MATH 340 FINAL EXAM
DECEMBER 18, 2004

1.(10) Evaluate $\int_C \mathbf{F} \cdot dr$ where $\mathbf{F}(x, y, z) = (2xy, x^2 + z^2, 2yz + z)$ and C is the helix $g(t) = (3\cos t, 3\sin t, 2t)$ for $0 \leq t \leq 2\pi$.

2.(20) Let Σ be the intersection of the ellipsoid $2x^2 + y^2 + z^2 = 10$ and the elliptic cylinder $x^2 + 5z^2 = 1$. The intersection is non-empty.

a) Explain why there exists points on Σ closest and farthest from the origin.
b) Use the Lagrange equations to write down the (5) equations in (5) unknowns that will help to locate these points.
c) Show that if $\mathbf{p}$ is any point on Σ then $3 \leq |\mathbf{p}| \leq \sqrt{10}$.
d) Find the equation of the tangent line to Σ at the point $(1, 2\sqrt{2}, 0)$.

3.(15) Consider a cylinderical can Σ of radius 1 with unspecified boundary $\partial\Sigma$ oriented counterclockwise when viewed from above. Let $\mathbf{F}(x, y, z) = (2y^2, x^3, z^2)$. Compute the line integral $\int_{\partial\Sigma} \mathbf{F} \cdot dr$.

4.(15) Consider the vector field $\mathbf{F}(x, y) = (F_1, F_2) = (-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2})$ defined in $\mathbb{R}^2 - \{(0, 0)\}$. It is easy to show $\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$. Let C be any closed curve in $\mathbb{R}^2$ that goes once around the origin in the counterclockwise direction. Show the value of the line integral $\int_C \mathbf{F} \cdot dr$ does not depend on the closed curve C and compute the value of the integral. The value is NOT zero.

To make the calculations in problems 5 and 6 somewhat easier it might be best to place the bowl upside down on the x, y-plane.

5.(15) A master spherical bowl maker makes bowls which have essentially no thickness. How much clay is needed to make a spherical bowl whose bottom diameter is 2 inches and whose top diameter is 4 inches. Note: bowls have bottoms but no top.

6.(10) How much water does the bowl in problem 5 hold? Recall that the volume of a sphere of radius R is $\frac{4}{3}\pi R^3$.

7.(15) Place the bowl from problem 1 in $\mathbb{R}^3$ with the center of the bottom of the bowl at the origin to obtain a surface Σ in $\mathbb{R}^3$. Orient Σ with the normal that points into the bowl. Consider the vector field $F(x, y, z) = (x, y, z)$ and compute the flux integral $\int_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS$.