MATH 340 EXAM 1
OCTOBER 6, 2009

1.(25) True/false. Justify by a short reason or counterexample.

a) Any 3 non-zero, distinct vectors in $\mathbb{R}^3$ are linearly independent.
b) Every non-zero $n \times n$ matrix A is invertible.
c) If A and B are square matrices, $\det(A+B) = \det A + \det B$.
d) If A is an $n \times m$ matrix and $\mathbf{0} \neq \mathbf{b} \in \mathbb{R}^n$ then the set of solutions of the equation $A\mathbf{x} = \mathbf{b}$ is a subspace of $\mathbb{R}^m$.
e) If V is a vector space and $W \subseteq V$ is a subspace with $\dim W = \dim V < \infty$, then $W = V$.

2.(15) Consider the plane $\mathcal{P}_1$ in $\mathbb{R}^3$ given by $x+y+z=0$ and the plane $\mathcal{P}_2$ given $g(s,t) = s(1,1,2) + t(0,1,1)$. Find the (cosine) of the acute angle between the planes and the equation of the line of intersection.

3.(10) Let $A = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$. Consider the linear map $L_A : \mathbb{R}^4 \to \mathbb{R}^3$ with $L_A(\mathbf{v}) = A\mathbf{v}$. Find bases for the image and the null space of L_A.

4.(10) Let $A = \begin{pmatrix} 1 & 3 \\ -1 & 1 \end{pmatrix}$. Show there is a unique line $ax + by = 0$ in $\mathbb{R}^2$ so that for every vector $\mathbf{v}$ on the line, $A\mathbf{v}$ is perpendicular to $\mathbf{v}$.

5.(15) Consider the vector $\mathbf{v} = (1,1,1) = \mathbf{i} + \mathbf{j} + \mathbf{k} \in \mathbb{R}^3$ and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the map defined by $T(\mathbf{u}) = \mathbf{v} \times \mathbf{u}$, the cross product of $\mathbf{v}$ and $\mathbf{u}$.

a) Explain why T is a linear map.

b) What is the 3×3 matrix A (with respect to the standard basis of $\mathbb{R}^3$) that corresponds to T? (Recall $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.)

c) Is the matrix A invertible? (Hint: Think about T.)

6.(10) Let $f, g : \mathbb{R} \to \mathbb{R}^3$ be differentiable. Show $\frac{d}{dt}(f \cdot g) = \frac{df}{dt} \cdot g + f \cdot \frac{dg}{dt}$.

7.(15) a) Let $h(x,y) = \sin(xy)$. Compute $2h_{xy} + h_{yx}$.

b) If $f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$, $(x,y) \neq (0,0)$; $f(0,0) = 0$. Does the partial derivative of f with respect to x exist at $(x,y) = (0,0)$? Explain.

c) Let $f(u,v) = (u+v, u^2+v^2, uv)$. For what (u,v) do the vectors $\frac{\partial f}{\partial u}(u,v)$ and $\frac{\partial f}{\partial v}(u,v)$ span a plane in $\mathbb{R}^3$?