Math 340, Jeffrey Adams

Test I, October 11, 2010 SOLUTIONS

(1)

(a) $1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$

(b) $(-1-0) * \vec{i} - (-1-1) * \vec{j} + (0-1)\vec{k} = (-1, 2, -1)$

(c) $|c\vec{v}| = |c||\vec{v}|$ so $|-2\vec{v}| = |-2||\vec{v}| = 2 * 5 = 10$.

(d) $\vec{n} = \vec{v}/|\vec{v}| = (-1, 2, -3)/\sqrt{15}$

(2)

(a) $\vec{v} = \vec{v}_1 \times \vec{v}_2 = (2, -1, -1)$, which is easily seen to be perpendicular to $\vec{v}_1$ and $\vec{v}_2$.

(b) The plane is $(x, y, z) \cdot \vec{v} = 0$, i.e. $2x - y - z = 0$.

(c) The translated plane has the same normal vector, but the equation is not homogeneous. That is P' is given by $(x, y, z) \cdot \vec{v} = C$ for some C. What is C? Well, $(2, 3, 5)$ has to be a solution, so $(2, 3, 5) \cdot (2, -1, -1) = C$, i.e. $C = -4$. So P' is given by $2x - y - z = -4$.

(3) The determinant is $(2-a)*3$, which can be seen by any of the formulas, for example expanding via the first row gives $2*3 - a*3 = (2-a)*3$. The matrix is invertible if and only if the determinant is nonzero. The determinant is zero if and only if $a = 2$.

Take $a = 2$: $\begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ The reduced form is $M = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which has null space (x, y, z) with $x + y = 0$ and $z = 0$. This is one-dimensional, with basis $(1, -1, 0)$ (or any nonzero multiple of this).

(4)

$$\begin{aligned} 2x - y &= -1 \\ x - y + 2z &= 0 \end{aligned}$$

(a) The matrix equation is

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

To get reduced form add -2 times the second row to the first:

$$\begin{pmatrix} 0 & 1 & -4 \\ 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

Now add the first row to the second:

$$\begin{pmatrix} 0 & 1 & -4 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

Now the corresponding homogenous equation is

$$\begin{pmatrix} 0 & 1 & -4 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

or $y - 4z = 0, x - 2z = 0$. So z is a free variable, and the general solution is $(2z, 4z, z)$, or all mulitples of $(2, 4, 1)$.

(b) The nonhomogeneous equation is $y - 4z = -1$ and $x - 2z = -1$. Take $z - 0$ so that $x = y = -1$. The particular solution is $(-1, -1, 0)$.

(c) The general solution is $\{(-1, -1, 0) + t(2, 4, 1)\}$ with $t \in \mathbb{R}$.

(5) The eigenvalues are given by $\det \begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = 0$. The determinant is $(2-\lambda)^2$, so $(2-\lambda)^2 = 0$ has the unique solution 2. The only eigenvalue is 2.

Plug in 2, to give $M = \begin{pmatrix} 2-2 & 1 \\ 0 & 2-2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The null space of this matrix is $0x + y = 0$ and $0 = 0$, i.e. $y = 0$, i.e. $\{(x, 0)\}$. This is one-dimensional, all multiples of $(1, 0)$. These are the only eigenvectors. (In this case $\mathbb{R}^2$ is not spanned by eigenvectors for the matrix.)

(6)

(1) What does it mean for S to be linearly independent? Suppose $a_1 f(\vec{v}_1) + a_2 f(\vec{v}_2) + \dots a_n f(\vec{v}_n) = 0$. To show that S is linearly independent we have to show this only happens if $a_1 = a_2 = \cdots = a_n = 0$. Let's see, applying linearity of f twice gives:

$$\begin{aligned} 0 &= a_1 f(\vec{v}_1) + a_2 f(\vec{v}_2) + \cdots + a_n f(\vec{v}_n) \\ &= f(a_1 \vec{v}_1) + f(a_2 f_2) + \cdots + f(a_n \vec{v}_n) \\ &= f(a_1 \vec{v}_1 + a_2 \vec{v}_2 + \cdots + a_n \vec{v}_n). \end{aligned}$$

Now f is injective, so $f(\vec{v}) = 0$ implies $\vec{v} = 0$. So $a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_n\vec{v}_n = 0$. But $\{\vec{v}_1, \dots, \vec{v}_n\}$ is linearly independent. This implies $a_1 = a_2 = \cdots = a_n = 0$ as required.

(2) Since $\{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis of V, $\dim(V) = n$. But S is a basis of W, and has n elements, so $\dim(W) = n$ also.

(3) As in (2) $|S| = n = \dim(V) < \dim(W)$. So S can't be a basis of W.