Math 744, Fall 2014
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Homework IV

(1) Find an explicit two-to-one map $SL(2,\mathbb{C}) \to SO(3,\mathbb{C})$.

(2) Find an explicit two-to-one map $SL(4,\mathbb{C}) \to SO(6,\mathbb{C})$.

(3) Find an explicit two-to-one map $Sp(4,\mathbb{C}) \to SO(5,\mathbb{C})$.

(4) Prove the following result. Suppose V is a vector space and $R \subset V$ is a finite subset which spans V. If $\alpha \neq 0 \in R$ there exists at most one pseudo-reflection s such that $sv = -v$ and $s(R) = R$. (Recall a pseudo-reflection is any linear map satisfying $sv = v$ for all v in a subspace of codimension 1, and $sw = -w$ for some w).

Hint: Suppose s, s' both satisfy the condition, so if $t = ss'$ then $t\alpha = \alpha$ and $tv = v + f(v)\alpha$ for some $f \in V^*$. Consider powers of t.

This Lemma says that a root system can be defined without use of a bilinear form: the map $R \ni \alpha \to \alpha^\vee \in V^*$ is uniquely determined.

(5) If R is an irreducible root system, and $\Pi = \{\alpha_1, \dots, \alpha_n\}$ is a set of simple roots, then R has a unique maximal root β (i.e. $\alpha > 0$ implies $\beta + \alpha \notin R$).

Set $\alpha_0 = -\beta$, and define integers a_i by $a_0 = 1$ and

$$\sum_{i=0}^{n} a_i \alpha_i = 0.$$

Let $\widehat{\Pi} = \{\alpha_0, \alpha_1, \dots, \alpha_n\}$. Define the extended Dynkin diagram in the same way as the ordinary Dynkin diagram, applied to $\widehat{\Pi}$. Label each node $0 \leq i \leq n$ of the extended diagram with a_i.

(a) Draw the extended Dynkin diagrams for the classical groups, including the labels.

(b) Suppose R is simply laced. Show that a_i is one-half the sum of the labels on all adjacent nodes.

(c) The extended Dynkin diagram of type E_8 has α_0 adjacent only to the end of the long arm (with bond 1). Use (b) to compute the labels. Show that $\sum_{i=0}^{n} a_i = 30$.

(d) For a classical group show that the number of nodes of the extended diagram labelled 1 is the order of the center of the simply connected group. (This is true in general.)

(6) The root system of type D_4 has an outer automorphism of order 3 which preserves a set of positive roots (corresponding to an automorphism of the Dynkin diagram). Write down this automorphism explicitly.

(7) Consider the following game on a simply laced Dynkin diagram. Color each node black or white. If a node is black, you can toggle the colors of all *adjacent* nodes. Two colorings are said to be equivalent if you can relate them by a series of such operations.

(a) Show that in type A_n every coloring (with at least one black node) is equivalent to one with exactly 1 black node.

(b) Show that in type E_8 there are exactly three equivalence classes of colorings, one with all white nodes, and the others with one black node.

(In fact in any simply laced Dynkin diagram every nontrivial coloring is equivalent to one with exactly one black node.)