Math 744, Fall 2014
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Homework II SOLUTIONS

(1) Let M be a smooth manifold. Prove the Jacobi identity for derivations of $C^\infty(M,\mathbb{R})$.
(2) Consider the exponential map from $M_n(\mathbb{C})$ to $GL(n,\mathbb{C})$.
(a) Show that $\det(\exp(X)) = \exp(\mathrm{Tr}(X))$
Solution: It is obvious that $\exp(gXg^{-1}) = g\exp(X)g^{-1}$, and we using this we can reduce to Jordan form. It then comes down to the identity for any Jordan block matrix with diagonal λ and 1 on the super-diagonal, for which it is obvious.
(b) A matrix is nilpotent if $X^n = 0$ for some n, and unipotent if $X - I$ is nilpotent. Prove that exp is a bijection from nilpotent to unipotent matrices.
Solution: If $X^n = 0$ then the power series for e^X is finite, and $I - e^X = X + \frac{1}{2}X^2 + \dots \frac{(n-1)!}{X}^{n-1}$. It is easy to see this is 0 when raised to the n^{th} power.

The Taylor series of $\log(1-x)$ is $\sum_1^\infty \frac{x^n}{n}$, which converges for $|x| < 1$. But never mind convergence, if X is nilpotent this is just a polynomial, and gives an inverse to exp. This gives the bijection.
(c) A matrix is semisimple if it is diagonalizable. Show that if X is semisimple then $\exp(X)$ is semsimple. What about the converse?
Solution: If $X = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$ with respect to some basis, then $e^X = (e^{\lambda_1},\dots,e^{\lambda_n})$ with respect to this basis.

The converse is also true. Suppose $A = E^X$ is diagonalizable. Write $X = X_s + X_n$ for its Jordan decomposition. Since X_s, X_n commute, $e^X = e^{X_s}e^{X_n}$, and this is the Jordan decomposition of A. We want to know that $X_n = 0$, so we've reduced to the question: does X nilpotent, $e^X = I$ imply $X = 0$? This is true, by reduction to Jordan form.
(d) Show that the exponential map from $M_n(\mathbb{C})$ to $GL(n,\mathbb{C})$ is surjective.
Solution: Since the image is conjugation invariant, it is enough to prove this for matrices in Jordan form, in which case it follows from (b).
(e) Show tha the exponential map from $M_n(\mathbb{R})$ to $GL(n,\mathbb{R})$ is not surjective. Describe its image.
Solution: Obviously $\det(e^X) = e^{\mathrm{trace}(X)} > 0$, so the image is contained in the matrices of positive determinant. Is it surjective onto this subgroup?

Consider $GL(2,\mathbb{R})$. It is enough to consider conjugacy classes. Every element of $GL(2,\mathbb{R})$ is conjugate *over* $\mathbb{C}$ to one of the form:

$$\begin{array}{cl} \mathrm{diag}(x,y) & x,y\in\mathbb{R}^* \\ \mathrm{diag}(z,\overline{z}) & z\in\mathbb{C} \\ \begin{pmatrix} x & 1 \\ 0 & x\end{pmatrix} & x\in\mathbb{R}^* \end{array}$$

Obviously if $x, y > 0$ then $\exp(\mathrm{diag}(\ln(x),\ln(y))) = \mathrm{diag}(x,y)$. Also, since

$\exp(A+B) = \exp(A)\exp(B)$ if $[A,B]=0$, then

$$\exp\begin{pmatrix} x & y \\ 0 & x \end{pmatrix} = \exp\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\exp\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^x & 0 \\ 0 & e^z \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^x & e^x y \\ 0 & e^x \end{pmatrix}$$

so matrices of the form $\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$ are in the image if $x > 0$.

This takes care of all matrices with positive real generalized eigenvalues. The matrix $\mathrm{diag}(z, \overline{z})$ is conjugate to one of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ with $a^2 - b^2 \neq 0$. This is isomorphic to $\mathbb{C}^*$, and using the exponential map $\exp : \mathbb{C} \twoheadrightarrow \mathbb{C}^*$ we see these matrices are all in the image of exp.

The determinant condition rules out $\mathrm{diag}(x, y)$ with $xy < 0$.

This leaves matrices conjugate to $\mathrm{diag}(x, y)$ with $x, y < 0$, and those in (3) with $x < 0$.

Note that

$$\exp\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} = \begin{pmatrix} \cos(a) & \sin(a) \\ -\sin(a) & \cos(a) \end{pmatrix}$$

In particular taking $a = \pi$ gives $\exp(*) = -I$. More generally

$$\exp\begin{pmatrix} a & \pi \\ -\pi & a \end{pmatrix} = \begin{pmatrix} -e^a & 0 \\ 0 & -e^a \end{pmatrix}$$

This leaves: $\mathrm{diag}(x, y)$ with x, y distinct and negative, and $\begin{pmatrix} x & 1 \\ 0 & x \end{pmatrix}$ with $x < 0$. These are not in the image: clearly the image of exp is *contained* in the set of squares, and an easy calculation shows that none of these matrices can be written as a square.

The conclusion is the image consists of all matrices *except* those with two distinct negative eigenvalues, or non-semisimple matrices with a single negative eigenvalue.

The general case reduces (with some effort) to this one. Here is the conclusion.

If $g \in GL(n, \mathbb{R})$ is semisimple, it is in the image of exp if and only if each negative eigenvalue has even multiplicity (no condition on non-real eigenvalues).

Any Jordan block with positive real eigenvalue is allowed. The following matrices are also in the image of exp:

$$\begin{pmatrix} A & I & 0 & \dots \\ 0 & A & I & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \dots & 0 & A & I \\ \dots & 0 & 0 & A \end{pmatrix}$$

where $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. In particular A can be $\mathrm{diag}(x, x)$ with $x < 0$.

The image of exp is matrices with "generalized" Jordan blocks of this type.
(3) Let $G = SO(n)$, the set of $n \times n$ real matrices satisfying $gg^t = I$ and $\det(G) = 1$.
(a) Show that the Lie algebra of G is $\mathfrak{g} = \{X \in M_n(\mathbb{R}) \mid X + X^t = 0\}$.
Solution: Using the exponential map: $e^{sX} \in G$ for all s if and only if $X \in \mathfrak{g}$. Then $e^{sX}(e^{sX})^t = e^{s(X+X^t)} = I + s(X + X^t) + \dots$. This equals I if and only if $X + X^t = 0$.
(b) Show that the exponential map from $\mathfrak{g}$ to G is surjective.
Solution: The simplest proof is using the fact that every element of $SO(n)$ is $SO(n)$-conjugate to a block diagonal matrix with $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ in each block (plus an extra 1 if n is odd). Then the result follows from $ge^X g^{-1} = e^{gXg^{-1}}$, and $\exp(\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$ gives the previous matrix.
(4) Compute the Lie algebras of the classical groups $SL(n, \mathbb{R}), SO(p, q)$, and $SU(p, q)$. What are their dimensions?
Solution:

$$\mathfrak{sl}(n, \mathbb{R}) = \{X \in M_n(\mathbb{R}) \mid \mathrm{trace}(X) = 0\}$$

of real dimension $n^2 - 1$.

$$\mathfrak{so}(p, q) = \{X \in M_{p+q}(\mathbb{R}) \mid XJ + JX^t = 0\}$$

where $J = \mathrm{diag}(I_p, -I_q)$. This can be written

$$\{\begin{pmatrix} A & B \\ -B^t & D \end{pmatrix} \mid A \in \mathfrak{so}(p), B \in \mathfrak{so}(q)\},$$

which shows the dimension in

$$\frac{1}{2}p(p-1) + \frac{1}{2}q(q-1) + pq = \frac{1}{2}(p+q)(p+q-1)$$

Note that this only depends on $p + q$.

$$\mathfrak{su}(p, q) = \{X \in M_{p+q}(\mathbb{C}) \mid XJ + JX^t = 0, \mathrm{trace}(X) = 0\}$$

where $J = \mathrm{diag}(I_p, I_q)$. This can be writte

$$\{\begin{pmatrix} A & B \\ -\overline{B}^t & D \end{pmatrix} \mid A \in \mathfrak{u}(p), B \in \mathfrak{su}(q), \mathrm{trace}(A) + \mathrm{trace}(D) = 0\}$$

which gives (real) dimension ($\mathfrak{u}(n)$ has dimension n^2):

$$p^2 + q^2 + 2pq - 1 = (p+q)^2 - 1.$$

Note that this only depends on $p + q$.
(5) Let $G = SU(2)$, and let $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g} = \{X \in M_2(\mathbb{C}) \mid X + \overline{X}^t = 0, \mathrm{Tr}(X) = 0\}$.

(a) Show that $\mathfrak{g}$ is a three dimensional real vector space, and $(X, Y) = \mathrm{Tr}(X\overline{Y}^t)$ is a definite symmetric bilinear form on $\mathfrak{g}$.
Solution: The algebra is $\{\begin{pmatrix} x & z \\ -\overline{z} & -x \end{pmatrix}$ with $x \in \mathbb{R}, z \in \mathbb{C}$, and

$$\langle X, X \rangle = -2(x^2 - |z|^2)$$

is *negative* definite.
(b) Let G act on $\mathfrak{g}$ by $g : X \to gXg^{-1}$. Show that this preserves the form $(,)$, so defines a map $\phi : SU(2) \to SO(3)$.
Solution: $(gX, gY) = \mathrm{trace}((gXg^{-1})(gYg^{-1})^t) = gXg^{-1}(g^{-1})^t Y g^t = \mathrm{trace}(gXY^t g^t) = \mathrm{trace}(XY^t g^t g) = \mathrm{trace}(XY)$.
(c) Show that ϕ is surjective, and identify its kernel.
Solution: Surjectivity follows from the fact that the exponential map is surjective. The kernel is $\pm I$.