

**Exact**  
9.1–9.2

New type of ODE: its solution is  $H(x, y) = c$

Differentiate along the solution  $y(x)$  (**chain rule**):

$$\frac{d}{dx}H(x, y(x)) = H_x + H_y y' = 0 \iff H_x dx + H_y dy = 0$$

**Ex**  $\underbrace{(2x + y)}_{M=H_x} dx + \underbrace{(x + 2y)}_{N=H_y} dy = 0, \quad y(0) = 1.$

$M dx + N dy$  is an **exact differential form** if  $M = H_x$ ,  $N = H_y$ .

**Exact?**  $H_{xy} = H_{yx}$  means  $M_y = N_x$

1. **Identify**  $M = 2x + y, \quad N = x + 2y$

2. **Exact?**  $M_y = 1, \quad N_x = 1$  equal: exact

3.  $H_x = M$   $H = \int (2x + y) dx = x^2 + xy + h(y)$

$h(y)$  is a **function** of  $y$ , not a constant!

4.  $H_y = N$   $x + h'(y) = x + 2y \implies h(y) = y^2$

5.  $H = C$   $x^2 + xy + y^2 = C$

6. **IVP**  $y(0) = 1 \implies C = 1$   $x^2 + xy + y^2 = 1$

**Integrating**  
**factor**  
9.3

$2y dx + x dy = 0: \quad M_y = 2 \neq 1 = N_x$  not exact.

Multiply by  $x$ :  $2xy dx + x^2 dy = 0: \quad M_y = 2x = N_x$  **exact!**

$\implies d(x^2 y) = 0, \quad x^2 y = C. \quad \mu = x$  is an **integrating factor**.

**Phase line**

**Ex** Consider  $y' = 3y^2(y + 2)$ .

(a) Find all stationary points.

(b) Draw the phase line; classify each as stable, unstable, or semistable.

(c) If  $y(0) = -1$ , find  $\lim_{t \rightarrow \infty} y(t)$ .

1. **Zeros**  $y = -2, 0$       2. **Signs**  $-, +, +$  ( $3y^2 \geq 0$ )

3. **Arrows**  $\begin{array}{c} - \quad \quad + \quad \quad + \\ \leftarrow \quad \bullet \quad \rightarrow \quad \bullet \quad \rightarrow \\ -2 \quad \quad \quad 0 \end{array} y$

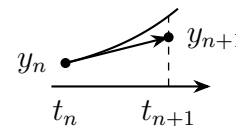
4. **Stability**  $-2$  unstable,  $0$  semistable

5. **Long time**  $y(0) = -1 \in (-2, 0)$  moves up  $\lim_{t \rightarrow \infty} y(t) = 0$

## Numerical Euler, error

**Ex**  $y' = t + y$ ,  $y(0) = 1$ ,  $h = 0.5$ . Estimate  $y(1)$ .

**Explicit Euler method:** follow the slope at the **current point** for one **step size**  $h$ :



$$y_{n+1} = y_n + h f(t_n, y_n)$$

**Step 1** at  $(t_0, y_0) = (0, 1)$ : slope  $f = 0 + 1 = 1$

$$y_1 = 1 + 0.5 \cdot 1 = 1.5, \quad t_1 = 0.5$$

**Step 2** at  $(t_1, y_1) = (0.5, 1.5)$ : slope  $f = 0.5 + 1.5 = 2$

$$y_2 = 1.5 + 0.5 \cdot 2 = 2.5, \quad t_2 = 1 \quad \boxed{y(1) \approx 2.5}$$

**Ex** A **second-order method** uses 20 steps. With 60 steps, how does the global error change?

**Global error**  $E = O(h^p)$ ,  $p = \text{order of accuracy}$

Fixed interval,  $N$  steps:  $h \propto 1/N$  so  $\boxed{E \propto N^{-p}}$

**1. Step size**  $N: 20 \rightarrow 60$  so  $h_{\text{new}} = h_{\text{old}}/3$

**2. Order**  $p = 2$ :  $E_{\text{new}}/E_{\text{old}} = (1/3)^2 = 1/9$

**3. Say it** The error **decreases by a factor of 9**.

(= new error is  $\frac{1}{9}$  of the old one; not “decreases by  $\frac{1}{9}$ ”)

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## Mixing

amount' = rate in – rate out, conc. out = amount / **current** volume

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