Computing Unipotent Representations of Real Groups using the Atlas Software

Jeffrey The Plumber Adams

October 19, 2008

INTRODUCTION

G: real points of connected, reductive algebraic group

G: real points of connected, reductive algebraic group

$\Psi : W_{\mathbb{R}} \times SL(2, \mathbb{C}) \to {}^L G$

G: real points of connected, reductive algebraic group

$\Psi : W_{\mathbb{R}} \times SL(2, \mathbb{C}) \to {}^{L}G$

$\Psi|_{\mathbb{C}^*} = 1$

G: real points of connected, reductive algebraic group

$\Psi : W_{\mathbb{R}} \times SL(2,\mathbb{C}) \rightarrow {}^LG$

$\Psi|_{\mathbb{C}^*} = 1$

$\Psi : \mathbb{Z}/2\mathbb{Z} \times SL(2,\mathbb{C}) \rightarrow {}^LG$

Conjecture: (Arthur) Associated to Ψ is a (finite) set Π_Ψ of representations, satisfying (…among other things)

G: real points of connected, reductive algebraic group

$\Psi : W_{\mathbb{R}} \times SL(2, \mathbb{C}) \rightarrow {}^L G$

$\Psi|_{\mathbb{C}^*} = 1$

$\Psi : \mathbb{Z}/2\mathbb{Z} \times SL(2, \mathbb{C}) \rightarrow {}^L G$

Conjecture: (Arthur) Associated to Ψ is a (finite) set Π_Ψ of representations, satisfying (…among other things)

1) $\sum_{\pi \in \Pi_\Psi} a_\pi \pi$ is stable for some integers a_π,

G: real points of connected, reductive algebraic group

$\Psi : W_{\mathbb{R}} \times SL(2,\mathbb{C}) \to {}^LG$

$\Psi|_{\mathbb{C}^*} = 1$

$\Psi : \mathbb{Z}/2\mathbb{Z} \times SL(2,\mathbb{C}) \to {}^LG$

Conjecture: (Arthur) Associated to Ψ is a (finite) set Π_Ψ of representations, satisfying (…among other things)

1) $\sum_{\pi\in\Pi_\Psi} a_\pi \pi$ is stable for some integers a_π,

2) each $\pi \in \Pi_\Psi$ is unitary

$$\Psi|_{SL(2,\mathbb{C})} \longleftrightarrow \mathcal{O}^{\vee}_{\mathbb{C}} \quad \text{(complex unipotent orbit)}$$

$\Psi|_{SL(2,\mathbb{C})} \longleftrightarrow \mathcal{O}^\vee_{\mathbb{C}}$ (complex unipotent orbit)

Weaker version (forget the $\mathbb{Z}/2\mathbb{Z}$):

$$\Pi_{\mathcal{O}^\vee_{\mathbb{C}}} = \cup_\Psi \Pi_\Psi \quad (\Psi|_{SL(2,\mathbb{C}} \leftrightarrow \mathcal{O}^\vee_{\mathbb{C}})$$

$\Psi|_{SL(2,\mathbb{C})} \longleftrightarrow \mathcal{O}^\vee_{\mathbb{C}}$ (complex unipotent orbit)

Weaker version (forget the $\mathbb{Z}/2\mathbb{Z}$):

$$\Pi_{\mathcal{O}^\vee_{\mathbb{C}}} = \cup_\Psi \Pi_\Psi \quad (\Psi|_{SL(2,\mathbb{C}} \leftrightarrow \mathcal{O}^\vee_{\mathbb{C}})$$

Barbasch/Vogan: gave an explicitly computable definition of $\Pi_{\mathcal{O}^\vee_{\mathbb{C}}}$.

$\Psi|_{SL(2,\mathbb{C})} \longleftrightarrow \mathcal{O}^\vee_{\mathbb{C}}$ (complex unipotent orbit)

Weaker version (forget the $\mathbb{Z}/2\mathbb{Z}$):

$$\Pi_{\mathcal{O}^\vee_{\mathbb{C}}} = \cup_\Psi \Pi_\Psi \quad (\Psi|_{SL(2,\mathbb{C}} \leftrightarrow \mathcal{O}^\vee_{\mathbb{C}})$$

Barbasch/Vogan: gave an explicitly computable definition of $\Pi_{\mathcal{O}^\vee_{\mathbb{C}}}$.

All of Arthur's conjectured properties are known to hold, except unitarity

Barbasch/Vogan/A: A definition of Π_Ψ, not explicitly computable.

Barbasch/Vogan/A: A definition of Π_Ψ, not explicitly computable.

Again all of Arthur's conjectured properties are known to hold, except unitarity

Barbasch/Vogan/A: A definition of Π_Ψ, not explicitly computable.

Again all of Arthur's conjectured properties are known to hold, except unitarity

Today: an Atlas-based algorithm to compute

$$\Pi_{\mathcal{O}^\vee_{\mathbb{C}}} = \cup_\Psi \Pi_\Psi$$

Barbasch/Vogan/A: A definition of Π_Ψ, not explicitly computable.

Again all of Arthur's conjectured properties are known to hold, except unitarity

Today: an Atlas-based algorithm to compute

$$\Pi_{\mathcal{O}^\vee_{\mathbb{C}}} = \cup_\Psi \Pi_\Psi$$

and how to try to compute Π_Ψ

COMPUTING $\Pi_{\mathcal{O}^\vee_{\mathbb{C}}}$ FOR $\mathcal{O}^\vee_{\mathbb{C}}$ EVEN

[Barbasch/Vogan + Atlas]

Computing $\Pi_{\mathcal{O}^\vee_{\mathbb{C}}}$ for $\mathcal{O}^\vee_{\mathbb{C}}$ even

[Barbasch/Vogan + Atlas]

Ingredients:

[Barbasch/Vogan + Atlas]

Ingredients:

0) real group G, dual group $G^\vee$ (type)

Computing $\Pi_{\mathcal{O}^\vee_{\mathbb{C}}}$ for $\mathcal{O}^\vee_{\mathbb{C}}$ even

[Barbasch/Vogan + Atlas]

Ingredients:

0) real group G, dual group $G^\vee$ (type)

1) orbit $\mathcal{O}^\vee_{\mathbb{C}}$ (partition, tables)

Computing $\Pi_{\mathcal{O}^\vee_{\mathbb{C}}}$ for $\mathcal{O}^\vee_{\mathbb{C}}$ even

[Barbasch/Vogan + Atlas]

Ingredients:

0) real group G, dual group $G^\vee$ (type)

1) orbit $\mathcal{O}^\vee_{\mathbb{C}}$ (partition, tables)

1a) (integral) infinitesimal character $\lambda = \lambda(\mathcal{O}^\vee_{\mathbb{C}})$; only need S = simple roots such that $\langle\lambda, \alpha^\vee\rangle = 0$
1b) Special representation $\sigma(\mathcal{O}^\vee)$ of W associated to $\mathcal{O}^\vee$

2) block $\mathcal{B} = \{\pi_0, \ldots, \pi_n\} = \{0, 1, \ldots, n\}$ of representations (`block`); carries a representation of W.

2) block $\mathcal{B} = \{\pi_0, \ldots, \pi_n\} = \{0, 1, \ldots, n\}$ of representations (`block`); carries a representation of W.

2a) For $j \in \mathcal{B}$, the tau-invariant $\tau(j) = \tau(\pi_j)$, a subset of the simple roots (`block`)

Computing $\Pi_{\mathcal{O}^\vee_{\mathbb{C}}}$ for $\mathcal{O}^\vee_{\mathbb{C}}$ even

2) block $\mathcal{B} = \{\pi_0, \ldots, \pi_n\} = \{0, 1, \ldots, n\}$ of representations (`block`); carries a representation of W.

2a) For $j \in \mathcal{B}$, the tau-invariant $\tau(j) = \tau(\pi_j)$, a subset of the simple roots (`block`)

2a) cell decomposition $\mathcal{B} = \cup C_i$ $(i = 1, \ldots, m)$ (`wcells`)

Computing $\Pi_{\mathcal{O}^\vee_\mathbb{C}}$ for $\mathcal{O}^\vee_\mathbb{C}$ even

2) block $\mathcal{B} = \{\pi_0, \dots, \pi_n\} = \{0, 1, \dots, n\}$ of representations (`block`); carries a representation of W.

2a) For $j \in \mathcal{B}$, the tau-invariant $\tau(j) = \tau(\pi_j)$, a subset of the simple roots (`block`)

2a) cell decomposition $\mathcal{B} = \cup C_i$ $(i = 1, \dots, m)$ (`wcells`)

Each cell carries a representation, with a unique
2b) Special representation σ_i in C_i $(i = 1, \dots, m)$ (`wcells` + character calculation for W)

$G, \mathcal{B} \longrightarrow G^\vee = G^\vee_{\mathbb{R}}, \mathcal{B}^\vee$ (block for a real form bof $G^\vee(\mathbb{C})$)

$G, \mathcal{B} \longrightarrow G^\vee = G^\vee_{\mathbb{R}}, \mathcal{B}^\vee$ (block for a real form bof $G^\vee(\mathbb{C})$)

Bijection $\mathcal{B} \ni \pi \to \pi^\vee \in \mathcal{B}^\vee$

Vogan Duality

$G, \mathcal{B} \longrightarrow G^\vee = G^\vee_{\mathbb{R}}, \mathcal{B}^\vee$ (block for a real form bof $G^\vee(\mathbb{C})$)

Bijection $\mathcal{B} \ni \pi \to \pi^\vee \in \mathcal{B}^\vee$

Vogan duality: carries a lot of information between $\mathcal{B}$ and $\mathcal{B}^\vee$; essential part of Atlas algorithm.

Computing $\Pi_{\mathcal{O}_{\mathbb{C}}^{\vee}}$ for $\mathcal{O}_{\mathbb{C}}^{\vee}$ even

Recall (associated variety of primitive ideals):

$$\pi \to \mathrm{AV}(\mathrm{Ann}(\pi)) = \overline{\mathcal{O}_{\mathbb{C}}} \quad \text{some } \mathcal{O}_{\mathbb{C}})$$

Computing $\Pi_{\mathcal{O}^\vee_{\mathbb{C}}}$ for $\mathcal{O}^\vee_{\mathbb{C}}$ even

Recall (associated variety of primitive ideals):

$$\pi \to \mathrm{AV}(\mathrm{Ann}(\pi)) = \overline{\mathcal{O}_{\mathbb{C}}} \quad \text{some } \mathcal{O}_{\mathbb{C}})$$

Given $\mathcal{O}^\vee_{\mathbb{C}}$:

COMPUTING $\Pi_{\mathcal{O}^\vee_{\mathbb{C}}}$ FOR $\mathcal{O}^\vee_{\mathbb{C}}$ EVEN

Recall (associated variety of primitive ideals):

$$\pi \rightarrow \mathrm{AV}(\mathrm{Ann}(\pi)) = \overline{\mathcal{O}_{\mathbb{C}}} \quad \text{some } \mathcal{O}_{\mathbb{C}})$$

Given $\mathcal{O}^\vee_{\mathbb{C}}$:

Definition (Barbasch/Vogan):

$$\Pi_{\mathcal{O}^\vee_{\mathbb{C}}} = \{\pi \mid \text{infinitesimal character } \lambda(\mathcal{O}^\vee_{\mathbb{C}}),\ AV(\mathrm{Ann}(\pi^\vee)) = \overline{\mathcal{O}^\vee_{\mathbb{C}}})\}$$

Computing $\Pi_{\mathcal{O}_{\mathbb{C}}^{\vee}}$ for $\mathcal{O}_{\mathbb{C}}^{\vee}$ even

Recall (associated variety of primitive ideals):

$$\pi \rightarrow \mathrm{AV}(\mathrm{Ann}(\pi)) = \overline{\mathcal{O}_{\mathbb{C}}} \quad \text{some } \mathcal{O}_{\mathbb{C}})$$

Given $\mathcal{O}_{\mathbb{C}}^{\vee}$:

Definition (Barbasch/Vogan):

$$\Pi_{\mathcal{O}_{\mathbb{C}}^{\vee}} = \{\pi \mid \text{infinitesimal character } \lambda(\mathcal{O}_{\mathbb{C}}^{\vee}),\ AV(\mathrm{Ann}(\pi^{\vee})) = \overline{\mathcal{O}_{\mathbb{C}}^{\vee}})\}$$

Algorithm:

Given $\mathcal{O}_{\mathbb{C}}^{\vee}, \lambda \rightarrow S, \mathcal{B} = \cup C_i, \sigma(\mathcal{O}^{\vee}), \sigma_i,$

Computing $\Pi_{\mathcal{O}^\vee_{\mathbb{C}}}$ for $\mathcal{O}^\vee_{\mathbb{C}}$ even

Recall (associated variety of primitive ideals):

$$\pi \to \mathrm{AV}(\mathrm{Ann}(\pi)) = \overline{\mathcal{O}_{\mathbb{C}}} \quad \text{some } \mathcal{O}_{\mathbb{C}})$$

Given $\mathcal{O}^\vee_{\mathbb{C}}$:

Definition (Barbasch/Vogan):

$$\Pi_{\mathcal{O}^\vee_{\mathbb{C}}} = \{\pi \mid \text{infinitesimal character } \lambda(\mathcal{O}^\vee_{\mathbb{C}}),\ AV(\mathrm{Ann}(\pi^\vee)) = \overline{\mathcal{O}^\vee_{\mathbb{C}}})\}$$

Algorithm:

Given $\mathcal{O}^\vee_{\mathbb{C}}, \lambda \to S, \mathcal{B} = \cup C_i, \sigma(\mathcal{O}^\vee), \sigma_i,$

For each cell C_i such that $\sigma_i = \sigma(\mathcal{O}^\vee_{\mathbb{C}}) \otimes sgn$, take all $j \in C_i$ such that $\tau(j) \cap S = \emptyset$. These π_j (translated to λ) are the contributions of $\mathcal{B}$ to $\Pi_{\mathcal{O}^\vee_{\mathbb{C}}}$. Run over all $\mathcal{B}$ (only some $\mathcal{B}$ if G is not simply connected).

Block (`block`)

```
 0( 0,6):  0  0  [i1,i1]   1   2   ( 6, *)  ( 4, *)
 1( 1,6):  0  0  [i1,i1]   0   3   ( 6, *)  ( 5, *)
 2( 2,6):  0  0  [ic,i1]   2   0   ( *, *)  ( 4, *)
 3( 3,6):  0  0  [ic,i1]   3   1   ( *, *)  ( 5, *)
 4( 4,4):  1  2  [C+,r1]   8   4   ( *, *)  ( 0, 2)   2
 5( 5,4):  1  2  [C+,r1]   9   5   ( *, *)  ( 1, 3)   2
 6( 6,5):  1  1  [r1,C+]   6   7   ( 0, 1)  ( *, *)   1
 7( 7,2):  2  1  [i2,C-]   7   6   (10,11)  ( *, *)   2,1,2
 8( 8,3):  2  2  [C-,i1]   4   9   ( *, *)  (10, *)   1,2,1
 9( 9,3):  2  2  [C-,i1]   5   8   ( *, *)  (10, *)   1,2,1
10(10,0):  3  3  [r2,r1]  11  10   ( 7, *)  ( 8, 9)   1,2,1,2
11(10,1):  3  3  [r2,rn]  10  11   ( 7, *)  ( *, *)   1,2,1,2
```

Cells (`wcells`):

```
// Cells and their vertices.
#0={0}
#1={1}
#2={2,4,8}
#3={3,5,9}
#4={6,7,11}
#5={10}

// Induced graph on cells.
#0:.
#1:.
#2:->#0.
#3:->#1.
#4:->#0,#1.
#5:->#2,#3,#4.

// Individual cells.
// cell #0:
0[0]: {}

// cell #1:
0[1]: {}

// cell #2:
0[2]: {1} --> 1
1[4]: {2} --> 0,2
2[8]: {1} --> 1

// cell #3:
0[3]: {1} --> 1
```

Orbits/Cells and λ

```
Sp(4,R)                 SO(3,2)
Special                 Special
Orbit   Cells           Dual Orbit  Cells  #O_R   lambda
4       0,1 (large DS)  11111       4,5    1      (0,0)
22      2,3,4           311         1,2,3  2      (1,0)
1111    5 (trivial)     5           0      1      (2,1)

(211 not special        221 not special)
```

Orbits/Cells/Unipotents for $Sp(4, \mathbb{R})$ (big block)

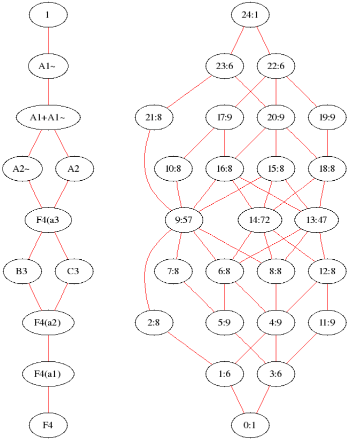
```
Sp(4,R)                 SO(3,2)
Special                 Special
Orbit   Cells           Dual Orbit  Cells  #O_R   lambda  Unipotents
4       0,1 (large DS)  11111       4,5    1      (0,0)   0,1 (large DS at 0)
22      2,3,4           311         1,2,3  2      (1,0)   2,3,6,8,9,11
1111    5 (trivial)     5           0      1      (2,1)   10 (trivial)

(211 not special        221 not special)
```

Special orbits and cells for F_4

EXAMPLE: F_4

Special orbits and cells for F_4



EXAMPLE: F_4

Orbits/Cells/λ for F_4 (big block)

Special Orbit	Cells	Special Dual Orbit	Cells	diagram
F4	0 (large DS)	0	24	0000
F4(a3)	9,13,14	F4(a3)	9,13,14	0200
C3	12	A2˜	18	0002
B3	2,6,7,8	A2	10,15,16,21	2000
A2	10,15,16,21	B3	2,6,7,8	2200
A1+A1˜	17,19,20	F4(a2)	4,5,11	0202
A1˜	22,23	F4(a1)	1,3	2202
0	24(trivial)	F4	0	2222
F4(a1)	1,3	A1˜	22,23	0001
F4(a2)	4,5,11	A1+A1˜	17,19,20	0100
A2˜	18	C3	12	1012

EXAMPLE: F_4

Orbits/Cells/λ for split F_4 (big block)

Special Orbit	Cells	Special Dual Orbit	Cells	diagram	Unipotents
F4	0 (large DS)	0	24	0000	7 (large DS at 0)
F4(a3)	9,13,14	F4(a3)	9,13,14	0200	34,81,98,147,161,191,192,193
					194,225,246,285,295,327
C3	12	A2~	18	0002	213
B3	2,6,7,8	A2	10,15,16,21	2000	68,208,251,324
A2	10,15,16,21	B3	2,6,7,8	2200	146,257,293,325
A1+A1~	17,19,20	F4(a2)	4,5,11	0202	267,309,333
A1~	22,23	F4(a1)	1,3	2202	291,297,314,334
0	24(trivial)	F4	0	2222	331 (trivial)
F4(a1)	1,3	A1~	22,23	0001	
F4(a2)	4,5,11	A1+A1~	17,19,20	0100	
A2~	18	C3	12	1012	

$\Psi : \mathbb{Z}_2 \times SL(2,\mathbb{C}) \to W_\mathbb{R}$

$\Psi : \mathbb{Z}_2 \times SL(2,\mathbb{C}) \to W_\mathbb{R}$

$\Psi|_{SL(2,\mathbb{C})} \leftrightarrow G^\vee(\mathbb{C})$-orbit $\mathcal{O}^\vee_\mathbb{C} \subset \mathfrak{g}^\vee$

Computing an Arthur packet Π_Ψ

$\Psi : \mathbb{Z}_2 \times SL(2,\mathbb{C}) \to W_\mathbb{R}$

$\Psi|_{SL(2,\mathbb{C})} \leftrightarrow G^\vee(\mathbb{C})$-orbit $\mathcal{O}^\vee_\mathbb{C} \subset \mathfrak{g}^\vee$

$y = \Psi(-1) \to \mathrm{int}(y)$: Cartan involution for $G^\vee(\mathbb{C})$

Computing an Arthur packet Π_Ψ

$\Psi : \mathbb{Z}_2 \times SL(2,\mathbb{C}) \to W_\mathbb{R}$

$\Psi|_{SL(2,\mathbb{C})} \leftrightarrow G^\vee(\mathbb{C})$-orbit $\mathcal{O}^\vee_\mathbb{C} \subset \mathfrak{g}^\vee$

$y = \Psi(-1) \to \mathrm{int}(y)$: Cartan involution for $G^\vee(\mathbb{C})$

$\Psi \longleftrightarrow G^\vee(\mathbb{R})$-orbit $\mathcal{O}^\vee_\mathbb{R} \subset \mathcal{O}^\vee_\mathbb{C} \cap \mathfrak{g}^\vee_\mathbb{R}$

Computing an Arthur packet Π_Ψ

$\Psi : \mathbb{Z}_2 \times SL(2,\mathbb{C}) \to W_\mathbb{R}$

$\Psi|_{SL(2,\mathbb{C})} \leftrightarrow G^\vee(\mathbb{C})$-orbit $\mathcal{O}^\vee_\mathbb{C} \subset \mathfrak{g}^\vee$

$y = \Psi(-1) \to \mathrm{int}(y)$: Cartan involution for $G^\vee(\mathbb{C})$

$\Psi \longleftrightarrow G^\vee(\mathbb{R})$-orbit $\mathcal{O}^\vee_\mathbb{R} \subset \mathcal{O}^\vee_\mathbb{C} \cap \mathfrak{g}^\vee_\mathbb{R}$

$\Pi_\Psi = \Pi_{\mathcal{O}^\vee_\mathbb{R}} \subset \Pi_{\mathcal{O}^\vee_\mathbb{C}}$

Associated Varieties/Cycles

$\pi \rightarrow \mathrm{AV}(\mathrm{Ann}(\pi)) = \overline{\mathcal{O}}_{\mathbb{C}}$

Associated Varieties/Cycles

$\pi \rightarrow \mathrm{AV}(\mathrm{Ann}(\pi)) = \overline{\mathcal{O}}_{\mathbb{C}}$

$\pi \rightarrow \mathrm{AV}(\pi) = \mathcal{O}^1_{\mathbb{R}} \cup \mathcal{O}^2_{\mathbb{R}} \cup \cdots \cup \mathcal{O}^m_{\mathbb{R}} \subset \mathcal{O}_{\mathbb{C}}$
(via Kostant-Sekiguchi correspondence)

Associated Varieties/Cycles

$\pi \to \mathrm{AV}(\mathrm{Ann}(\pi)) = \overline{\mathcal{O}}_{\mathbb{C}}$

$\pi \to \mathrm{AV}(\pi) = \mathcal{O}^1_{\mathbb{R}} \cup \mathcal{O}^2_{\mathbb{R}} \cup \cdots \cup \mathcal{O}^m_{\mathbb{R}} \subset \mathcal{O}_{\mathbb{C}}$
(via Kostant-Sekiguchi correspondence)

Also:
$\pi \to \mathrm{AC}(\pi) = \sum_{i=1}^m a(\mathcal{O}^i_{\mathbb{R}}, \pi)\mathcal{O}^i_{\mathbb{R}}$

Given: $\mathcal{O}^\vee_\mathbb{R} \subset \mathcal{O}^\vee_\mathbb{C} \cap \mathfrak{g}^\vee_\mathbb{R}$

Given: $\mathcal{O}_{\mathbb{R}}^\vee \subset \mathcal{O}_{\mathbb{C}}^\vee \cap \mathfrak{g}_{\mathbb{R}}^\vee$

Recall

$$\Pi_{\mathcal{O}_{\mathbb{C}}^\vee} = \{\pi \mid \text{infinitesimal character } \lambda(\mathcal{O}_{\mathbb{C}}^\vee), \text{AV}(\text{Ann}(\pi^\vee)) = \mathcal{O}_{\mathbb{C}}^\vee\}$$

Definition: (ABV):

$$\Pi_\Psi = \{\pi \mid \text{infinitesimal character } = \lambda(\mathcal{O}_{\mathbb{C}}^\vee),\ \mathcal{O}_{\mathbb{R}}^\vee \subset \text{AV}(\pi^\vee)\}$$

Computing an Arthur packet Π_Ψ

Given: $\mathcal{O}^\vee_{\mathbb{R}} \subset \mathcal{O}^\vee_{\mathbb{C}} \cap \mathfrak{g}^\vee_{\mathbb{R}}$

Recall

$$\Pi_{\mathcal{O}^\vee_{\mathbb{C}}} = \{\pi \mid \text{infinitesimal character } \lambda(\mathcal{O}^\vee_{\mathbb{C}}), \mathrm{AV}(\mathrm{Ann}(\pi^\vee)) = \mathcal{O}^\vee_{\mathbb{C}}\}$$

Definition: (ABV):

$$\Pi_\Psi = \{\pi \mid \text{infinitesimal character } = \lambda(\mathcal{O}^\vee_{\mathbb{C}}),\ \mathcal{O}^\vee_{\mathbb{R}} \subset \mathrm{AV}(\pi^\vee)\}$$

$$\Pi_{\mathcal{O}^\vee_{\mathbb{C}}} = \cup_i \Pi_{O^{\vee i}_{\mathbb{R}}} \quad \text{(not disjoint)}$$

Problem: No closed algorithm to compute $\mathrm{AV}(\pi)$

Given: $\mathcal{O}^\vee_{\mathbb{R}} \subset \mathcal{O}^\vee_{\mathbb{C}} \cap \mathfrak{g}^\vee_{\mathbb{R}}$

Recall

$$\Pi_{\mathcal{O}^\vee_{\mathbb{C}}} = \{\pi \mid \text{infinitesimal character } \lambda(\mathcal{O}^\vee_{\mathbb{C}}), \text{AV}(\text{Ann}(\pi^\vee)) = \mathcal{O}^\vee_{\mathbb{C}}\}$$

Definition: (ABV):

$$\Pi_\Psi = \{\pi \mid \text{infinitesimal character } = \lambda(\mathcal{O}^\vee_{\mathbb{C}}),\ \mathcal{O}^\vee_{\mathbb{R}} \subset \text{AV}(\pi^\vee)\}$$

$$\Pi_{\mathcal{O}^\vee_{\mathbb{C}}} = \cup_i \Pi_{O^{\vee i}_{\mathbb{R}}} \quad \text{(not disjoint)}$$

Problem: No closed algorithm to compute $\text{AV}(\pi)$

Not hopeless, lots of cases where it is known. . .

Given: $\mathcal{O}^\vee_{\mathbb{R}}$

Recall:

$$\Pi_{\mathcal{O}^\vee_{\mathbb{R}}} = \{\pi_1, \dots, \pi_n\}$$

STABILITY

Given: $\mathcal{O}_{\mathbb{R}}^\vee$
Recall:

$$\Pi_{\mathcal{O}_{\mathbb{R}}^\vee} = \{\pi_1, \ldots, \pi_n\}$$

Recall
$\mathrm{AC}(\pi) = \sum_{i=1}^{m} a(\mathcal{O}_{\mathbb{R}}^i, \pi)\mathcal{O}_{\mathbb{R}}^i$

Given: $\mathcal{O}^\vee_{\mathbb{R}}$
Recall:

$$\Pi_{\mathcal{O}^\vee_{\mathbb{R}}} = \{\pi_1, \dots, \pi_n\}$$

Recall
$\mathrm{AC}(\pi) = \sum_{i=1}^m a(\mathcal{O}^i_{\mathbb{R}}, \pi)\mathcal{O}^i_{\mathbb{R}}$

Definition:

$$\Theta(\mathcal{O}^\vee_{\mathbb{R}}) = \sum_i \epsilon_i a(\mathcal{O}_{\mathbb{R}}, \pi_i)\pi_i$$

($\epsilon_i = \pm 1$ explicit)

This is the stable virtual character of $\Pi_{\mathcal{O}^\vee_{\mathbb{R}}}$ conjectured by Arthur:

This is the stable virtual character of $\Pi_{\mathcal{O}^\vee_{\mathbb{R}}}$ conjectured by Arthur:

Theorem: (ABV) $\Theta(\mathcal{O}^\vee_{\mathbb{R}})$ is stable.

This is the stable virtual character of $\Pi_{\mathcal{O}^\vee_{\mathbb{R}}}$ conjectured by Arthur:

Theorem: (ABV) $\Theta(\mathcal{O}^\vee_{\mathbb{R}})$ is stable.

Problem: Can't compute $\Pi_{\mathcal{O}^\vee_{\mathbb{R}}}, \Theta(\mathcal{O}^\vee_{\mathbb{R}})$

This is the stable virtual character of $\Pi_{\mathcal{O}^\vee_{\mathbb{R}}}$ conjectured by Arthur:

Theorem: (ABV) $\Theta(\mathcal{O}^\vee_{\mathbb{R}})$ is stable.

Problem: Can't compute $\Pi_{\mathcal{O}^\vee_{\mathbb{R}}}, \Theta(\mathcal{O}^\vee_{\mathbb{R}})$

Part of the Atlas philosophy: (almost) everything follows from the Kazhdan-Lusztig-Vogan polynomials.

This is the stable virtual character of $\Pi_{\mathcal{O}_{\mathbb{R}}^\vee}$ conjectured by Arthur:

Theorem: (ABV) $\Theta(\mathcal{O}_{\mathbb{R}}^\vee)$ is stable.

Problem: Can't compute $\Pi_{\mathcal{O}_{\mathbb{R}}^\vee}, \Theta(\mathcal{O}_{\mathbb{R}}^\vee)$

Part of the Atlas philosophy: (almost) everything follows from the Kazhdan-Lusztig-Vogan polynomials.

Example: we can compute the space of stable virtual characters.

This is the stable virtual character of $\Pi_{\mathcal{O}^\vee_{\mathbb{R}}}$ conjectured by Arthur:

Theorem: (ABV) $\Theta(\mathcal{O}^\vee_{\mathbb{R}})$ is stable.

Problem: Can't compute $\Pi_{\mathcal{O}^\vee_{\mathbb{R}}}$, $\Theta(\mathcal{O}^\vee_{\mathbb{R}})$

Part of the Atlas philosophy: (almost) everything follows from the Kazhdan-Lusztig-Vogan polynomials.

Example: we can compute the space of stable virtual characters.

Idea: Use stability to try to compute $\Pi_{\mathcal{O}^\vee_{\mathbb{R}}}$, and therefore also $\mathrm{AV}(\pi)$, $\mathrm{AC}(\pi)$

Algorithm to try to compute $\Pi_{\mathcal{O}^\vee_\mathbb{R}}$

Algorithm to try to compute $\Pi_{\mathcal{O}^\vee_{\mathbb{R}}}$

Input: G, $G^\vee$, block $\mathcal{B}$, dual block $\mathcal{B}^\vee$, cell decomposition of $\mathcal{B}$ (as before)

Algorithm to try to compute $\Pi_{\mathcal{O}^\vee_{\mathbb{R}}}$

Input: G, $G^\vee$, block $\mathcal{B}$, dual block $\mathcal{B}^\vee$, cell decomposition of $\mathcal{B}$ (as before)

Also: $\mathcal{O}^\vee_{\mathbb{C}}$, special representation $\sigma\,(\mathcal{O}^\vee_{\mathbb{C}})$ (as before)

Algorithm to try to compute $\Pi_{\mathcal{O}^\vee_{\mathbb{R}}}$

Input: G, $G^\vee$, block $\mathcal{B}$, dual block $\mathcal{B}^\vee$, cell decomposition of $\mathcal{B}$ (as before)

Also: $\mathcal{O}^\vee_{\mathbb{C}}$, special representation $\sigma(\mathcal{O}^\vee_{\mathbb{C}})$ (as before)

New: real form $\mathcal{O}^\vee_{\mathbb{R}}$ of $\mathcal{O}^\vee_{\mathbb{C}}$ (signed permutations, tables)

Algorithm to try to compute $\Pi_{\mathcal{O}^\vee_\mathbb{R}}$

Input: G, $G^\vee$, block $\mathcal{B}$, dual block $\mathcal{B}^\vee$, cell decomposition of $\mathcal{B}$ (as before)

Also: $\mathcal{O}^\vee_\mathbb{C}$, special representation $\sigma(\mathcal{O}^\vee_\mathbb{C})$ (as before)

New: real form $\mathcal{O}^\vee_\mathbb{R}$ of $\mathcal{O}^\vee_\mathbb{C}$ (signed permutations, tables)

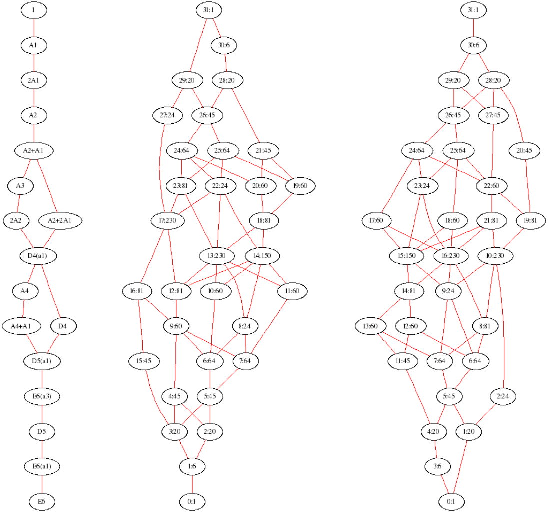
Look for stable sums in $\Pi_{\mathcal{O}^\vee}$.

EXAMPLE: E_6

Special complex orbits, cells and dual cells for for E_6 (split sc/ad quaternionic)

EXAMPLE: E_6

Special complex orbits, cells and dual cells for for E_6 (split sc/ad quaternionic)



E_6 split (large block)

```
split                quaternionic
special              special
orbit    cells       dualorbit cells           diagram
0        31 (trivial) E6       0   (large DS)  222222   1
A1       30          E6(a1)    3               222022   1
2A1      28,29       D5        1,4             220202   2
A2       21,26       E6(a3)    5,11            200202   2
A3       18,23       A4        8,14            220002   2
2A2      22,27       D4        2,9             020200   2
D4(a1)   13,14,17    D4(a1)    10,15,16        000200   3
D4       8           2A2       23              200002   1
E6(a3)   4,5,15      A2        20,26,27        020000   3
E6       0 (large FS) 0        31 (trivial)    000000   1

A2+A1    24,25       D5(a1)    6,7             121011
A2+2A1   19,20       A4+A1     12,13           121011
A4       12,16       A3        19,21           120001
A4+A1    9,10,11     A2+2A1    17,18,22        001010
D5(a1)   6,7         A2+A1     24,25           110001
D5       2,3         2A1       28,29           100001
E6(a1)   1           A1        30              010000
```

In this case (for each even orbit):

In this case (for each even orbit):

number of real forms of $\mathcal{O}^\vee_\mathbb{C}$

=

number of cells associated to $\mathcal{O}^\vee_\mathbb{C}$

In this case (for each even orbit):

number of real forms of $\mathcal{O}^\vee_{\mathbb{C}}$

=

number of cells associated to $\mathcal{O}^\vee_{\mathbb{C}}$

Conclusion: Can compute $\Pi_{\mathcal{O}_{\mathbb{R}}}$

EXAMPLE: E_6

Arthur packets for E_6 split (large block)

split special orbit	cells	quaternionic special dualorbit	cells	diagram		
0	31 (trivial)	E6	0 (large DS)	222222	1	31 (trivial)
A1	30	E6(a1)	3	222022	1	1865
2A1	28,29	D5	1,4	220202	2	1778,1878
A2	21,26	E6(a3)	5,11	200202	2	1171+1703, 1503,1867
A3	18,23	A4	8,14	220002	2	1574,1861
2A2	22,27	D4	2,9	020200	2	1540,1873
D4(a1)	13,14,17	D4(a1)	10,15,16	000200	3	499+1465, 421+1193+1851, 981+1538
D4	8	2A2	23	200002	1	1027
E6(a3)	4,5,15	A2	20,26,27	020000	3	4+820, 18+911, 596+1874
E6	0 (large FS)	0	31 (trivial)	000000	1	0 (large FS at 0)
A2+A1	24,25	D5(a1)	6,7	121011		
A2+2A1	19,20	A4+A1	12,13	121011		
A4	12,16	A3	19,21	120001		
A4+A1	9,10,11	A2+2A1	17,18,22	001010		
D5(a1)	6,7	A2+A1	24,25	110001		
D5	2,3	2A1	28,29	100001		
E6(a1)	1	A1	30	010000		

EXAMPLE: F_4

Special Orbit	Cells	Special Dual Orbit	Cells	diagram	#real forms of #(even) dual orbit
F4	0(LDS)	0	24	0000	1
F4(a1)	1,3	A1˜	22,23	0001	
F4(a2)	4,5,11	A1+A1˜	17,19,20	0100	
F4(a3)	9,13,14	F4(a3)	9,13,14	0200	3
C3	12	A2˜	18	0002	1
B3	2,6,7,8	A2	10,15,16,21	2000	3
A2	10,15,16,21	B3	2,6,7,8	2200	2
A2˜	18	C3	12	1012	
A1+A1˜	17,19,20	F4(a2)	4,5,11	0202	2
A1˜	22,23	F4(a1)	1,3	2202	2
0	24(trivial)	F4	0	2222	1

EXAMPLE: F_4

Special Orbit	Cells	Special Dual Orbit	Cells	diagram	#real forms of #(even) dual orbit
F4(a3)	9,13,14	F4(a3)	9,13,14	0200	3

Basis of stable characters expressed as sums of irreducibles
34,81,98,147,161,191,192,193,194,225,246,285*,295*,327*:

0	-1	0	1	0	1	0	0	0	0	0	0	0	1
0	0	0	0	0	1	1	0	0	0	0	0	1	0
0	0	0	1	0	1	1	0	0	0	0	1	0	0
1	1	0	0	0	-1	0	0	0	0	1	0	0	0
1	0	0	0	0	-1	-1	0	0	1	0	0	0	0
0	1	0	0	0	0	1	0	1	0	0	0	0	0
-1	-1	0	1	0	1	0	1	0	0	0	0	0	0
0	1	0	-1	1	0	0	0	0	0	0	0	0	0
-1	-1	1	0	0	0	0	0	0	0	0	0	0	0

(*): basepoints

Three real forms of orbit, three cells $\Rightarrow$ three disjoint Arthur packets:
cell 9: 81,191,192,194,295*:
cell 13: 34,147,193,246,327*:
cell 14: 98,161,225,285*:
(*): basepoint

From the data we can conclude for some ($a \geq 1, b \geq 2$):

$$\begin{aligned}
\Theta_{\Psi_9} &= a\pi_{81} + \pi_{191} + (a+1)\pi_{192} + a\pi_{194} + \pi_{295^*} \\
\Theta_{\Psi_{13}} &= \pi_{34} + b\pi_{147} + (b-1)\pi_{193} + b\pi_{246} + \pi_{327^*} \\
\Theta_{\Psi_{14}} &= \pi_{98} + \pi_{161} + \pi_{225} + \pi_{285^*}
\end{aligned}$$

(*): basepoints (know coefficient =1)