

**Constant
coefficients**
4.1

$$y'' + a_1 y' + a_2 y = 0, \quad a_1, a_2 \text{ constants.}$$

Characteristic polynomial $p(z) = z^2 + a_1 z + a_2$

Recipe z_0 is a root of $p \implies y = e^{z_0 t}$ is a solution

Recipe
4.5

root of p	solutions
real z_0 , mult. m	$e^{z_0 t}, t e^{z_0 t}, \dots, t^{m-1} e^{z_0 t}$
$\mu \pm i\nu$, mult. m	$t^k e^{\mu t} \cos \nu t, t^k e^{\mu t} \sin \nu t \quad (k = 0, \dots, m-1)$

Count: each real root gives m , each pair $\mu \pm i\nu$ gives $2m \implies$ total = n .

**Simple real
roots**
4.2.1

Ex $y'' - 3y' + 2y = 0, \quad y(0) = 1, y'(0) = 0.$

1. $p(z)$ $z^2 - 3z + 2 = (z-1)(z-2)$
2. **Roots** $z = 1, 2$ real, distinct
3. **Solutions** $y_1 = e^t, y_2 = e^{2t}$
4. **General** $y = c_1 e^t + c_2 e^{2t}$
5. **IVP** $c_1 + c_2 = 1, c_1 + 2c_2 = 0 \implies c_1 = 2, c_2 = -1$
 $y = 2e^t - e^{2t}$

Why it works

Plug in $y = e^{zt}$ ($y' = z e^{zt}, y'' = z^2 e^{zt}$):

$$z^2 e^{zt} + a_1 z e^{zt} + a_2 e^{zt} = p(z) e^{zt}$$

$e^{zt} \neq 0$, so e^{zt} is a solution $\iff p(z) = 0$.

**Multiple
real roots**
4.2.2

Ex $y''' - y'' - y' + y = 0.$

1. $p(z)$ $z^3 - z^2 - z + 1 = z^2(z-1) - (z-1) = (z-1)^2(z+1)$
2. **Roots** $z = 1$ (double), $z = -1$ (simple)
3. **Solutions** $z = 1$: $e^t, t e^t$ $z = -1$: e^{-t}
4. **General** $y = (c_1 + c_2 t) e^t + c_3 e^{-t}$

Why it works

Plugging e^{zt} into the ODE gives, for **every** z :

$$(e^{zt})''' - (e^{zt})'' - (e^{zt})' + e^{zt} = p(z) e^{zt}$$

Differentiate both sides **in** z ($\frac{d}{dz} e^{zt} = t e^{zt}$, and $\frac{d}{dz}$ passes through $\frac{d}{dt}$):

$$\underbrace{(t e^{zt})''' - (t e^{zt})'' - (t e^{zt})' + t e^{zt}}_{\text{ODE with } y=t e^{zt}} = \underbrace{p'(z)}_{=0} e^{zt} + \underbrace{p(z)}_{=0} t e^{zt}$$

= 0 if z is a **double** root $\implies t e^{zt}$ is a solution.

Multiplicity m : $p, p', \dots, p^{(m-1)}$ vanish at $z_0 \implies e^{z_0 t}, t e^{z_0 t}, \dots, t^{m-1} e^{z_0 t}.$

Complex extension 4.3

$p(z)$ has real coefficients but may have **complex roots**: $z = \mu \pm i\nu$
 Plugging in e^{zt} still gives $p(z) e^{zt}$, $z \in \mathbb{C}$

Euler's formula: $e^{(\mu+i\nu)t} = e^{\mu t}(\cos \nu t + i \sin \nu t)$

Euler's formula proof

Maclaurin series: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$

Put $x = i\theta$ ($i^2 = -1$, $i^3 = -i$, $i^4 = 1, \dots$):

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \dots$$

Split into **real** and **imaginary** parts:

$$e^{i\theta} = \underbrace{\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right)}_{\cos \theta} + i \underbrace{\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)}_{\sin \theta}$$

$$e^{i\theta} = \cos \theta + i \sin \theta$$

Then $e^{(\mu+i\nu)t} = e^{\mu t} e^{i\nu t} = e^{\mu t}(\cos \nu t + i \sin \nu t)$ ($\theta = \nu t$).

Simple complex roots 4.4.1

Ex $y'' + 2y' + 5y = 0$.

1. $p(z)$ $z^2 + 2z + 5 = 0$
2. **Roots** $z = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm 2i$ ($\mu = -1$, $\nu = 2$)
3. **Solutions** $e^{-t} \cos 2t$, $e^{-t} \sin 2t$ real and imaginary parts
4. **General** $y = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$

Why it works

Root $-1 + 2i$ gives a **complex** solution:

$$y = e^{(-1+2i)t} = \underbrace{e^{-t} \cos 2t}_u + i \underbrace{e^{-t} \sin 2t}_v$$

Plug $y = u + iv$ into the ODE (1, 2, 5 are real):

$$\underbrace{(u'' + 2u' + 5u)}_{\text{real}} + i \underbrace{(v'' + 2v' + 5v)}_{\text{real}} = 0$$

Real + $i \cdot$ real = 0 $\implies$ each part = 0 $\implies$ u and v are each a **real** solution. The i is gone.

$-1 - 2i$ gives $u - iv$: the same u, v , so one root of the pair is enough.

Multiple complex roots 4.4.2

Ex $y^{(4)} + 2y'' + y = 0$.

$p(z) = z^4 + 2z^2 + 1 = (z^2 + 1)^2$, $z = \pm i$ each **double**

$$y = (c_1 + c_2 t) \cos t + (c_3 + c_4 t) \sin t$$

Why it works

Double root z (§4.2.2) $\implies e^{zt}$ and te^{zt} are solutions. This did not use that z is real, so it holds for $z = \mu + i\nu$ too.

Plug in $e^{(\mu+i\nu)t}$ and $te^{(\mu+i\nu)t}$, and split into real and imaginary parts (the coefficients are real, so each part is a solution):

$$e^{\mu t} \cos \nu t, \quad e^{\mu t} \sin \nu t, \quad t e^{\mu t} \cos \nu t, \quad t e^{\mu t} \sin \nu t$$

- 1 **Linear** 1st order: integrating factor. IVP + **interval**
 - 2 **Separable** IVP + interval; **existence & uniqueness**: find the largest interval $\ni t_0$ where coefficients, RHS are continuous
 - 3 **Phase line**: stationary points, stable / unstable / semistable, $\lim_{t \rightarrow \infty}$ from $y(0)$, basin of attraction
 - 4 **Modeling** (set up, don't solve): mixing, volume + salt balance $S' = \text{in} - \text{out}$; population $M' = rM - \text{const}$, doubling time gives r
 - 5 **Exact**: $M_y = N_x$, build H , $H = C$, IC gives implicit solution
 - 6 **Integrating factor**: not exact, find μ , check $\mu M dx + \mu N dy$ is exact (no need to solve)
 - 7 **Wronskian**: plug in to check solutions, $W \neq 0 \implies$ fundamental set, natural fundamental set at $t = 0$
 - 8 **Matlab** (syms, dsolve, fplot: read the IVP); **numerical order** $E = O(h^p)$: Euler $p=1$, Runge-trapezoidal $p=2$, RK4 $p=4$
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Exam 1
one line each