

Before class **Matlab 2** due on Monday
Midterm 1 next week (Tue Oct 6)
IVP **don't forget the initial condition!**

Exact Solution is $\boxed{H(x, y) = C}$
 (recall) $\frac{d}{dx}H = H_x + H_y y' = 0 \iff H_x dx + H_y dy = 0$
 $M dx + N dy = 0$ is **exact** $\iff \boxed{M_y = N_x}$

Integrating
factor
9.3

Ex $\underbrace{x^2}_M dx + \underbrace{(x^3 + y^6 + 2y^5)}_N dy = 0.$

$M_y = 0, \quad N_x = 3x^2 \quad M_y \neq N_x$: **not exact**.

Multiply by $\mu(x, y)$ so that it becomes exact (**integrating factor**).

Check $\boxed{(\mu M)_y = (\mu N)_x}$ this time we only try $\mu(x)$ or $\mu(y)$

(i) $\mu(x)$ $(\mu M)_y = 0, \quad (\mu N)_x = 3\mu x^2 + \mu'(x^3 + y^6 + 2y^5)$
 $0 = 3\mu x^2 + \mu'(x^3 + y^6 + 2y^5)$
 $\frac{\mu'}{\mu} = \frac{-3x^2}{x^3 + y^6 + 2y^5}$ depends on $y \implies$ **not good**

(ii) $\mu(y)$ $(\mu M)_y = \mu' x^2, \quad (\mu N)_x = 3\mu x^2$
 $\mu' x^2 = 3\mu x^2 \implies \frac{\mu'}{\mu} = 3$ **separable**

$\boxed{\mu = e^{3y}}$

$e^{3y} x^2 dx + e^{3y} (x^3 + y^6 + 2y^5) dy = 0$ **is exact!**

$\mu(x)$ works $\iff \frac{M_y - N_x}{N}$ depends on x only

$\mu(y)$ works $\iff \frac{N_x - M_y}{M}$ depends on y only

Higher order ODE

Ex $t^2 y'' + 4t y' + 2y = 0, \quad y(1) = 1, y'(1) = 3.$

Given $y_1 = \frac{1}{t}, y_2 = \frac{1}{t^2}$: (a) interval, (b) verify, (c) W , (d) solve.

Type: 2nd order, **linear, homogeneous** $\implies$ **2** initial values.

Normal form: $y'' + \frac{4}{t} y' + \frac{2}{t^2} y = 0$

Theorem: coefficients (and f) **continuous** in normal form $\implies$ **unique** solution on $I \ni t_0$.

(a) interval

Singular at $t = 0, t_0 = 1$: $I = (0, \infty)$ no need to solve the ODE!

(b) verify

$$\begin{aligned} y_1 = t^{-1} & \quad y_1' = -t^{-2}, y_1'' = 2t^{-3} \implies (2 - 4 + 2)t^{-1} = 0 \\ y_2 = t^{-2} & \quad y_2' = -2t^{-3}, y_2'' = 6t^{-4} \implies (6 - 8 + 2)t^{-2} = 0 \end{aligned}$$

The solutions are **given**: how to find them comes later.

Superposition (linear, homogeneous): $y = \frac{c_1}{t} + \frac{c_2}{t^2}$ is a solution too.

Is this every solution, or just some?

(c) Wronskian

$$W(t) = \begin{vmatrix} t^{-1} & t^{-2} \\ -t^{-2} & -2t^{-3} \end{vmatrix} = -2t^{-4} + t^{-4} = \boxed{-t^{-4}} \neq 0 \quad \text{on } I$$

$\implies$ **fundamental set**: generates **every** solution

$\implies$ **general solution** $y = c_1/t + c_2/t^2$

Why det? At $t = 1$: $\begin{pmatrix} y(1) \\ y'(1) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \det = -1 \neq 0 \implies$ any initial data gives unique c_1, c_2 .

Abel: $W' + \frac{4}{t} W = 0 \implies W = Ct^{-4}$ (matches). W is never 0 or always 0: **one point is enough**.

(d) IVP

$$\begin{aligned} y' = -c_1/t^2 - 2c_2/t^3. \quad \text{At } t = 1: \\ c_1 + c_2 = 1, \quad -c_1 - 2c_2 = 3 \implies c_1 = 5, c_2 = -4 \end{aligned}$$

$$\boxed{y(t) = \frac{5}{t} - \frac{4}{t^2}, \quad t > 0}$$

Why the solution, not just a solution? existence & uniqueness: back to (a).

Story

2nd order linear homogeneous, initial values and 2 given solutions $\rightarrow$ check solution is valid for ODE $\rightarrow W \neq 0 \rightarrow$ fundamental set (linear independence) $\rightarrow y = c_1 y_1 + c_2 y_2$
 $\rightarrow$ initial data give c_1, c_2