Math 341, Jeffrey Adams

Test II, May 9, 2011 SOLUTIONS

Each problem is worth 20 points

(1)

The flow into tank 1 is $1.5\frac{gal}{min} \times .1\frac{lbs}{gal} = .15\frac{lbs}{min}$. Also $-3(x/40)$ outflow into tank 2, and $1.5(y/40)$ inflow from tank 2. So

$$x' = -\frac{3}{40}x + \frac{1.5}{40}y + .15$$

Similarly

$$y' = \frac{3}{40}x + \frac{1.5}{40}y + .3$$

The matrix equation is

$$\begin{pmatrix} -\frac{3}{40} & \frac{1.5}{40} \\ -\frac{3}{40} & -\frac{4}{40} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} .15 \\ .3 \end{pmatrix}$$

(2) Since $BAB^{-1} = C$ where $C = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$, $A = B^{-1}CB$, and $e^{tA} = B^{-1}e^{tC}B$, i.e.

$$e^{tA} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

and this works out to

$$e^{tA} = \begin{pmatrix} 2e^{-2t} - e^{t} & -2e^{-2t} + 2e^{t} \\ e^{-2t} - e^{t} & -e^{-2t} + 2e^{t} \end{pmatrix}$$

The columns are solutions, so

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} 2e^{-2t} - e^{t} \\ e^{-2t} - e^{t} \end{pmatrix} + c_2 \begin{pmatrix} -2e^{-2t} + 2e^{t} \\ -e^{-2t} + 2e^{t} \end{pmatrix}$$

The initial conditions give $x(0) = c_1 = 1$ and $y(0) = c_2 = 3$, which gives

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -4e^{-2t} + 5e^{t} \\ -2e^{-2t} + 5e^{t} \end{pmatrix}$$

Alternatively, find the eigenvectors $(1,1)$ and $(2,1)$ (directly, or as the columns of B^{-1}), and write

$$y = d_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + d_2 e^{-2t} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and solve to find $d_1 = 5$ and $d_2 = -2$.

(3)

(a) The only eigenvalue is 2, so this is unstable. The Jordan form is not $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, so it must be $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$. This flow lines are curves, see the last diagram on page 649.

(b) The eigenvalues are ± 2; this is an unstable saddle point.

(c) The eigenvalues are $\frac{5}{2} \pm i\frac{\sqrt{3}}{2}$, which gives an unstable spiral.

(d) The **Jacobian** is the matrix given in part (c). So the behavior is the same as in part (c), provided both eigenvalues have real part of the same sign. This is the case, so this is also unstable. See the Linearization Theorem 4.2, page 655.

(4) The eigenvalues are $\frac{1}{2}(3 \pm i\sqrt{3})$. The $\frac{1}{2}(3 + i\sqrt{3})$ vector is the nullspace of $\begin{pmatrix} 1 - \frac{1}{2}(3 + i\sqrt{3}) & -1 \\ 1 & -1 - \frac{1}{2}(3 + i\sqrt{3}) \end{pmatrix}$, which is $(1, \frac{1}{2}(1 - i\sqrt{3}))$. The $\frac{1}{2}(3 - i\sqrt{3})$ eigenvector is the complex conjugate of this one, i.e. $(1, \frac{1}{2}(1 + i\sqrt{3})$. Therefore the general solution of the homogeneous equation is

$$c_1 e^{\frac{1}{2}(3+i\sqrt{3})t} \begin{pmatrix} 1 \\ \frac{1}{2}(1 - i\sqrt{3}) \end{pmatrix} + c_2 e^{\frac{1}{2}(3-i\sqrt{3})t} \begin{pmatrix} 1 \\ \frac{1}{2}(1 + i\sqrt{3}) \end{pmatrix}$$

Never mind finding real valued solutions.

As for the particular solution of the nonhomogenous equation, the exponential generating function method is hopeless. **Use Undetermined Coefficients**. An obvious guess is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a + bt \\ c + dt \end{pmatrix}$$

Plugging this in gives

$$\begin{pmatrix} b \\ d \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} a + bt \\ c + dt \end{pmatrix} \begin{pmatrix} -5 - 2t \\ -4t \end{pmatrix}$$

This gives

$$\begin{aligned} b &= 2(a + bt) - (c + dt) + (-5 - 2t) \\ d &= (a + bt) + (c_d t) - rt \end{aligned}$$

or

$$\begin{aligned} (2b - d - 2)t + (2a - c - 5 - b) &= 0 \\ (b + d - 4)t + (a_c - d) &= 0. \end{aligned}$$

The coefficients of t give $wb-d-2=0, b+d-4=0$, and adding these give $3b-6=0$, $b=2$, and $d=2$. The constant terms give $a=3, c=1$. So the particular solution is

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3+2t \\ -1+2t \end{pmatrix}$$

(5) Let $y=\sum_{n=0}^{\infty} a_n x^n$. Then $y'=\sum n a_n x^n$, $y''=\sum n(n-1)a_n x^{n-2}$, and the equation is

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0.$$

Letting $m=n-2$, the first sum can be written

$$\begin{aligned}\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} &= \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} \\ &= \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m \\ &= \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n.\end{aligned}$$

In the last step we simply changed m to n. So the equation is

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} 2a_n x^n = 0.$$

Grouping terms gives

$$\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (n+2)a_n]x^n = 0$$

So the recurrence is $(n+2)(n+1)a_{n+2}+(n+2)a_n=0$ for all $n\geq 0$. Since $n+2>0$ we can divide by this to give $(n+2)a_{n+2}+a_n=0$ for all $n\geq 0$. Finally $n+1>0$ so divide by this to get

$$a_{n+2} = -\frac{a_n}{n+1}$$

The first few terms are

$$
\begin{aligned}
a_0, a_1 &\text{arbitrary} \\
a_2 &= -a_0 \\
a_3 &= -a_1/2 \\
a_4 &= -a_2/3 = +a_0/3 \\
a_5 &= -a_3/4 = +a_1/8
\end{aligned}
$$

The general solution is

$$y = a_0(1 - x^2 + \frac{1}{3}x^4 + \dots) + a_1(1 - \frac{1}{2}x^3 + \frac{1}{8}x^5 + \dots)$$

The singularities are where $x(1-x^2)^2 = 0$, i.e. $0, \pm 1$. Note that $(1-x^2)^2 = (1-x)^2(1+x)^2$.

Singularity at 0:

$x\frac{(1-x)}{x(1-x^2)^2} = \frac{(1-x)}{(1-x^2)^2}$ has a limit at 0;

$x^2\frac{(1+x)}{x(1-x^2)^2} = \frac{x(1+x)}{(1-x^2)^2}$ has a limit at 0, so 0 is a regular singular point.

Singularity at 1:

$(1-x)\frac{(1-x)}{x(1-x^2)^2} = \frac{(1-x)^2}{x(1-x)^2(1+x)^2} = \frac{1}{x(1+x)^2}$ has a limit at 1;

$(1-x)^2\frac{(1+x)}{x(1-x^2)^2} = \frac{(1-x)^2(1+x)}{x(1-x)^2(1+x)^2} = \frac{(1+x)}{x(1+x)^2}$ has a limit at 1; so 1 is a regular singular point.

Singularity at -1:

$(1+x)\frac{(1-x)}{x(1-x^2)^2} = \frac{(1+x)(1-x)}{x(1-x)^2(1+x)^2} = \frac{(1-x)}{x(1-x)^2(1+x)}$ does **not** have a limit at -1, so this is an irregular singular point.