homomorphism. More recent results of a somewhat similar nature may be found in [22, 21, 41].

Given the Kirchberg-Phillips result, one is still left with the question of determining when two Kirchberg algebras are KK-equivalent. But those of "Cuntz type" (like $\mathcal{O}_n$)[4] lie in $\mathcal{B}$, and Kirchberg and Phillips show that $\forall$ abelian groups G_0 and G_1 and $\forall g \in G_0$, there is a nonunital Kirchberg algebra $A \in \mathcal{B}$ with these K-groups, and there is a unital Kirchberg algebra $A \in \mathcal{B}$ with these K-groups and with $[1_A] = g$. By the UCT, these algebras are classified by their K-groups.

The original work on the Elliott program dealt with the opposite extreme: stably finite algebras. Here again, KK can play a useful role. Here is a typical result from the vast literature:

Theorem 3.11 (Elliott [23])**.** *If A and B are C^*-algebras of real rank 0 which are inductive limits of certain "basic building blocks", then any $x \in KK(A, B)$ preserving the "graded dimension range" can be lifted to a $*$-homomorphism. If x is a KK-equivalence, it can be lifted to an isomorphism.*

This theorem applies for example to the irrational rotation algebras A_θ, because of an amazing result by Elliott and Evans [24] that shows that these algebras are indeed inductive limits of the required type.

[4] This is the fundamental example of a Kirchberg algebra, invented by Cuntz [16]. It is the universal C^*-algebra generated by n isometries whose range projections are orthogonal and add to 1. Cuntz proved that it is simple, and showed that $\mathcal{O}_n \otimes \mathcal{K}$ is a crossed product of a UHF algebra (an inductive limit of matrix algebras) by an action of $\mathbb{Z}$. But crossed products by $\mathbb{Z}$ preserve the category $\mathcal{B}$, because of the arguments in Sections 2.4 and 2.5. Thus $\mathcal{O}_n$ lies in $\mathcal{B}$.

Lecture 4. A fundamental example in noncommutative geometry: topology and geometry of the irrational rotation algebra

4.1. **Basic facts about A_θ.** We previously mentioned the algebra A_θ, defined to be the crossed product $C(\mathbb{T}) \rtimes_{\alpha_\theta} \mathbb{Z}$, where $\mathbb{T}$ is the circle group (thought of as the unit circle in $\mathbb{C}$) and where α_θ sends the generator $1 \in \mathbb{Z}$ to multiplication by $e^{2\pi i\theta}$, i.e., rotation of the circle by an angle of $2\pi\theta$. This makes sense for any $\theta \in \mathbb{R}$, but of course only the class of θ mod $\mathbb{Z}$ matters, so we might as well take $\theta \in [0, 1)$. This algebra has two standard names: a *rotation algebra* (with parameter θ), or *irrational rotation algebra* in the most important case of $\theta \notin \mathbb{Q}$, or a *noncommutative* (2-)*torus*, because of the fact that when $\theta = 0$, we get back simply $C(\mathbb{T}^2)$, the continuous functions on the usual 2-torus. It is no exaggeration to say that *these C^*-algebras are the most important examples in (C^*-algebraic) noncommutative geometry.*

In this section we'll try to lay out the basic facts about these algebras, without attempting to prove everything or to explain the history of every result. The standard references for a lot of this material are the fundamental papers of Rieffel [48, 51]. A more extensive survey on this material can be found in [53].

We can describe the algebra A_θ quite concretely, using the definition of the crossed product in Section 2.2. The algebra has two unitary generators U and V, one of them generating $C(\mathbb{T})$ and the other corresponding to the generator of $\mathbb{Z}$. They satisfy the commutation relation $UV = e^{2\pi i\theta} VU$. The algebra A_θ is the completion of the noncommutative polynomials in U and V. But because of the commutation relation, we can move all U's to the left and all V's to the right in any noncommutative monomial in U and V, at the expense of a scalar factor of modulus 1. Thus A_θ is the completion of the polynomials $\sum_{m,n} c_{m,n} U^m V^n$ (with only finitely many non-zero coefficients). In fact, every element of A_θ is represented by a *formal* such infinite sum, but it is not so easy to describe the C^*-algebra norm in terms of the sequence of *Fourier coefficients* $\{c_{m,n}\}$. The one thing we can say, since $\|U\| = \|V\| = 1$, is that the C^*-norm is bounded by the L^1-norm, so that if the coefficients converge absolutely, then the corresponding infinite sum does represent an element of A_θ. (But the converse is false. This is classical when $\theta = 0$, and amounts to the fact that there are continuous functions whose Fourier series do not converge absolutely.)

The algebra A_θ has a canonical *trace* τ, i.e., a bounded linear functional with $\tau(ab) = \tau(ba)$ for all $a, b \in A_\theta$. We normalize by taking $\tau(1) = 1$. Usually we add the condition that τ should send self-adjoint elements to real values, though when θ is irrational, this is automatic. When $\theta = 0$, τ is just integration with respect to Haar measure on $\mathbb{T}^2$ (normalized to be a probability measure).

There is a basic dichotomy between two cases. If θ is irrational, then no different powers of $e^{2\pi i\theta}$ coincide. It is not too hard to show from this that A_θ is simple and that there is a *unique* trace in this case, defined by the condition that $\tau(U^m V^n) = 0$ if $m \neq 0$ or $n \neq 0$. (Recall we do require $\tau(1) = 1$.) So τ simply picks out the $(0, 0)$ coefficient $c_{0,0}$ from $\sum_{m,n} c_{m,n} U^m V^n$. On the other hand, if $\theta = \frac{p}{q} \in \mathbb{Q}$, then A_θ has a big center, and in fact A_θ is the algebra of sections of a bundle of matrix algebras over T^2. In fact one can show in this case that $A_\theta \cong \operatorname{End}_{T^2}(V)$, the bundle endomorphisms of any complex line bundle V over T^2 with first Chern class $\equiv p \pmod q$ (times the usual generator of $H^2(T^2, \mathbb{Z})$). The algebra has many traces in this case, but it's still convenient to let τ be the one with $\tau(U^m V^n) = 0$ if $m \neq 0$ or $n \neq 0$. (This along with the condition that $\tau(1) = 1$ then determines τ uniquely.)

The K-theory of A_θ can be computed from the Pimsner-Voiculescu sequence of Section 2.5. In fact, the main motivation of Pimsner and Voiculescu for developing this sequence was to compute $K_*(A_\theta)$. Since α_θ is isotopic to the trivial action, regardless of the value of θ, the map $1 - \alpha(1)_*$ in (2.1) is always 0. Hence, just as abelian groups, one always has $K_0(A_\theta) \cong K_1(A_\theta) \cong \mathbb{Z}^2$. But one wants more than this; one wants a description of the generators. Tracing through the various maps involved shows that one summand in K_0 is generated by the rank-one free module (or the projection 1), and that the two summands in K_1 are generated by U and V, respectively. But the interesting feature is the *order structure* on K_0, which comes from the inclusions of projective modules. Note that the trace gives a homomorphism from $K_0(A_\theta)$ to $\mathbb{R}$, sending a projective module to the trace of a self-adjoint projection (in some matrix algebra) representing it. (It's a fact that every idempotent in a C^*-algebra is similar to a self-adjoint one; see for example [6, §4.6]. Since the trace takes real values on self-adjoint elements, the dimension of a projection is real-valued.)

Theorem 4.1. *If $\theta \notin \mathbb{Q}$, the trace τ induces an isomorphism of $K_0(A_\theta)$ with $\mathbb{Z}+\theta\mathbb{Z}$* as ordered groups. *If $\theta \in \mathbb{Q}$, then τ still sends $K_0(A_\theta)$ to $\mathbb{Z}+\theta\mathbb{Z}$ (which is equal to $\theta\mathbb{Z}$ in this case), but is no longer an isomorphism.*

The original proof of this theorem was nonconstructive, i.e., it did not exhibit a projective module of dimension θ that should be the missing generator of K_0. We will talk about this issue later in Section 4.3.

It follows from Theorem 4.1 that the irrational rotation algebras must split into uncountably many Morita equivalence classes, since it is easy to see that Morita equivalence preserves the ordering on K_0, and since there are uncountably many order isomorphism classes of subgroups of $\mathbb{R}$ of the form $\mathbb{Z}+\theta\mathbb{Z}$. In fact, any order isomorphism $\mathbb{Z}+\theta\mathbb{Z} \to \mathbb{Z}+\theta'\mathbb{Z}$ must be given by multiplication by some $t \neq 0$ in $\mathbb{R}$, with the property that $t \in \mathbb{Z}+\theta'\mathbb{Z}$ and $t\theta \in \mathbb{Z}+\theta'\mathbb{Z}$. If we write $t = c\theta' + d$ and $t\theta = a\theta' + b$, $a, b, c, d \in \mathbb{Z}$, then

$$\theta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \theta'$$

for the usual action of 2×2 matrices by linear fractional transformations. Since the Morita equivalence must be invertible, we also have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z}).$$

So Morita equivalences of irrational rotation algebras correspond to the action of $GL(2,\mathbb{Z})$ by linear fractional transformations. The converse is also true.

Theorem 4.2 (Rieffel)**.** *Any unital C^*-algebra Morita equivalent to an irrational rotation algebra A_θ is a matrix algebra over $A_{\theta'}$ with θ' in the orbit of θ for the action of $GL(2,\mathbb{Z})$ on $\mathbb{RP}^1$ by linear fractional transformations. Every matrix in $GL(2,\mathbb{Z})$ gives rise to such a Morita equivalence.*

This is not true by "general nonsense" but requires an explicit construction, which arises from the following theorem of Rieffel:

Theorem 4.3 (Rieffel [50])**.** *If G is a locally compact group with closed subgroups H and K, then $H \ltimes (G/K)$ and $(H\backslash G) \rtimes K$ are Morita equivalent.*

If we apply this with $G = \mathbb{R}$, $H = 2\pi\mathbb{Z}$, and $K = 2\pi\theta\mathbb{Z}$, then $H\backslash G$ is the usual model of $\mathbb{T}$ and $(H\backslash G) \rtimes K$ is A_θ, while $H \ltimes (G/K)$ is $A_{1/\theta}$. The Morita equivalence bimodule between these two algebras is a completion of $\mathcal{S}(\mathbb{R})$, with the two generators of each algebra acting by translation and by multiplication by an exponential, respectively. The reason why the two actions commute is that translation by $\mathbb{Z}$ commutes with multiplication by $e^{2\pi i s}$, while translation by $\frac{1}{\theta}\mathbb{Z}$ commutes with multiplication by $e^{2\pi i\theta s}$.

The other Morita equivalences required by the theorem can be constructed similarly.

4.2. **Basic facts about A_θ^∞.** One of the interesting things about A_θ is that it behaves in many ways like a smooth manifold. That means that we should have an analogue of the C^∞ functions inside the algebra of "continuous" functions A_θ. To find this, note that A_θ carries an action of the compact Lie group $\mathbb{T}^2$ via $(z,w)\cdot U = zU$, $(z,w)\cdot V = wV$, for $z, w \in \mathbb{T}$ (viewed as complex numbers of modulus 1). This is analogous to the action of $\mathbb{T}^2$ on itself by translations. The *smooth subalgebra* A_θ^∞ is defined to be the set of C^∞ vectors for this action, i.e., the elements a for which $(z,w) \mapsto (z,w)\cdot a$ is C^∞ as a map $\mathbb{T}^2 \to A_\theta$. Alternatively, we can describe A_θ^∞ as the intersection of the domains of all polynomials in δ_1 and δ_2, the (unbounded) derivations obtained by differentiating the action. Since it is obvious that $\delta_1(U) = 2\pi i U$ and $\delta_2(U) = 0$, while $\delta_2(V) = 2\pi i V$ and $\delta_1(V) = 0$, one readily sees (as in the smooth case) that A_θ^∞ is a subalgebra and that it can be described as

$$A_\theta^\infty = \left\{ \sum_{m,n} c_{m,n} U^m V^n \mid c_{m,n} \text{ is rapidly decreasing} \right\},$$

where "rapidly decreasing" means decreasing faster than the reciprocal of any positive polynomial in m and n. Thus A_θ^∞ is isomorphic as a topological vector space (not as an algebra) to $\mathcal{S}(\mathbb{Z}^2)$ and then by Fourier transform to $C^\infty(\mathbb{T}^2)$.

Proposition 4.4. *The inclusion of A_θ^∞ into A_θ is "isospectral"* (*i.e., an element of the subalgebra is invertible in the subalgebra if and only if it has an inverse in the larger algebra*)*, and thus the inclusion $A_\theta^\infty \hookrightarrow A_\theta$ induces an isomorphism on K-theory.*

Proof. Isospectral inclusions preserve K_0 and (topological) K_1, by the "Karoubi density theorem," so it is enough to prove the first statement. But this follows from the characterization of A_θ^∞ in terms of derivations, and the familiar identity $\delta_j(a^{-1}) = -a^{-1}\delta_j(a)a^{-1}$, iterated many times. □

From this Proposition, as well as the fact that there is no essential difference between smooth and purely topological manifold topology in dimension 2, one might be tempted to guess that A_θ and A_θ^∞ behave similarly in all important respects. But a deep fact is that *this is false*; $\mathrm{Aut}(A_\theta)$ and $\mathrm{Aut}(A_\theta^\infty)$ are quite different from one another.

Theorem 4.5. *If θ is irrational, every automorphism of A_θ^∞ is "orientation-preserving," i.e., the determinant of the induced map on $K_1(A_\theta) \cong \mathbb{Z}^2$ is $+1$. On the other hand, A_θ has orientation-reversing automorphisms.*

Comment: The first part of this is due to [19]. The second part is due to Elliott and Evans [24, 23].

4.3. **Geometry of vector bundles.** In classical topology, vector bundles play an important role in studying compact manifolds M. Recall *Swan's Theorem* ([6, §1.7] or [20, §1.3.3]): there is an equivalence of categories between topological (respectively, smooth) vector bundles over M and finitely generated projective modules over $C(M)$ (resp., $C^\infty(M)$). Thus in noncommutative geometry, finitely generated projective modules play the same role as vector bundles. Because of Proposition 4.4, when it comes to irrational rotation algebras, the "vector bundle" theory is essentially the same in both the continuous and C^∞ cases, in that every finitely generated projective module over A_θ is extended from a finitely generated projective module over A_θ^∞, which is unique up to isomorphism.

Now K-theory gives the *stable* classification of vector bundles. The *unstable* classification is always more delicate, but for A_θ, this, too is known:

Theorem 4.6 (Rieffel [51]). *For A_θ with θ irrational, complete cancellation holds for finitely generated projective modules, i.e., if $P \oplus Q \cong P' \oplus Q$ as A_θ-modules, for any finitely generated projective A_θ-modules P, P', Q, then P and P' are isomorphic. The isomorphism classes of projective submodules of a free A_θ-module of rank n are distinguished by the trace, and are given exactly by elements of $K_0(A_\theta) \cong \mathbb{Z} + \theta\mathbb{Z}$ between 0 and n (inclusive).*

Once one knows the classification of the "vector bundles," in both the smooth and continuous categories, a natural next step is to study "geometry" on them. In his fundamental paper [14], Alain Connes explained how the theory of connections and curvature in differential geometry can be carried over to the noncommutative case, at least when one has an algebra A like A_θ with an action of a Lie group G for which the "smooth subalgebra" A^∞ is the set of C^∞-vectors for the G-action on A. (This of course applies here with $G = \mathbb{T}^2$ acting as we described above.) Then if V is a finitely generated (right) A^∞-module, a connection on V is a map $\nabla\colon V \to V \otimes \mathfrak{g}^*$ ($\mathfrak{g}$ the Lie algebra of G) satisfying the usual Leibniz rule

$$\nabla_X(v \cdot a) = \nabla_X(v) \cdot a + v \cdot (X \cdot a), \quad v \in V,\ a \in A^\infty,\ X \in \mathfrak{g}.$$

Usually one requires a connection to be compatible with an inner product also. Connections always exist and have a curvature 2-form $\Theta \in \mathrm{End}_A(V) \otimes \bigwedge^2 \mathfrak{g}^*$ defined as usual by

$$\Theta(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}.$$

Theorem 4.7 (Connes [14]). *Every finitely generated projective module over A_θ^∞ admits a connection of constant curvature (i.e., with the curvature in $i \bigwedge^2 \mathfrak{g}^*$). The curvature can be taken to be 0 if and only if the module is free. More precisely, on the projective module with "dimension" $p + q\theta > 0$, $p, q \in \mathbb{Z}$, the constant curvature connections have curvature*

$$\Theta(\delta_1, \delta_2) = \frac{2\pi i q}{p + q\theta}.$$

Connes and Rieffel defined the notion of *Yang-Mills energy* of a connection, precisely analogous to the classical case for smooth vector bundles over manifolds. This is defined by

$$\mathrm{YM}(\nabla) = -\tau_{\mathrm{End}(V)}\big(\{\Theta_\nabla, \Theta_\nabla\}\big),$$

where $\{\_\_, \_\_\}$ is the natural bilinear form on 2-forms.

Theorem 4.8 (Connes and Rieffel [15, 52])**.** *If V is a finitely generated projective module over A_θ^∞, a connection ∇ on V gives a minimum for* YM *if and only if it has constant curvature, and gives a critical point for* YM *if and only if it is a direct sum of constant curvature connections (i.e., V has a decomposition $V_1 \oplus \cdots V_n$ with respect to which ∇ has a similar decomposition into connections of constant curvature).*

As we mentioned earlier, the original calculation of $K_0(A_\theta)$ was nonconstructive, and the problem remained of explicitly exhibiting representatives for the the finitely generated projective modules. One answer is already implicit in what we have explained: if P is a finitely generated projective A-module, then it gives rise to a Morita equivalence between A and $\mathrm{End}_A(P)$, so constructing all possible P's is equivalent to finding all Morita equivalence bimodules for A. In the case of A_θ, they are all similar to the bimodule we mentioned before between A_θ and $A_{1/\theta}$. But one could ask for another answer to the problem, namely to give explicit representatives for all the equivalence classes of projections in A_θ (or in matrix algebras over it). Here two good solutions have been proposed, one by Rieffel [48] and one by Boca [7]. Rieffel constructed explicit projections in A_θ of the form $Uf + g + \overline{f}U^*$, where f and g are functions of V. Boca instead constructed projections in terms of theta-functions which can be described as follows: if X is an A-B Morita equivalence bimodule as above, with $A = A_\theta$, and if one can find an element $\psi \in X$ with $\langle \psi, psi\rangle_B = 1_B$, then ${}_A\langle \psi, psi\rangle$ will be a projection in A. Boca's projections come from choosing ψ closely related to a Gaussian function in $\mathcal{S}(\mathbb{R})$.

4.4. **Miscellaneous other facts about A_θ.** Here we just mention a few other things about the algebras A_θ. The work of Elliott and Evans [24, 23], which we mentioned before, has more detailed implications for automorphisms and endomorphisms of A_θ. Assuming θ is irrational, given any $A \in GL(2,\mathbb{Z})$, there is an automorphism of A_θ inducing the map A on $K_1(A_\theta) \cong \mathbb{Z}^2$, and given any $B \in \mathrm{End}(\mathbb{Z}^2)$ (including 0!), there is a unital endomorphism of A_θ inducing the identity on $K_0(A_\theta)$ and the map B on $K_1(A_\theta)$. Furthermore, the connected component of the identity in $\mathrm{Aut}(A_\theta)$ is topologically simple, and $\mathrm{Aut}(A_\theta)$ is just an extension of this connected group by $GL(2,\mathbb{Z})$ [25]. All of this seems quite strange from the perspective of ordinary manifold topology, since a self-map $T^2 \to T^2$ inducing the identity on $K^0(T^2)$ is of degree 1, and thus cannot induce the 0-map on $K^{-1}(T^2) \cong H^1(T^2)$.

However, the endomorphisms constructed by Elliott's procedure are unlikely to be smooth. Kodaka [38] did construct some special smooth proper unital endomorphisms of irrational rotation algebras, but only when θ lies in a real quadratic number field.

And one more structural fact about the algebras A_θ: they have *real rank zero*, that is, finite linear combinations of projections are dense in the set of self-adjoint elements.

Lecture 5. Applications of the irrational rotation algebra in number theory and physics

5.1. **Applications to number theory.**

5.2. **Applications to physics.**

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