Math 744, Fall 2014
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Homework III

(1) Let $\mathbb{F}_q$ be the field with q elements.

(a) Show that $GL(2,\mathbb{F}_q)$ acts transitively on the projective space of lines in $\mathbb{F}_q^2$. Use this to compute the order of $GL(2,\mathbb{F}_q)$.

(b) Compute the order of $PGL(2,\mathbb{F}_q) = GL(2,\mathbb{F}_q)/\{xI\}$, $SL(2,\mathbb{F}_q) = \{g \in GL(2,\mathbb{F}_q)| \mid \det(g) = 1\}$, and $PSL(2,\mathbb{F}_q) = SL(2,\mathbb{F}_q)/\pm I$.

(c) Show that $PSL(2,2) \simeq S_3, PSL(2,3) \simeq A_4$, and $PSL(2,5) \simeq A_5$.

(2) Suppose $\mathfrak{g}$ is a semisimple Lie algebra, let $(\ ,\ )$ be the Killing form, let $\{X_i\}$ be a basis of $\mathfrak{g}$, and let $\{Y_i\}$ be the dual basis with respect to $(,)$ (i.e. $(X_i, Y_j) = \delta_{i,j}$. Finally let (π, V) be a representation of $\mathfrak{g}$, and let

$$C = \sum_i \pi(X_i)\pi(Y_i) \in \mathrm{End}(V)$$

(a) Show that C is independent of the choice of basis $\{X_i\}$.

(b) Show that $C\pi(X) = \pi(X)C$ for all $X \in \mathfrak{g}$.

C is called the Casimir element of π.

(4) Show that the only three-dimensional simple complex Lie algebra is $\mathfrak{sl}(2,\mathbb{C})$ (up to isomorphism).

(5) Compute the root system of $\mathfrak{s}\mathfrak{O}(2n+1,\mathbb{C})$. Describe a choice of a set of positive roots. What is the order of the Weyl group?

(6) Do the same for $\mathfrak{sp}(2n,\mathbb{C})$.

(7) We defined a root system to have the property: if $\alpha \in R$, then $-\alpha \in R$, and no other multiple of α is in R. (This is actually a *reduced* root system). Assume only that $\alpha \in R$ implies $-\alpha \in R$. Show that if $\alpha \in R$ at there are at most 4 multiples of α contained in R. Give an example of a root system where this holds.

(8) Calculate the Cartan matrices in types A_n, B_n, C_n, D_n. By induction, calculate their determinants.