Math 744, Fall 2014
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Homework I

(1) Consider the action of $SO(n+1)$ acting on $S^n \subset \mathbb{R}^{n+1}$.
(a) Show this action is transitive.
(b) Compute $\mathrm{Stab}_G(v)$ where $v=(1,0,\dots,0)$.
(c) Show there is an isomorphism $SO(n+1)/SO(n)\simeq S^n$ (it is enough to give the bijection).
(2)
(a) Show that $\{(z,w)\in\mathbb{C}^2 \mid z^2+w^2=1\}\simeq\mathbb{C}^*$
(b) Show that $SO(2,\mathbb{C})\simeq\mathbb{C}^*$
(c) Show that $SO(2,\mathbb{R})\simeq S^1$
(d) Show that $SO(1,1)\simeq\mathbb{R}^*$. Recall $SO(1,1)$ is the group preserving a symmetric bilinear form on $\mathbb{R}^2$ of signature $(1,1)$.
(e) Show that $O(2)$ contains $SO(2)$ as a subgroup of index 2, that $O(2)$ is no abelian, and the elements of $O(2)-SO(2)$ constitute a single conjugacy class.
(3) Show that the proper algebraic subsets of the one dimensional vector space $\mathbb{C}$ are the finite sets.

(4) Show that the Euclidean topology on $\mathbb{C}^n$ is finer than the Zariski topology.

(5) Show that $\mathrm{Hom}_{\mathrm{alg}}(G_m,G_m)\simeq\mathbb{Z}$; the left hand side is the set of morphisms from G_m to G_m (as algebraic varieties) which are also group homomorphisms.

(6) Recall an action of an algebraic group G on an algebraic variety X is a morphism of varieties $G\times X\to X$, $(g,x)\to g\cdot x$, satisfying $g\cdot(h\cdot x)=(gh)\cdot x$, and $e\cdot x=x$.
(a) Consider the action of $GL(n,K)$ on K^n (K is any field). Determine the orbits of $GL(n,K)$ and $SL(n,K)$ on K^n.
(b) Show that $GL(2,K)$ acts transitively on P^1, the set of lines through the origin in K^2. Compute the stabilizer of a point. Compute the orbits of $GL(2,K)$ on $P^1\times P^1$.