Atlas of Lie Groups and Representations

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Automorphisms

The Contragredient and Hermitian Duals

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Honk Please

Overview

$G(F)$ reductive, F local

$$\tau \circlearrowright \widehat{G}(F)_{adm} \longleftrightarrow \{\phi : W_F' \to {}^L G\} \circlearrowleft \gamma$$

Examples:

1) $\tau(\pi) = \pi^*$ (contragredient)

2) $\tau(\pi) = \pi^h$ Hermitian dual, and variants of this

3) γ: algebraic automorphism of $G^\vee$

4) γ: automorphism of $G^\vee(\mathbb{C})$ viewed as a *real* group

Closely related: D. Prasad (recent), D. Prasad/Ramakrishnan

The Contragredient

π^* = contragredient of π

Question: What is $\pi \to \pi^*$ in terms of L-homomorphisms?

(thanks to Kevin Buzzard for asking)

$$\phi : W'_F \to {}^L G \rightsquigarrow \Pi(\phi) \quad \text{(conjectural)}$$

Question: Given: $\pi \in \Pi(\phi)$. Find ϕ^* so that $\pi^* \in \Pi(\phi^*)$.

The Contragredient

$GL(n,F)$ (F p-adic)

$\phi \rightsquigarrow \pi(\phi)$ (singleton)

$\phi =$ representation of $W_F' \rightsquigarrow \phi^* = {}^t\phi^{-1}$ (contragradient)

Theorem (Harris/Taylor/Henniart):

LLC for $GL(n,F)$ commutes with the contragredient:

$$\pi(\phi^*) = \pi(\phi)^*$$

(tied up with L, epsilon, and especially gamma factors)

The Contragredient

General G

C = Chevalley automorphism of $G^\vee(\mathbb{C})$:

1) $C(h) = h^{-1}$, $h \in H^\vee$, $C^2 = 1$;

2) $C(h) \sim h^{-1}$ for all semisimple elements h,

3) C is unique up to conjugation by an inner automorphism,

4) C is the Cartan involution of the split real form of $G^\vee$,

5) C defined in terms of the pinning $(H_0^\vee, B_0^\vee, \{X_{\alpha^\vee}\})$ defining ${}^LG$,

6) C extends to ${}^LG$, trivial on the Galois group.

The Contragredient

Conjecture: Assume the local Langlands classification is known for π and π^*. Then

$$\pi \in \Pi(\phi) \Leftrightarrow \pi^* \in \Pi(C \circ \phi)$$

i.e.

$$\boxed{\Pi(\phi)^* = \Pi(C \circ \phi)}$$

(implies $\Pi(\phi)^*$ is an L-packet)

GL(n): $C(g) = {}^t g^{-1} \Rightarrow$ true for $GL(n, F)$ p-adic

D. Prasad: stronger version of the same conjecture

The Contragredient

Theorem: (A/Vogan) The conjecture holds over $\mathbb{R}$ and $\mathbb{C}$

Sketch of proof (comes down to a characterization of LLC)

Fix $H_0, H_0^\vee, X^*(H_0) = X_*(H_0^\vee)$ defining $G^\vee$

$$\phi(z) = z^{\lambda}\overline{z}^{\lambda'} \quad (\lambda, \lambda' \in X_*(H_0^\vee) \otimes \mathbb{C})$$

$$G_{rad} \hookrightarrow G \rightsquigarrow {}^L G \overset{p}{\twoheadrightarrow} {}^L G_{rad}$$

The Contragredient

Definition: $\phi : W_{\mathbb{R}} \to {}^L G$

$$\chi_{inf}(\phi) = \lambda \in X^*(H_0) \otimes \mathbb{C}$$

$$\chi_{rad}(\phi) = \pi(p \circ \phi) \in \widehat{G_{rad}(\mathbb{R})} \quad \text{(from the torus case)}$$

Definition: $\chi_{inf}(\pi), \chi_{rad}(\pi)$

(infinitesimal character and radical characters of π)

The Contragredient

Theorem:

The correspondence $\phi \to \Pi(\phi)$ is uniquely determined by:

1) $\Pi(\phi)$ has infinitesimal character $\chi_{inf}(\phi)$

2) $\Pi(\phi)$ has radical character $\chi_{rad}(\phi)$,

3) compatibility with parabolic induction:

roughly:

$$
\begin{array}{ccccc}
W_{\mathbb{R}} & \xrightarrow{\phi_M} & {}^L M & \longrightarrow & \Pi_M(\phi_M) \\
 & \searrow^{\phi} & \downarrow \iota & & \downarrow Ind \\
 & & {}^L G & \longrightarrow & \Pi_G(\phi)
\end{array}
$$

The Contragredient

Note: A discrete series L-packet is determined by an infinitesimal and radical character
(don't need the full central character, embedding G in a group with connected center, etc.)

Lemma A:

1) $\chi_{inf}(\pi^*) = -\chi_{inf}(\pi)$

2) $\chi_{rad}(\pi^*) = \chi_{rad}(\pi)^*$

3) $\mathrm{Ind}_M^G(\pi_M^*) \simeq \mathrm{Ind}_M^G(\pi_M)^*$

Lemma B:

1) $\chi_{inf}(C \circ \phi) = -\chi_{inf}(\phi)$

2) $\chi_{rad}(C \circ \phi) = \chi_{rad}(\phi)^*$ (torus case)

3) $C_G|_M = C_M$

$\Rightarrow$ the theorem

The Contragredient

Theorem is a special case of:

$F = \mathbb{R}$, G arbitrary:

$\tau \in \mathrm{Aut}(G) = Aut_{alg=hol}(G)$, $\tau\theta = \theta\tau$
τ acts on $(\mathfrak{g}, K)$-modules

$$\mathrm{Aut}(G) \to \mathrm{Out}(G) \simeq \mathrm{Out}(G^\vee) \hookrightarrow \mathrm{Aut}({}^L G)$$

$$\tau \longrightarrow \tau^t$$

Theorem

$$\boxed{\Pi(\phi)^\tau = \Pi(\tau^t \circ \phi)}$$

($\tau = C \Rightarrow$ contragredient Theorem)

Digression: version without packets?

For simplicity assume: G is adjoint, simply connected, and $Aut(G) = 1$.

G has real forms $G_1(\mathbb{R}), \dots, G_n(\mathbb{R})$,
$K_1, \dots, K_n$ (complexified maximal compacts)

$$\mathcal{X} = \bigcup_i K_i \backslash G/B$$

$$\mathcal{X}^\vee = \bigcup_j K_j^\vee \backslash G^\vee/B^\vee$$

Digression: version without packets?

Theorem (atlas algorithm): There is a canonical bijection:

$$\{(x,y) \in \mathcal{X} \times \mathcal{X}^\vee\}_0 \longleftrightarrow \bigcup_i \widehat{G_i(\mathbb{R})}_\rho$$

[$\{\ \}_0$: subset of (x,y) satisfying $\theta_x^t = -\theta_y$]

[General statement: fix an inner class; strong real forms; other infinitesimal characters]

$y \rightsquigarrow \phi$, $x \rightsquigarrow \pi$ in $\Pi(\phi)$

involution of $\{(x,y)\}$?

Contragredient: $(x,y) \to (w_0 x, w_0 C(y))$

Different version: D. Prasad

The Hermitian Dual

G: complex reductive, $\theta=$ involution, $K = G^\theta \leftrightarrow G(\mathbb{R})$

$(\pi, V) = (\mathfrak{g}, K)$-module (everything here is complex)
correspond to representations of $G(\mathbb{R})$

Definition: The Hermitian dual (π^h, V^h) of (π, V):

V^h: K-finite, conjugate-linear functionals $V \to \mathbb{C}$

$$\pi^h(X)(f)(v) = -\pi(f(X)v) \quad (X \in \mathfrak{g}_0)$$

better:

$$\boxed{\pi^h(X)(f)(v) = -\pi(f(\sigma(X))v) \quad (X \in \mathfrak{g})}$$

$(\mathfrak{g}^\sigma = \mathfrak{g}_0)$

The Hermitian Dual

(π, V) irreducible

Lemma: π has an invariant Hermitian form

$$\langle \pi(X)v, w\rangle + \langle v, \pi(\sigma(X))w\rangle = 0$$

if and only if $(\pi, V) \simeq (\pi^h, V^h)$.

Do not assume $\langle \, , \, \rangle$ is definite.

Unitary dual: subset of the fixed points of the $\pi \to \pi^h$

(those for which the form is definite).

The Hermitian Dual

Question: What is $\pi \to \pi^h$ on the level of L-homomorphisms?

Guess: since π^h involves $\sigma \ldots$ use an anti-holomorphic involution of ${}^L G$? Which one?

Digression on real forms

G complex

θ (holomorphic involution), $K = G^\theta$

σ (antiholomorphic involution), $G(\mathbb{R}) = G^\sigma$

Fix σ_c, $G_c(\mathbb{R}) = G^{\sigma_c}$ is compact (compact real form)

$$\theta \leftrightarrow \sigma : \quad \theta = \sigma\sigma_c, \sigma = \theta\sigma_c$$

$K(\mathbb{R}) = K \cap G(\mathbb{R})$ is a maximal compact subgroup of $G(\mathbb{R})$

Digression on real forms

Some standard real forms

σ	θ	real form
σ_s	$\theta_s = C$	split
σ_{qs}	principal	quasisplit
σ	θ	$G(\mathbb{R})$
σ_{qc}	distinguished	quasicompact
σ_c	$\theta_c = 1$	compact

quasisplit: most split in the inner class (σ_{qs} fixes a Borel)

quasicompact: most compact in the inner class (θ_{qc} distinguished)

(Distinguished: preserves a splitting datum $(G, H, \{X_\alpha\})$)

(Yu: "quasianisotropic")

Digression on real forms

Example:

$SO(5,5)$	split=quasisplit
$SO(7,3)$	$G(\mathbb{R})$
$SO(9,1)$	quasicompact
$SO(10)$	compact

$SO(5,5)$	split
$SO(6,4)$	quasisplit
$SO(8,2)$	$G(\mathbb{R})$
$SO(10)$	compact=quasicompact

The Hermitian Dual

$G,\ \theta \to {}^LG = G^\vee \rtimes \delta^\vee$

$\delta^\vee$ distinguished

$\delta^\vee$ think of as a Cartan involution $\rightsquigarrow \sigma^\vee_{qc}$ quasicompact real form, (antiholomorphic automorphism of $G^\vee, {}^LG$)

Theorem: For $F = \mathbb{R}$, G arbitrary:

$$\boxed{\Pi(\phi)^h = \Pi(\sigma^\vee_{qc} \circ \phi)}$$

Note: $\sigma^\vee_{qc} = \sigma^\vee_c$ iff $G(\mathbb{R})$ is split

Note: This is a kind of functoriality for antiholomorphic automorphisms of ${}^LG$ ...what about when F is p-adic?

The Hermitian Dual: $GL(n)$

$GL(n,F)$, F local, characteristic 0

$\delta^\vee = 1$, $\sigma^\vee$ is the compact real form, $\sigma^\vee(g) = {}^t\overline{g}^{-1}$

$GL(n,\mathbb{C})^{\sigma^\vee} = U(n)$

$\phi : W'_F \to GL(n,\mathbb{C})$, n-dimensional representation

Hermitian dual of ϕ: $\phi^h = {}^t\overline{\phi}^{-1}$

1) ϕ preserves a Hermitian form $\Leftrightarrow \phi \simeq \phi^h$

2) ϕ is unitary $\leftrightarrow \phi = \phi^h$

Hermitian dual of π defined as over $\mathbb{R}$

Theorem: (A/Ciubotaru) $GL(n,F)$ for F local, characteristic 0

1) LLC commutes with the Hermitian dual:

$$\pi(\phi^h) = \pi(\phi)^h$$

2) ϕ is Hermitian if and only if $\pi(\phi)$ is Hermitian

3) ϕ is unitary if and only if $\pi(\phi)$ is tempered

Sketch of proof in p-adic case: supercuspidal, discrete series, relative discrete series, induction

Digression: KLV for forms

Kazhdan-Lusztig-Vogan picture

λ = regular infinitesimal character

$S = \{\gamma\}$ parameter set (finite) for irreducible representations with infinitesimal character λ

$\gamma \rightsquigarrow \pi(\gamma)$: irreducible representation

$\gamma \rightsquigarrow I(\gamma)$: standard representation

Digression: KLV for forms

Character theory:

$$\pi(\gamma) = \sum_{\delta \in S} (-1)^{\ell(\gamma)-\ell(\delta)} P_{\gamma,\delta}(1) I(\delta)$$

$P_{\gamma,\delta}$: Kazhdan-Lusztig-Vogan polynomial

Version for representations equipped with Hermitian forms?

$$(\pi(\gamma), \langle\, ,\, \rangle) \overset{?}{=} \sum_{\delta \in S} (-1)^{\ell(\gamma)-\ell(\delta)} M^h_{\gamma,\delta}(I(\delta), \langle\, ,\, \rangle)$$

some $M^h_{\gamma,\delta} \in \mathbb{Z}[s]$ $(s^2 = 1)$, (presumably given by some kind of KLV polynomial)

$(a + bs)(\pi, \langle\, ,\, \rangle)$ means: $a(\pi, \langle\, ,\, \rangle) + b(\pi, -\langle\, ,\, \rangle)$

Digression: KLV for forms

Problem:

1) $\pi(\gamma)$ may not have an invariant form

2) $\boxed{\text{there is no canonical choice of } \langle\,,\,\rangle \text{ versus } -\langle\,,\,\rangle}$

$\Rightarrow M^h_{\gamma,\delta}$ not well defined

The c-Hermitian Dual

Recall (π^h, V^h):

$$\pi^h(X)(f)(v) = -f(\pi(\sigma(X)v))$$

Suppose σ': any conjugate-linear automorphism of $\mathfrak{g}$

Definition: $\pi^{h,\sigma'}(X)(f)(v) = -f(\pi(\sigma'(X)v))$

Proposition: $(\pi^{h,\sigma'}, V^h)$ is a $(\mathfrak{g}, K)$-module

(even though σ' is unrelated to $\sigma, \mathfrak{g}_0$)

$\pi^{h,\sigma'} = \sigma'$–Hermitian dual

The c-Hermitian Dual

Check linearity:

$$\begin{aligned}
\pi^{h,\sigma'}(\lambda X)(f)(v) &= -\pi(f(\sigma'(\lambda X)v)) \\
&= -\pi(f(\overline{\lambda}\sigma'(X)v)) \quad (\sigma' \text{ is conj. linear}) \\
&= -\pi(\lambda f(\sigma'(X)v)) \quad (f \text{ is conj. linear}) \\
&= \lambda\pi^{h,\sigma'}(X)(f)(v)
\end{aligned}$$

Entirely trivial…

Remark: σ' inner to $\sigma \Rightarrow \pi^{h,\sigma'} \simeq \pi^{h,\sigma}$

The c-Hermitian Dual

Definition: The c-Hermitian dual is the Hermitian dual defined with respect to the compact form σ_c:

$$\boxed{\pi^{h,c} = \pi^{h,\sigma_c}}$$

$\pi^{h,c}$ is a $(\mathfrak{g}, K)$-module

Definition: c-invariant form $\langle\ ,\ \rangle_c$:

$$\langle \pi(X)v, w\rangle_c + \langle X, \pi(\sigma_c(X))\rangle_c = 0$$

The c-Hermitian Dual

Theorem: (A, Trapa, Vogan, van Leeuwen, Yee)

π irreducible, real infinitesimal character (in $X^*(H)\otimes\mathbb{R}$)

1) $\pi\simeq\pi^{h,c}$; π has a c-invariant form

2)

$$\boxed{\pi \text{ has a canonical c-invariant form}}$$

positive definite on all lowest K-types.

The c-Hermitian Dual

Sketch of proof:

1) π discrete series

$\sigma = \theta\sigma_c$, θ is inner

$\Rightarrow \pi^{h,\sigma_c} = \pi^{h,\sigma} = \pi \qquad (\pi \text{ is unitary})$

2) H split torus, on $X^*(H) \otimes \mathbb{R}$:

$\sigma_c = \theta\sigma = (-1)(+1) = -1$

$\chi \in X^*(H)$, $\chi^{h,c} = -\chi^c = \chi$

3) G split, H = split,

$$\mathrm{Ind}_{HN}^{G}(\chi)^{h,\sigma_c} = \mathrm{Ind}_{HN}^{G}(\chi^{h,\sigma_c}) = \mathrm{Ind}_{HN}^{G}(\chi)$$

The c-Hermitian Dual

Corollary: π irreducible, real infinitesimal character:

$$\boxed{\pi^h \simeq \pi^\theta}$$

since

$$\pi^{h,\sigma} = \pi^{h,\theta\sigma_c} = (\pi^{h,\sigma_c})^\theta = \pi^\theta$$

Corollary: $G(\mathbb{R})$ equal rank $\Rightarrow$

Every representation with real infinitesimal character has an invariant Hermitian form.

($\delta = 1$, θ is inner)

The c-Hermitian Dual

Corollary: The theory of

$$\text{pairs } ((\pi, V), \langle\, , \,\rangle_c)$$

$((\mathfrak{g}, K)$-module, c-Hermitian form)

is equivalent to (twisted theory)

$$(\mathfrak{g}, K \rtimes \theta)\text{-modules}$$

(θ acts on $(\mathfrak{g}, K)$-module π by an intertwining operator $\pi \to \pi^{h,c}$)

$$\boxed{\text{new class of KLV polynomials } P^c_{\gamma,\delta} \in \mathbb{Z}[s][q]}$$

Equal rank case: $P^c_{\gamma,\delta}(q) = P_{\gamma,\delta}(qs)$

(only new in the unequal rank case)

See Lusztig/Vogan, arXiv

Digression: Hodge Theory

Schmid/Vilonen; also Milicic/Hecht

The c-invariant form appears naturally in Saito's theory of mixed Hodge modules

Saito $\Rightarrow$ (π, V) has a canonical filtration $F_p(V)$

Conjecture: (Schmid/Vilonen) The c-Hermitian form satisfies

$$\text{the sign of the form is } (-1)^p \text{ on } F_p(V) \cap F_{p-1}(V)^{\perp}$$

The c-Hermitian Dual

Natural Question: What is the c-Hermitian dual in terms of L-homomorphisms?

Recall: $\Pi(\phi)^h = \Pi(\sigma_{qc}^\vee \circ \phi)$

Theorem: $\sigma_s^\vee$ = split real form of $G^\vee$:

$$\boxed{\Pi(\phi)^{h,c} = \Pi(\sigma_s^\vee \circ \phi)}$$

Questions

1) $F = \mathbb{R}$: what is the meaning of $\phi \to \sigma^\vee \circ \phi$ for any conjugate-linear involution of ${}^L G$? ($\sigma^\vee$ inner to $\sigma_{qc}^\vee, \sigma_s^\vee$ give Hermitian dual, c-Hermitian dual)

2) F p-adic: What is the Hermitian dual on the ${}^L G$ side?

(should be $\sigma_c^\vee \circ \phi$ if $G(F)$ is split)

3) F p-adic: What is the meaning of $\phi \to \sigma_s^\vee \circ \phi$?

(should be some analogue of the c-Hermitian dual; answer is probably known on the level of affine Hecke algebras (given a type) (A/Ciubotaru))

4) $(\pi, V) \to (\overline{\pi}, \overline{V})$, relation with real representations, symplectic/orthogonal indicator (?)