

Metrical Combinatorics and the Real Half of the Fibonacci Sequence

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Languages with stress group syllables into metrical feet (Halle and Idsardi 1995, Hayes 1995)—non-exhaustive groups of contiguous syllables. The size of feet in natural languages ranges from unary (a single syllable) to unbounded (as many syllables as possible); in addition syllables can also remain unfooted. Under these conditions, the number of possible metrical footings for a string of n syllables is known to be $\text{Fib}(2n)$ (Idsardi 2008), where $\text{Fib}(n)$ is the Fibonacci sequence, as in (1)¹ where the n^{th} element is the sum of the previous two elements (for example, $13 = 8 + 5$).

(1)	n :	0	1	2	3	4	5	6	7	...
	$\text{Fib}(n)$:	<u>1</u>	1	<u>2</u>	3	<u>5</u>	8	<u>13</u>	21	...

For example, a string of two syllables (here notated with 'x's) can be non-exhaustively footed in five ways (= $\text{Fib}(4)$): (xx), (x)(x), (x)x, x(x), and xx. In contrast, if footing were required to be exhaustive (that is, if every syllable had to belong to some foot) then a string of two syllables could only be footed in two ways: (xx) and (x)(x). It is easy to see from the bracketed grid representations that the number of possible exhaustive footings of a string of n syllables must be 2^{n-1} as every exhaustive footing must begin and end with foot-boundaries and between each pair of x's we have a binary choice between having a foot juncture and not having one. Since there are two choices for each space between x's and there are $n-1$ spaces between n x's, it follows directly that there are 2^{n-1} distinct exhaustive footings.

As a consequence, only half of the Fibonacci numbers (those underlined in (1): 1, 2, 5, 13, ...) are solutions to the task of creating non-exhaustive footings; the other half (3, 8, 21, ...) are not. An intriguing question is: Why is it the one half of the sequence and not the other? We venture some speculations about potential answers.

In 1680, Cassini (1733) discovered a relation among successive members of the Fibonacci sequence, expressed in (2):²

$$(2) \quad \text{Fib}(n)^2 - \text{Fib}(n-1) \cdot \text{Fib}(n+1) = (-1)^n$$

¹ The Fibonacci sequence can also be defined to start with 0: 0, 1, 1, 2, 3, ...

² This relation was independently discovered by Simson (1753).



That is, the square of any Fibonacci number is equal to the product of the two flanking Fibonacci numbers, give or take one. For example, $\text{Fib}(4)^2 - \text{Fib}(3) \cdot \text{Fib}(5) = 25 - 3 \cdot 8 = 1 = (-1)^4$ and $\text{Fib}(5)^2 - \text{Fib}(4) \cdot \text{Fib}(6) = 82 - 5 \cdot 13 = -1 = (-1)^5$. Rearranging (2) gives (3):

$$(3) \quad \text{Fib}(n)^2 - (-1)^n = \text{Fib}(n-1) \cdot \text{Fib}(n+1)$$

The left-hand side of (3) has two possible expansions, depending on whether n is odd or even, as in (4):

$$(4) \quad \begin{array}{ll} \text{a.} & n \text{ is even: } \text{Fib}(n)^2 - 1 \\ \text{b.} & n \text{ is odd: } \text{Fib}(n)^2 + 1 \end{array}$$

We can now see that (4a) has the form $(x^2 - 1)$ and (4b) has the form $(x^2 + 1)$. Elementary algebraic polynomial factorization (Herstein 1977) shows that (4a) has real-valued roots, (5a), whereas (4b) only has complex-valued roots, (5b):

$$(5) \quad \begin{array}{ll} \text{a.} & \text{Fib}(n)^2 - 1 = [\text{Fib}(n) - 1][\text{Fib}(n) + 1] \\ \text{b.} & \text{Fib}(n)^2 + 1 = [\text{Fib}(n) - i][\text{Fib}(n) + i] \text{ (where } i^2 = -1) \end{array}$$

Thus, for example, $\text{Fib}(3) \cdot \text{Fib}(5) = 3 \cdot 8 = 24 = 4 \cdot 6 = [\text{Fib}(4) - 1][\text{Fib}(4) + 1]$. Only the even-numbered Fibonacci numbers (here, $\text{Fib}(4)$) show up in the real-valued roots, and this is the same Fibonacci subset that characterizes the number of valid metrical groupings of strings of n syllables.

In conclusion, the ‘metrical’ half of the Fibonacci sequence is also the ‘real-valued’ half of the sequence (in the sense of (5)). Evidently, the Fibonacci character of footing arises just when we allow for non-exhaustive footing, as exhaustive footings can be counted as a simple set of independent binary choices. Generally, the Fibonacci sequence is associated with a number of ‘edge of chaos’ effects, especially systems which illustrate *dynamical frustration* (Binder 2008); systems in which opposing forces cannot reach an equilibrium solution. We speculate that the ‘forces’ operative here in defining non-exhaustive footings could be the local coherence of syllables into feet clashing with word-level properties of footing. Another potential view of the emergent complexity observed here would be that sequences of footed syllables can be metrically distinct — for example, $(x)(x) \neq (xx)$ — whereas all sequences of unfooted syllables are the same; thus we have asymmetric growth patterns in the footed and unfooted portions of syllabic strings resulting in Fibonacci complexity.

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