On mysteriously missing T-duals, H-flux and the T-duality group

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A general formula for the topology and H-flux of the T-duals of type II string theories with H-flux on toroidal compactifications is presented here. It is known that toroidal compactifications with H-flux do not necessarily have T-duals which are themselves toroidal compactifications. A big puzzle has been to explain these mysterious "missing T-duals", and our paper presents a solution to this problem using noncommutative topology. We also analyze the T-duality group and its action, and illustrate these concepts with examples.

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T-duality is a symmetry of type II string theories that involves exchanging a theory compactified on a torus with a theory compactified on the dual torus. The T-dual of a type II string theory compactified on a circle, in the presence of a topologically nontrivial NS 3-form H-flux, was analyzed in special cases in [2, 5, 6, 7]. There it was observed that T-duality changes not only the H-flux, but also the spacetime topology. A general formalism for dealing with T-duality for compactifications arising from a free circle action was developed in [8]. This formalism was shown to be compatible with two physical constraints: (1) it respects the local Buscher rules [1], and (2) it yields an isomorphism on twisted K-theory, in which the Ramond-Ramond charges and fields take their values [11, 12, 13]. It was shown in [8] that T-duality exchanges the first Chern class with the fiberwise integral of the H-flux, thus giving a formula for the T-dual spacetime topology. In this note we will present an account for physicists of the results in [16], consisting of a formula for the T-dual of a toroidal compactification, that is a theory compactified via a free torus action, with H-flux. One striking new feature that occurs for higher dimensional tori is that not every toroidal compactification with H-flux has a T-dual; moreover, even if it has a T-dual, then the T-dual need not be another toroidal compactification with H-flux. A big puzzle has been to explain these mysterious "missing T-duals", and our paper presents a solution to this problem using noncommutative topology. A similar phenomenon was noticed in [15] in the special case of the trivial $\mathbb{T}^2$ bundle over $\mathbb{T}$ with non-trivial H-flux. We also show that the generalized T-duality group $GO(n, n; \mathbb{Z})$, n being the rank of the torus, acts to generate the complete list of T-dual pairs related to a given toroidal compactification with H-flux. We will explain these results by providing examples and applications.

In this letter we will consider type II string theories on target d-dimensional manifolds X, which are assumed to admit free, rank n torus actions. While for most physical applications one wants $d = 10$, we do not need to assume this, and in fact X could represent a partial reduction of the original 10-dimensional spacetime after preliminary compactification in $10-d$ dimensions. The space of orbits of the torus action on X is given by a $(d-n)$-dimensional manifold, which we call Z. The freeness of the action implies that each orbit is a torus and that none of these tori degenerate. As a result X is a principal torus bundle over the base Z, and so its topology is entirely determined by the topology of the base Z together with the first Chern class c of the bundle $X \xrightarrow{p} Z$ in $H^2(Z, \mathbb{Z}^n)$. This viewpoint is useful in that it automatically identifies some gauge equivalent configurations, excludes configurations not satisfying some equations of motion and imposes the Dirac quantization conditions. The Chern class c is represented by a vector valued closed 2-form with integral periods, the curvature F. We will discuss conditions under which the pair $(X \xrightarrow{p} Z, H)$ has a T-dual, either another pair $(X \xrightarrow{p^\#} Z, H^\#)$ with the same base Z (the "classical" case) or a more general non-commutative object (the "nonclassical" case). In both cases, there should be a sense in which string theory on the original space X (with H-flux H) is equivalent to a theory on the T-dual.

Basic setup: **Let $p\colon X \to Z$ be a principal T-bundle as above, where $T = (S^1)^n = \mathbb{T}^n$ is a rank n torus. Let $H \in H^3(X, \mathbb{Z})$ be an H-flux on X satisfying $\iota^* H = 0$, $\iota^*\colon H^3(X, \mathbb{Z}) \to H^3(T, \mathbb{Z})$, where $\iota\colon T \hookrightarrow X$ is the inclusion of a fiber. (This condition is automatically satisfied when $n \leq 2$.)**

The simplest case when the condition $\iota^*(H) = 0$ does *not* apply is $X = \mathbb{T}^3$, when considered as a rank 3, principal torus bundle over a point, with H-flux a nonzero integer multiple of the volume 3-form on $\mathbb{T}^3$. When $\iota^*(H) \neq 0$, there is no T-dual in the sense we are considering, even in what we call the "nonclassical" sense.

It turns out that nontrivial bundles are always T-dual to trivial bundles with non-zero H-flux. Therefore we will need to include the fluxes H and $H^\#$ in our toroidal compactifications, which are then topologically determined by the triples (Z, c, H) and $(Z, c^\#, H^\#)$, where H and $H^\#$ are closed three-forms on the total spaces X and $X^\#$ respectively.

Our result on classical T-duals: **Suppose that we are in the basic setup as above. Choose a basis $\{\mathbb{T}^2_j\}_{j=1}^k$, $k = \binom{n}{2}$ for $H_2(T, \mathbb{Z})$ consisting of 2-tori, and push this forward into $H_2(X, \mathbb{Z})$ via ι_*. We**

can consider the cohomology classes

$$\int_{\mathbb{T}_j^2} H = H \cap \iota_*(\mathbb{T}_j^2) \in H^1(X, \mathbb{Z}).$$

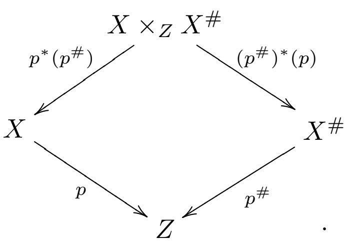
These classes restrict to 0 **on the fibers, since** $\iota^*(H) = 0$**. Using the following exact sequence, derived from the spectral sequence of the torus bundle,**

$$0 \to H^1(Z, \mathbb{Z}) \xrightarrow{p^*} H^1(X, \mathbb{Z}) \xrightarrow{\iota^*} H^1(T, \mathbb{Z}) \to \cdots, \quad (1)$$

we see that the classes $\int_{\mathbb{T}_j^2} H = H \cap \iota_*(\mathbb{T}_j^2) \in H^1(X, \mathbb{Z})$ **come from unique classes** $\{\beta_j\}_{j=1}^k$ **in** $H^1(Z, \mathbb{Z})$**. Set**

$$p_!(H) = \Big(\beta_1, \ldots, \beta_k\Big) \in H^1(Z, \mathbb{Z}^k). \quad (2)$$

If $p_!(H) = 0 \in H^1(Z, \mathbb{Z}^k)$**, and in particular if** Z **is simply connected, then there is a classical T-dual to** (p, H)**, consisting of** $p^\# \colon X^\# \to Z$**, which is another principal** T**-bundle over** Z**, and** $H^\# \in H^3(X^\#, \mathbb{Z})$**, the T-dual H-flux on** $X^\#$**. One obtains a commuting diagram of the form**



(3)

In this case, the compactifications topologically specified by (Z, c, H) **and** $(Z, c^\#, H^\#)$ **are T-dual if** c, $c^\# \in H^2(Z, \mathbb{Z}^n)$ **are related as follows:**

Let c_j**,** $j = 1, \cdots, n$**, be the components of** c**. Let** $X_j \xrightarrow{\pi_j} Z$ **be the principal** $\mathbb{T}^{n-1}$ **subbundle of** X **obtained by deleting** c_j**, i.e. the Chern class of** X_j **is**

$$c(\pi_j) = (c_1, \ldots, \hat{c}_j, \ldots, c_n).$$

Then $X \xrightarrow{p_j} X_j$ **is a principal** S^1**-bundle whose Chern class is equal to** $\pi_j^*(c_j)$**. Define** $X_j^\# \xrightarrow{\pi_j^\#} Z$**,** $X^\# \xrightarrow{p_j^\#} X_j^\#$ **etc. similarly. Then we have**

$$(\pi_j)^*(c_j^\#) = (p_j)_!(H) \quad \textbf{and} \quad (\pi_j^\#)^*(c_j) = (p_j^\#)_!(H^\#).$$

Here the correspondence space $X \times_Z X^\#$ is the submanifold of $X \times X^\#$ consisting of pairs of points (x, y) such that $p(x) = p^\#(y)$, and has the property that it implements the T-duality between (p, H) and $(p^\#, H^\#)$. It also turns out that $p_!^\#(H^\#) = 0 \in H^1(Z, \mathbb{Z}^k)$ and that the T-dual of $(p^\#, H^\#)$ is (p, H). So in this case, T-duality *exchanges the integral of the H-flux* (over a basis of circles in the fibers) *with the first Chern class.* The condition in the result above determines, at the level of cohomology, the curvatures F and $F^\#$. However the NS field strengths are only determined up to the addition of a three-form on the base Z, because the integral of such a form over a basis of circles in the fibers vanishes. This settles a conjecture in [8], and was also considered by [9].

The simplest higher rank example is $X = S^2 \times \mathbb{T}^2$, considered as the trivial $\mathbb{T}^2$ bundle over $Z = S^2$, with H-flux equal to $H = k_1 a \wedge b_1 + k_2 a \wedge b_2$, where we use the Künneth theorem to identify $H^3(S^2 \times \mathbb{T}^2, \mathbb{Z})$ with $H^2(S^2, \mathbb{Z}) \otimes H^1(\mathbb{T}^2, \mathbb{Z})$, and a is the generator of $H^2(S^2, \mathbb{Z}) \cong \mathbb{Z}$, b_1, b_2 are the generators of $H^1(\mathbb{T}^2, \mathbb{Z}) \cong \mathbb{Z}^2$ and $k_1, k_2 \in \mathbb{Z}$. Since S^2 is simply connected, $p_!(H) = 0$ and the T-dual of $(S^2 \times \mathbb{T}^2, H)$ is the nontrivial rank 2 torus bundle P over S^2 with Chern class $c_1(P) = (k_1 a, k_2 a) \in H^2(S^2, \mathbb{Z}) \oplus H^2(S^2, \mathbb{Z}) = H^2(S^2, \mathbb{Z}^2)$, and with H-flux equal to zero. This example generalizes easily by taking the Cartesian product with a manifold M, and pulling back the H-flux to the product and arguing as before, we see that the T-dual of $(M \times S^2 \times \mathbb{T}^2, H)$ is $(M \times P, 0)$.

Our result on nonclassical T-duals: **Suppose that we are in the basic setup as above. If** $p_!(H) \neq 0 \in H^1(Z, \mathbb{Z}^k)$**, then there is *no* classical T-dual to** (p, H)**; however, there is a nonclassical T-dual consisting of a continuous field of (stabilized) noncommutative tori** A_f **over** Z**, where the fiber over the point** $z \in Z$ **is equal to the rank** n **noncommutative torus** $A_{f(z)}$ **(see Figure 1 below). Here** $f \colon Z \to \mathbb{T}^k$ **is a continuous map representing** $p_!(H)$**.**

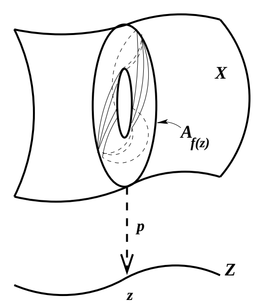


FIG. 1: In the diagram, the fiber over $z \in Z$ is the noncommutative torus $A_{f(z)}$, which is represented by a foliated torus, with foliation angle equal to $f(z)$.

This suggests an unexpected link between classical string theories and the "noncommutative" ones, obtained by "compactifying" matrix theory on tori, as in [4] (cf. also [19, §§6–7]). We now recall the definition of the rank n noncommutative torus A_θ, cf. [18]. This algebra (stabilized by tensoring with the compact operators $\mathcal{K}$) occurs geometrically as the foliation algebra associated to Kronecker foliations on the torus [3]. In [4], the same algebra occurs naturally from studying the field equations of the IKKT (Ishibashi-Kawai-Kitazawa-Tsuchiya) model compactified on n-tori, or from the study of BPS states of the BFSS (Banks-Fisher-Shenker-Susskind) model. (The IKKT and BFSS models are both large-N matrix models in which Poisson brackets in the Lagrangian are replaced by matrix commutators.) For each $\theta \in \mathbb{T}^k$, identified with a hermitian matrix $\theta = (\theta_{ij}),\ i, j = 1, \ldots n$, $\theta_{ij} \in S^1$ with 1's down the diagonal, the *noncommutative torus* A_θ is defined abstractly as the C^*-algebra generated by n unitaries $U_j,\ j = 1, \ldots, n$ in an infinite dimensional Hilbert space satisfying the commutation relation $U_iU_j = \theta_{ij}U_jU_i,\ i, j = 1, \ldots, n$. Elements in A_θ can be represented by infinite power series

$$f = \sum_{m\in\mathbb{Z}^n} a_m U^m, \tag{4}$$

where $a_m \in \mathbb{C}$ and $U^m = U_1^{m_1} \ldots U_n^{m_n}$, for all $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$.

A famous example of a principal torus bundle with non-T-dualizable H-flux is provided by $\mathbb{T}^3$, considered as the trivial $\mathbb{T}^2$-bundle over $\mathbb{T}$, with H given by k times the volume form on $\mathbb{T}^3$. H is non T-dualizable in the classical sense since $p_!(H) \neq 0 \in H^1(\mathbb{T}, \mathbb{Z})$. Alternatively, there are no non-trivial principal $\mathbb{T}^2$-bundles over $\mathbb{T}$, since $H^2(\mathbb{T}, \mathbb{Z}^2) = 0$, that is, there is no way to dualize the H-flux by a (principal) torus bundle over $\mathbb{T}$, cf. [7]. This is an example of a mysteriously missing T-dual. This example is covered by our result on nonclassical T-duals above. The T-dual is realized by a field of stabilized *noncommutative tori* fibered over $\mathbb{T}$. Let $\mathcal{H} = L^2(\mathbb{T})$ and consider the the projective unitary representation $\rho_\theta \colon \mathbb{Z}^2 \to \mathrm{PU}(\mathcal{H})$ in which the generator of the first $\mathbb{Z}$ factor acts by multiplication by z^k (where $\mathbb{T}$ is thought of as the unit circle in $\mathbb{C}$) and the generator of the second $\mathbb{Z}$ factor acts by translation by $\theta \in \mathbb{T}$. Then the Mackey obstruction of ρ_θ is $\theta^k \in \mathbb{T} \cong H^2(\mathbb{Z}^2, \mathbb{T})$. Let $\mathcal{K}(\mathcal{H})$ denote the algebra of compact operators on $\mathcal{H}$ and define an action α of $\mathbb{Z}^2$ on continuous functions on the circle with values in compact operators, $C(\mathbb{T}, \mathcal{K}(\mathcal{H}))$, given at the point θ by ρ_θ. Define the C^*-algebra B, which is obtained by inducing the $\mathbb{Z}^2$ action to an action of $\mathbb{R}^2$ on $B = \mathrm{Ind}_{\mathbb{Z}^2}^{\mathbb{R}^2}(C(\mathbb{T}, \mathcal{K}(\mathcal{H})), \alpha)$, i.e. $B = \{f : \mathbb{R}^2 \to C(\mathbb{T}, \mathcal{K}(\mathcal{H})) : f(t + g) = \alpha(g)(f(t)),\ t \in \mathbb{R}^2, g \in \mathbb{Z}^2\}$. Then B is a continuous-trace C^*-algebra having spectrum $\mathbb{T}^3$ and Dixmier-Douady invariant H. B also has an action of $\mathbb{R}^2$ whose induced action on the spectrum of B is the trivial bundle $\mathbb{T}^3 \to \mathbb{T}$. Then our noncommutative T-dual is the crossed product algebra $B \rtimes \mathbb{R}^2 \cong C(\mathbb{T}, \mathcal{K}(\mathcal{H})) \rtimes \mathbb{Z}^2 = A_f$, which has fiber over $\theta \in \mathbb{T}$ given by $\mathcal{K}(\mathcal{H}) \rtimes_{\rho_\theta} \mathbb{Z}^2 \cong A_\theta \otimes \mathcal{K}(\mathcal{H},)$ where A_θ is the noncommutative 2-torus. In fact, the crossed product $B \rtimes \mathbb{R}^2$ is isomorphic to the (stabilized) group C^*-algebra $C^*(H_\mathbb{Z}) \otimes \mathcal{K}$, where $H_\mathbb{Z}$ is the integer Heisenberg-type group,

$$H_\mathbb{Z} = \left\{ \begin{pmatrix} 1 & x & \frac{1}{k}z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} : x, y, z \in \mathbb{Z} \right\}. \tag{5}$$

In summary, the nonclassical T-dual of $(\mathbb{T}^3, H = k)$ is $A_f = C^*(H_\mathbb{Z}) \otimes \mathcal{K}$. As required in order to match up RR charges, the K-theory of this algebra is the same as the K-theory of $\mathbb{T}^3$ with twist given by our H-flux, or k times the volume form.

This example generalizes easily by taking the Cartesian product with a manifold M. Pulling back the H-flux to the product and arguing as before, we see that $(M \times \mathbb{T}^3, H = k)$ is T-dual to $C(M) \otimes C^*(H_\mathbb{Z}) \otimes \mathcal{K}$. For instance, if the dimension of M is seven, then $M \times \mathbb{T}^3$ is ten dimensional, yielding examples of spacetime manifolds that are relevant to type II string theory.

Our results on the T-duality group:

It is important to realize that a fixed space X can sometimes be given the structure of a principal torus bundle over Z in many different ways. For example, given a free action of a torus $T = \mathbb{T}^n$ on X, with quotient space $Z = X/T$, we can for every element $g \in \mathrm{Aut}(\mathbb{T}^n) = GL(n, \mathbb{Z})$ define a new free action of T on X, twisted by g, by the formula $x \cdot_g t = x \cdot g(t)$. (Here $t \in T$, $\cdot$ is the original free right action of T on X, and $\cdot_g$ is the new twisted action.) If $c \in H^2(Z, \mathbb{Z}^n)$ was the Chern class of the original bundle, the Chern class of the g-twisted bundle is $g \cdot c$, with g acting via the action of $GL(n, \mathbb{Z})$ on $\mathbb{Z}^n$.

The group $GL(n, \mathbb{Z})$ embeds in $O(n, n; \mathbb{Z})$, the subgroup of $GL(2n, \mathbb{Z})$ preserving the quadratic form defined by $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$, via $a \mapsto \begin{pmatrix} a & 0 \\ 0 & (a^t)^{-1} \end{pmatrix}$ (see [10, §2.4]). This larger group $O(n, n; \mathbb{Z})$ is often called the *T-duality group*. In fact we will consider the still larger *generalized T-duality group* $GO(n, n; \mathbb{Z}) = O(n, n; \mathbb{Z}) \rtimes (\mathbb{Z}/2)$ of matrices in $GL(2n, \mathbb{Z})$ preserving the form $\begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}$ up to sign. Good references for the T-duality group include [10] (for the state of the theory up to 1994) and [14] for more current developments.

Suppose that we are in the basic setup as above, with Z simply connected, so that one is always guaranteed to have a classical T-dual. Then the generalized T-duality group $GO(n, n; \mathbb{Z})$ acts on the set of T-dual pairs (p, H) and $(p^\#, H^\#)$ to generate all related T-dual pairs. *All of these pairs are physically equivalent.* The restriction of the action to $GL(n, \mathbb{Z})$ (as embedded above) corresponds to twisting of the action on the same underlying space as above.

When Z is not simply connected and $p_!(H) \neq 0$, it is not clear that one has an action of the full T-duality group. But the action of $GL(n, \mathbb{Z})$ always sends the pair consisting of (p, H) and its nonclassical T-dual to another nonclassical pair, involv-

ing continuous fields of (stabilized) noncommutative tori over Z.

We illustrate the action of the generalized T-duality group in the simplest case of circle bundles with H-flux, in which case the generalized T-duality group reduces to $GO(1,1;\mathbb{Z})$, a dihedral group of oder 8.

Consider the example of the 3 dimensional lens space $L(1,p) = S^3/\mathbb{Z}_p$, with H-flux $H = q$ times the volume form, cf. [17]. Here $p, q \in \mathbb{Z}$, and initially we take $p, q > 0$. Then $L(1,p)$ is a circle bundle over the 2-dimensional sphere S^2 and has first Chern class equal to p times the volume form of S^2. Then, as shown in [8, §4.3], $(L(1,p), H = q)$ and $(L(1,q), H = p)$ are T-dual to each other, and the element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of $O(1,1;\mathbb{Z})$ interchanges them. The element $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ of the T-duality group $O(1,1;\mathbb{Z})$ lies in the subgroup $GL(1,\mathbb{Z})$, embedded as above, and acts by twisting the S^1 action on $L(1,p)$. This twisted action makes $L(1,p)$ into a circle bundle over S^2 having first Chern class equal to $-p$ times the volume form of S^2. This bundle is denoted $L(1,-p)$, and its total space is diffeomorphic to $L(1,p)$, though by an orientation-reversing diffeomorphism. Therefore the action of $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ on the pair $(L(1,p), H = q)$ and $(L(1,q), H = p)$ gives rise to a new T-dual pair $(L(1,-p), H = -q)$ and $(L(1,-q), H = -p)$. The group $GO(1,1;\mathbb{Z})$ is generated by the two elements of $O(1,1;\mathbb{Z})$ just discussed and by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, which replaces the original T-dual pair by the pair consisting of $(L(1,p), H = -q)$ and $(L(1,-q), H = p)$. Here we have tacitly assumed $p, q \geq 2$; we can extend things to other values of p and q by making the convention that $L(1,1) = S^3$ and $L(1,0) = S^2 \times S^1$. This refines the T-duality in [8, §4.3]. Thus in general there are 8 different (bundle, H-flux) pairs with equivalent physics, corresponding to $(\pm p, \pm q)$ and $(\pm q, \pm p)$.

This example generalizes easily by taking the Cartesian product with a manifold M. For instance, if the dimension of M is seven, then we obtain 8 different (bundle, H-flux) pairs in the same $GO(1,1;\mathbb{Z})$-orbit as $M \times L(1,p)$. All of these are ten-dimensional spacetime manifolds relevant to type II string theory.

We end with some open problems. A critical verification of any proposed duality is that the anomalies should match on both sides. This was checked for T-duality involving circle bundles with H-flux in [8], but remains to be analyzed in the general torus bundle case with H-flux. It also remains to be determined whether or not the group $GO(n,n;\mathbb{Z})$ also operates in the nonclassical case. Another problem is to extend our results to non-free torus actions [20], in which case it could be relevant to mirror symmetry.

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