Math 744, Fall 2014
Jeffrey Adams
Homework II

(1) Let M be a smooth manifold. Prove the Jacobi identity for derivations of $C^\infty(M,\mathbb{R})$.

(2) Consider the exponential map from $M_n(\mathbb{C})$ to $GL(n,\mathbb{C})$.

(a) Show that $\det(\exp(X)) = \exp(\mathrm{Tr}(X))$

(b) A matrix is nilpotent if $X^n = 0$ for some n, and unipotent if $X - I$ is nilpotent. Prove that exp is a bijection from nilpotent to unipotent matrices.

(c) A matrix is semisimple if it is diagonalizable. Show that if X is semisimple then $\exp(X)$ is semsimple. What about the converse?

(d) Show that the exponential map from $M_n(\mathbb{C})$ to $GL(n,\mathbb{C})$ is surjective.

(e) Show tha the exponential map from $M_n(\mathbb{R})$ to $GL(n,\mathbb{R})$ is not surjective. Describe its image.

(3) Let $G = SO(n)$, the set of $n \times n$ real matrices satisfying $gg^t = I$ and $\det(G) = 1$.

(a) Show that the Lie algebra of G is $\mathfrak{g} = \{X \in M_n(\mathbb{R}) \mid X + X^t = 0\}$.

(b) Show that the exponential map from $\mathfrak{g}$ to G is surjective.

(4) Compute the Lie algebras of the classical groups $SL(n,\mathbb{R}), SO(p,q)$, and $SU(p,q)$. What are their dimensions?

(5) Let $G = SU(2)$, and let $\mathfrak{g}$ be its Lie algebra, $\mathfrak{g} = \{X \in M_2(\mathbb{C}) \mid X + \overline{X}^t = 0, \mathrm{Tr}(X) = 0\}$.

(a) Show that $\mathfrak{g}$ is a three dimensional real vector space, and $(X,Y) = \mathrm{Tr}(X\overline{Y}^t)$ is a positive definite symmetric bilinear form on $\mathfrak{g}$.

(b) Let G act on $\mathfrak{g}$ by $g : X \to gXg^{-1}$. Show that this preserves the form $(,)$, so defines a map $\phi : SU(2) \to SO(3)$.

(c) Show that ϕ is surejective, and identify its kernel.