

Math 340, Jeffrey Adams

Test I, December 6, 2010

TAKE HOME - Due Wednesday December 8 in class

Solutions

Question 1 This is obviously infinite.

Question 2 Surprisingly it is easier to do it in vertical slices.

$$\begin{aligned} & \int_{x=-\frac{1}{3}}^{x=0} \int_{y=-\sqrt{3x+1}}^{y=\sqrt{3x+1}} xy \, dy \, dx + \int_{x=0}^{x=5} \int_{y=x-1}^{y=\sqrt{3x+1}} xy \, dy \, dx \\ &= \int_{-\frac{1}{3}}^0 x \frac{1}{2} y^2 \Big|_{-\sqrt{3x+1}}^{\sqrt{3x+1}} + \int_0^5 x \frac{1}{2} y^2 \Big|_{x-1}^{\sqrt{3x+1}} \\ &= \frac{1}{2} \int_{-\frac{1}{3}}^0 x((3x+1) - (3x+1)) \, dx + \frac{1}{2} \int_0^5 x((2x+1) - (x-1)^2) \, dx \\ &= 0 + \frac{1}{2} \int_0^5 x(3x+1 - x^2 + 2x - 1) \, dx \\ &= \frac{1}{2} \int_0^5 5x^2 - 3x \, dx = \frac{1}{2} \left[\frac{5}{3} x^3 - \frac{3}{2} x^2 \right]_0^5 \\ &= \frac{1}{2} \left[\frac{5}{3} 5^3 - \frac{3}{2} 5^2 \right] - \frac{1}{2} [0] \\ &= \frac{1}{6} 5^4 - \frac{1}{8} 5^4 = 5^4 \left(\frac{1}{6} - \frac{1}{8} \right) = \frac{625}{24} \end{aligned}$$

Question 3 The region in polar coordinates is $0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \phi \leq \frac{\pi}{2}, 0 \leq r \leq$

1. The Jacobian is $r^2 \sin(\phi)$. So:

$$\begin{aligned} \int_{\theta=0}^{\frac{\pi}{2}} \int_{\phi=0}^{\frac{\pi}{2}} \int_{r=0}^1 r^4 r^2 \sin(\phi) \, dr \, d\phi \, d\theta &= \left(\frac{1}{7} r^7 \Big|_0^1 \right) (-\cos(\phi) \Big|_0^{\frac{\pi}{2}}) \left(\theta \Big|_0^{\frac{\pi}{2}} \right) \\ &= \frac{1}{7} (-0 + 1) \left(\frac{\pi}{2} - 0 \right) = \frac{\pi}{14} \end{aligned}$$

Question 4 In polar coordinates $0 \leq \theta \leq \frac{\pi}{2}$ and $1 \leq r \leq \sqrt{2}$ (note $1 \leq r^2 \leq 2$, so $r \leq \sqrt{2}$ not 2). So

$$\begin{aligned} \int_0^{\frac{\pi}{2}} \int_1^{\sqrt{2}} \int_1^{\sqrt{2}} e^{-r^2} r \, dr \, d\theta &= -\frac{1}{2} e^{-r^2} \Big|_1^{\sqrt{2}} (\theta) \Big|_0^{\frac{\pi}{2}} \\ &= -\frac{1}{2} (e^{-2} - e^{-1}) \frac{\pi}{2} = \frac{\pi}{4} \left(\frac{1}{3} - \frac{1}{e} \right) = \frac{\pi}{4} \frac{e-1}{e^2} \end{aligned}$$

Question 5 Obviously the center of mass is on the axis, so you only need to find the distance along the axis. Place the cone with the vertex at the origin, pointing to the right, so the base is at $x = h$. At x the radius of the circle $r(x)$ satisfies $\frac{r(x)}{x} = \frac{R}{h}$, so $r(x) = \frac{R}{h}x$. The integral is

$$\begin{aligned}\int_0^h x \pi r(x)^2 dx &= \pi \int_0^h \left(\frac{R}{h}x\right)^2 x dx \\ &= \frac{\pi R^2}{h^2} \int_0^h x^3 dx = \frac{\pi R^2}{h^2} \frac{1}{4} x^4 \Big|_0^h \\ &= \frac{\pi R^2}{h^2} \frac{h^4}{4} = \frac{\pi R^2 h^2}{4}.\end{aligned}$$

For the moment divide by the mass, $\frac{1}{3}\pi R^2 h$ to give

$$\frac{\frac{\pi R^2 h^2}{4}}{\frac{1}{3}\pi R^2 h} = \frac{3h}{4}.$$

This is $3/4$ along the axis, closer to the base.

Note: it is necessary to draw a picture or describe the coordinates. Otherwise it isn't clear what $\frac{3h}{4}$ means, versus $\frac{h}{4}$ if measured from the other end.

Question 6 This is much easier than most people made it. Since $2 < \pi$, $\sin(2) > 0$. More generally, since $0 \leq x+y \leq 2$ on the region, $0 \leq \sin(x+y) \leq 1$. The integral is obviously positive. The area of the region is 1, so the integral is less than $\int \int 1 dx dy = 1$.

The integral computes to $2 \sin(1) - \sin(2)$. It is pretty clear this is between 0 and 1, but this isn't easy to show, and most answers on the exam fell slightly short in this regard.

Question 7 The lines are $y = \frac{1}{a}x - 1$ and $y = -\frac{1}{a}x + 1$. The integral for the x -component of the center of mass is

$$\begin{aligned}\int_{x=0}^a \int_{y=\frac{1}{a}x-1}^{y=-\frac{1}{a}x+1} x dy dx &= \int_0^a x [y]_{\frac{1}{a}x-1}^{-\frac{1}{a}x+1} dx = \int_0^a x \left(-\frac{1}{a}x + 1 - \frac{1}{a}x + 1\right) dx \\ &= \int_0^a x \left(2 - \frac{2}{a}x\right) dx = x^2 \Big|_0^a - \frac{2}{3a} x^3 \Big|_0^a \\ &= a^2 - \frac{2a^3}{3a} = a^2 - \frac{2}{3}a^2 = \frac{1}{3}a^2\end{aligned}$$

The mass is $\frac{1}{2}2a = a$, so the x component of the center of mass is $\frac{1}{3}a^2/a = \frac{a}{3}$. We want this to equal 3, so $a = 9$.

Question 8 The integral is $\int_0^1 \int_0^1 \vec{n} \cdot (\vec{x} - \vec{x}_0) dx dy$ where $\vec{x} = (x, y)$, $\vec{x}_0 = (1, 1)$ and $\vec{n}$ is the unit normal to the line. The line has slope c , so $\vec{n}$ is in the direction of $(-1, c)$, and being length 1 gives $\vec{n} = \frac{1}{\sqrt{c^2+1}}(-1, c)$.

The integral is

$$\begin{aligned}
 & \frac{1}{\sqrt{c^2+1}} \int_0^1 \int_0^1 (-1, c) \cdot ((x-1, y-1)) dx dy \\
 &= \frac{1}{\sqrt{c^2+1}} \int_0^1 \int_0^1 (-x+1-cy-c) dx dy \\
 &= \frac{1}{\sqrt{c^2+1}} \int_0^1 \left[-\frac{1}{2}x^2 + (1+cy-c)x \right]_0^1 dy \\
 &= \frac{1}{\sqrt{c^2+1}} \int_0^1 \left[-\frac{1}{2} + (1+cy-c) - 0 \right] dy \\
 &= \frac{1}{\sqrt{c^2+1}} \left[\left(\frac{1}{2} - c \right) y + \frac{1}{2} cy^2 \right]_0^1 \\
 &= \frac{1}{\sqrt{c^2+1}} \left[\left(\frac{1}{2} - c \right) + \frac{1}{2} c \right] \\
 &= \frac{1}{\sqrt{c^2+1}} \frac{1-c}{2} = \frac{1-c}{2\sqrt{c^2+1}}
 \end{aligned}$$