The Real Chevalley Involution

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The Chevalley Involution

G: connected, reductive, H : Cartan subgroup

Theorem

(1) There is an involution C of G satisfying: $C(h) = h^{-1}$ $(h \in H)$;

(2) $C(g) \sim g^{-1}$ for all semisimple elements g;

(3) Any two such involutions are conjugate by an inner automorphism;

(4) C is the Cartan involution of the split real form of $G(\mathbb{C})$.

C is the Chevalley involution of G

The Contragredient

suppose G defined over F (local field)

π: irreducible admissible representation of $G(F)$

$\pi^* =$ contragredient of π

Question

What is $\pi \to \pi^*$ in terms of L-homomorphisms?

$\phi : W_F' \to {}^LG$, $\pi \in \Pi(\phi) \quad \longrightarrow \phi^*$ s.t. $\pi^* \in \Pi(\phi^*)$

Question is: what is the map $\phi \to \phi^*$?

The Contragredient

C: Chevalley involution of ${}^LG$

Conjecture (A/Vogan)

$\phi^* = C \circ \phi$

i.e.

$$\boxed{\Pi(\phi)^* = \Pi(C \circ \phi)}$$

(true for $GL(n, F)$, F p-adic)

Theorem (A/Vogan)

True $F = \mathbb{R}$.

(Mumbai 2012, arXiv 1201.0496)

The Contragredient

Corollary

Every L-packet is self-dual if and only if $-1 \in W(G,H)$

$(W(G,H) = W(G(\mathbb{C}), H(\mathbb{C})))$

The Contragredient

Today: the group side

Question

What about realizing π^* via an involution of $G(\mathbb{R})$?

(1) Is the Chevalley involution defined over $\mathbb{R}$?

(2) Does it satisfy $C(g) \sim_{G(\mathbb{R})} g^{-1}$ for all $g \in G(\mathbb{R})$?

Note: (1) only implies $C(g) \sim_{G(\mathbb{C})} g^{-1}$

Motivation

General question: automorphisms of G, on the dual side

Hermitian dual, applications to unitarity

Character theory

Frobenius-Schur (symplectic/orthogonal) indicator

Applications to L-functions (contragredient)

recent paper of D. Prasad and Ramakrishnan

The Contragredient

Example (D. Prasad)

$G = F_4, G_2, E_8$, F p-adic, $G(F)$ split

There are Chevalley involutions C of G, defined over F

None of them satisfy: $C(g) \sim_{G(F)} g^{-1}$

(only $C(g) \sim_{G(\overline{F})} g^{-1}$)

(since every automorphism of $G(F)$ is inner, and $G(F)$ has non-self dual representations)

Example

$G = SL(2, \mathbb{R})$

$x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$\tau(g) = xgx^{-1}$

$\tau(g) \sim g^{-1}$ ($g \in$ split Cartan subgroup)

But $\tau(g) \not\sim g^{-1}$ ($g \in$ compact Cartan)

$y = \mathrm{diag}(i, -i)$

$C(g) = ygy^{-1}$, $C(g) \sim g^{-1}$ for all g

Moral: Focus on the fundamental (most compact) Cartan subgroup

The Real Chevalley Involution

G defined over $\mathbb{R}$, $\theta =$ Cartan involution

H is fundamental if the split rank of $H_f(\mathbb{R})$ is minimal

Definition

A Chevalley involution is fundamental if $C(g) = g^{-1}$ for all g in some fundamental Cartan subgroup of G.

The Real Chevalley Involution

Theorem

(1) There is a fundamental Chevalley involution C of G;

(2) C is defined over $\mathbb{R}$, $C : G(\mathbb{R}) \to G(\mathbb{R})$;

(3) $C(g) \sim_{G(\mathbb{R})} g^{-1}$ ($g \in G(\mathbb{R})$ semisimple)

(3) Any two fundamental Chevalley involutions are conjugate by an inner automorphism of $G(\mathbb{R})$.

Sketch of proof of the Theorem

Existence of C:

Pinning: $\mathcal{P} = (B, H, \{X_\alpha\})$

Line everything up with respect to $\mathcal{P}$

$C(X_\alpha) = X_{-\alpha}, \quad \sigma_c(X_\alpha) = -X_{-\alpha}$ (G^{σ_c} compact)

δ: distinguished automorphism (preserving $\mathcal{P}$), $\quad x \in H^\delta$

$\theta(X_\alpha) = \alpha(x) X_{\delta(\alpha)}$

$\sigma = \theta\sigma_c, \quad G(\mathbb{R}) = G^\sigma$

Sketch of proof of the Theorem

Lemma

$$\theta\sigma_c = \sigma_c\theta$$
$$C\theta = \theta C$$
$$C\sigma = \sigma C$$

Digression

Proposition (Lusztig)

F algebraically closed $\Rightarrow$

$$C(g) \sim_G g^{-1} \text{ for all } g$$

Question

C = *fundamental Chevalley involution*

$$C(g) \sim_{G(\mathbb{R})} g^{-1} \text{ for all } g?$$

Note added after the talk: Lusztig's proof generalizes readily to answer the question affirmatively. Binyong Sun proved this for (certain) classical groups, and Sun, Vogan and I saw how to generalize Lusztig and Sun's proofs to the general real case.

When is every π self-dual?

since $C(g) \sim_{G(\mathbb{R})} g^{-1}$ (g semisimple)

Corollary

$$\pi \textit{ irreducible } \Rightarrow \pi^C \simeq \pi^*$$

Corollary

Every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if C is inner for $G(\mathbb{R})$

Necessary but not sufficient: $-1 \in W(G, H)$

When is every π self-dual?

$H(\mathbb{R})$ fundamental

$W(G(\mathbb{R}),H(\mathbb{R})) = \mathrm{Norm}_{G(\mathbb{R})}(H(\mathbb{R}))/H(\mathbb{R}) \hookrightarrow W(G,H)$

Proposition

Every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if

$$-1 \in W(G(\mathbb{R}),H(\mathbb{R}))$$

(easy consequence of the Theorem)

When is every π self-dual?

G, $G(\mathbb{R}) = G\sigma$, $K = G^\theta$ (K is complex)

$H_K = H \cap K \subset H$: Cartan subgroup of K

Equal rank case: $H_K = H$

$W(K, H) \simeq W(G(\mathbb{R}), H(\mathbb{R}))$

When is every π self-dual?

Corollary

Every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if

$$-1 \in W(K,H)$$

Dangerous Bend In the unequal rank case

$$W(K,H) \simeq W(K,H_K)$$

right hand side: Weyl group of a (disconnected) reductive group

but -1 has different meaning on the two sides
$x \in \mathrm{Norm}_K(H) = \mathrm{Norm}_K(H_K)$,

$$xhx^{-1} = h^{-1} \quad (h \in H_K) \not\Rightarrow xhx^{-1} = h^{-1} \quad (h \in H)$$

When is every π self-dual?

Proposition

Every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if every irreducible representation of K is self-dual, and, in the unequal rank case, $-1 \in W(G, H)$

(equal rank case: $-1 \in W(K, H_K) \Rightarrow -1 \in W(G, H)$)

When is every π self-dual?

Proposition

$G(\mathbb{R})$ is simple: every irreducible representation of $G(\mathbb{R})$ is self-dual if and only if $-1 \in W(G,H)$ and, in the equal rank case, $G(\mathbb{R})$ is a pure real form.

pure: $\theta = \text{int}(x)$, $x^2 = 1$

($-1 \in W(G,H) \Rightarrow Z(G) =$ two-group $\Rightarrow$ purity independent of the choice of x) "Purity Of Essence"

Key point: $g \in \text{Norm}_G(H)$ representative of $-1 \in W(G,H)$:

$$-1 \in W(K,H) \Leftrightarrow xgx^{-1} = g \Leftrightarrow x^2 g = g \Leftrightarrow x^2 = 1$$

When is every π self-dual?

Corollary

G adjoint, $-1 \in W(G,H) \Rightarrow$

every irreducible representation of $G(\mathbb{R})$ is self-dual

List of simple $G(\mathbb{R})$, with all π self-dual

(1) A_n: $SO(2,1)$, $SU(2)$ and $SO(3)$.

(2) B_n: Every real form of the adjoint group, $Spin(2p, 2q+1)$ (p even).

(3) C_n: Every real form of the adjoint group, $Sp(p,q)$.

(4) D_{2n+1}: none.

(5) D_{2n}, unequal rank: all real forms

(6) D_{2n}, equal rank (various cases...)

(7) E_6: none.

(8) E_7: Every real form of the adjoint group, simply connected compact.

(9) G_2, F_4, E_8: every real form.

(10) complex groups of type $A_1, B_n, C_n, D_{2n}, G_2, F_4, E_7, E_8$

Frobenius-Schur indicator

$T : \pi \simeq \pi^* \to \langle v, w \rangle = (Tv)(w)$

$\langle , \rangle$ bilinear, symmetric or antisymmetric:

$$\langle v, w \rangle = \epsilon_\pi \langle w, v \rangle \quad (\epsilon_\pi = \pm 1)$$

ϵ_π = Frobenius-Schur indicator

Problem

How do you compute ϵ_π?

(interesting invariant of self-dual representations)

Frobenius-Schur indicator: Finite Dimensional Representations

$G(\mathbb{R})$, $\pi \simeq \pi^*$ finite dimensional,

χ_π : central character

$$z(\rho^\vee) = \exp(2\pi i \rho^\vee) \in Z(G)$$

(fixed by all automorphisms)

Proposition (Bourbaki)

$$\epsilon_\pi = \chi_\pi(z(\rho^\vee))$$

Frobenius-Schur indicator: Finite Dimensional Representations

Key ingredient of proof:

$$w_0 \in W = W(G,H)(\text{ long element})) \to g \in \text{Norm}_H(G) \quad (\text{mapping to } w_0)$$
$$\to g^2 \in H$$

Lemma

We can choose g so that

$$g^2 = z(\rho^\vee),$$

If $-1 \in W$ this is independent of all choices.

(proof: uses the Tits group)

Remark: Same fact (dual side): discrete series are parametrized by $X^*(H) + \rho$

Frobenius-Schur indicator: Finite Dimensional Representations

proof of Proposition:

$$
\begin{aligned}
\chi_\pi(g^2)\langle v, \pi(g)v\rangle &= \langle \pi(g^2)v, \pi(g)v) \\
&= \langle \pi(g)v, v\rangle \\
&= \epsilon(\pi)\langle v, \pi(g)v\rangle
\end{aligned}
$$

i.e.

$$
\boxed{\chi_\pi(g^2)\langle v, \pi(g)v\rangle = \epsilon(\pi)\langle v, \pi(g)v\rangle}
$$

Take $v \in V_\lambda$ (highest weight space), $\pi(g)v \in V_{-\lambda}$, $\langle v, \pi(g)v\rangle \neq 0$

(also see [Prasad, IMRN 1999])

Frobenius-Schur indicator

Suppose every irreducible π (infinite dimensional) is self-dual

μ : lowest K-type, multiplicity one, self-dual (by previous lemma)

$$\epsilon_\pi = \epsilon_\mu$$

Example: Assume K is connected

Take π finite dimensional

(1) $\epsilon_\pi = \chi_\pi(z(\rho_G^\vee))$ (result applied to G)

(2) $\epsilon_\pi = \epsilon_\mu = \chi_\mu(z(\rho_K^\vee))$ (result applied to K)

How can this be?

Frobenius-Schur indicator

This implies

$$K \text{ connected}, -1 \in W(K,H) \Rightarrow z(\rho_G^\vee) = z(\rho_K^\vee)$$

alternative proof:

$$W(G,H) \ni -1 \to g \to g^2 = z(\rho_G^\vee)$$
$$\text{view } g \in W(K,H), -1 \to g^2 = z(\rho_K^\vee)$$

Surprise:

Lemma

Assume $-1 \in W(K,H)$. *Then*

$$z(\rho_G^\vee) = z(\rho_K^\vee)$$

Example: $G = SL(2)/PGL(2)$

$G(\mathbb{R}) = SL(2,\mathbb{R})/PGL(2,\mathbb{R}) : z(\rho_G^\vee) = -I$

$K = SO(2)/O(2)$: $z(\rho_K^\vee) = I$

$SL(2,\mathbb{R})$: $z(\rho_G^\vee) = -I \neq I = z(\rho_K^\vee)$ $(-1 \notin W(K,H))$

$PGL(2,\mathbb{R})$: $z(\rho_G^\vee) = -I = I = z(\rho_K^\vee)$ $(-1 \in W(K,H))$

Frobenius-Schur indicator

Hint of proof (equal rank case):

$\theta(g) = xgx^{-1}$

h : coxeter number

$$c = \begin{cases} h-1 & \text{Hermitian symmetric case} \\ \frac{h}{2} & \text{otherwise} \end{cases}$$

$-1 \in W(G,H) \Rightarrow c \in \mathbb{Z}$

Fact: WLOG

$$x = \exp\left(\frac{\pi i}{c}(\rho_K^\vee - \rho_G^\vee)\right)$$

Then (by the purity lemma) $1 = x^2 \Rightarrow$

$$z(\rho_K^\vee)/z(\rho_G^\vee) = x^{2c} = (x^2)^c = 1$$

Frobenius-Schur indicator

Theorem

Every irreducible representation self-dual implies

$$\epsilon_\pi = \chi_\pi(z(\rho^\vee))$$

Proof: $z(\rho_K^\vee) = z(\rho_G^\vee)$, minimal K-type $\mu\dots$

Done if K is connected

delicate argument about the disconnectedness of K

Key point: $\mu|_{K^0}$ has multiplicity one (branching law!)

Reduce to K^0 or $\langle K^0, C\rangle$.

Frobenius-Schur indicator

Corollary

$-1 \in W(G,H)$, G adjoint implies every irreducible representation of $G(\mathbb{R})$ is self-dual and orthogonal.

Problem

Consider the Frobenius-Schur indicator in general

(some of the same ideas apply)