

Before class	Matlab 1	due on Monday
	Quiz 1	returned — check: plug your answer into the ODE
	Quiz 2	next week (no make-up)

Application word problem	Identify the variables → Reduce to an IVP → Solve it			
	Mixture in tank	Population	Motion	Loan
	salt	num. of people	speed	balance

Ex (problems shown on slides)

Tank $S' = 15 - \frac{S}{100 + 2t}$ $S(0) = 2000$ $0 \leq t \leq 200$

Pop $P' = \frac{\ln 2}{4} P - 100$ $P(0) = 1000$ $t \geq 0$

Motion $v' = 9.8 - 0.005 v^2$ $v(0) = 5$ $t \geq 0$, $v_\infty \approx 44.3$

Loan $B' = 0.04 B - 1800$ $B(0) = 8000$ $0 \leq t \leq T$ s.t. $B(T) = 0$

Numerical Euler

Explicit $y_{n+1} = y_n + \Delta t f(t_n, y_n)$

Implicit $y_{n+1} = y_n + \Delta t f(t_{n+1}, y_{n+1})$ solve for y_{n+1}

Ex 1 $y' + y = t$, $y(0) = 1$, $h = \frac{1}{2}$. What is $y(1)$?

Exp $y' = t - y$

$y(\frac{1}{2}) = 1 + \frac{1}{2}(0 - 1) = \frac{1}{2}$

$y(1) = \frac{1}{2} + \frac{1}{2}(\frac{1}{2} - \frac{1}{2}) = \boxed{\frac{1}{2}}$

Imp

$y(\frac{1}{2}) = 1 + \frac{1}{2}(\frac{1}{2} - y(\frac{1}{2}))$
 $\implies \frac{3}{2} y(\frac{1}{2}) = \frac{5}{4} \implies y(\frac{1}{2}) = \frac{5}{6}$

$y(1) = \frac{5}{6} + \frac{1}{2}(1 - y(1))$
 $\implies \frac{3}{2} y(1) = \frac{4}{3} \implies y(1) = \boxed{\frac{8}{9}}$

Error

$$Y(t+h) = \underbrace{Y(t) + h Y'(t)}_{\text{Euler}} + \underbrace{\frac{h^2}{2} Y''(t) + \cdots}_{O(h^2)} \quad (\text{Taylor})$$

→ $O(h^2)$ error in **one step** (**local**)

→ **overall:** $N = \frac{T}{h}$ steps, $\boxed{N \times O(h^2) = O(h)}$ (**global**)

Application Problems — Answer Key

1. Tank mixture

Let $S(t)$ = amount of salt in the tank (g).

Volume:

$$V(t) = 100 + (3 - 1)t = 100 + 2t \text{ L.}$$

Rate in: $(3 \text{ L/min})(5 \text{ g/L}) = 15 \text{ g/min.}$

Rate out: well mixed $\implies$ concentration is S/V :

$$(1 \text{ L/min}) \cdot \frac{S}{100 + 2t} \text{ g/L} = \frac{S}{100 + 2t} \text{ g/min.}$$

Initial: $S(0) = (100 \text{ L})(20 \text{ g/L}) = 2000 \text{ g.}$

$$S' = 15 - \frac{S}{100 + 2t}, \quad S(0) = 2000$$

Interval: tank is full when

$$100 + 2t = 500 \implies t = 200, \text{ so}$$

$$0 \leq t \leq 200 \text{ min}$$

2. Population exponential + harvest

Let $P(t)$ = population after t hours.

Without treatment, exponential growth:

$$P' = rP \implies P = P(0)e^{rt}.$$

Doubles every 4 hours: $P(4) = 2P(0)$, so

$$P(0)e^{4r} = 2P(0) \implies e^{4r} = 2 \quad (P(0) \neq 0)$$

$$\implies 4r = \ln 2 \implies r = \frac{\ln 2}{4}$$

Treatment removes 100 per hour; initially 1000:

$$P' = \frac{\ln 2}{4} P - 100, \quad P(0) = 1000$$

3. Motion turbulent drag

Let $v(t)$ = downward velocity; **downward = positive.**

Gravity: downward acceleration $g = 9.8 \text{ m/s}^2$.

Drag: magnitude kv^2 with $k = 0.005 \text{ m}^{-1}$, opposite to motion.

Newton: $v' = g - kv^2$, released at $v_0 = 5 \text{ m/s}$:

$$v' = 9.8 - 0.005 v^2, \quad v(0) = 5$$

Terminal velocity = equilibrium, $v' = 0$:

$$9.8 - 0.005 v_{\infty}^2 = 0 \implies v_{\infty}^2 = \frac{9.8}{0.005} = 1960$$

Downward is positive, so

$$v_{\infty} = \sqrt{1960} \approx 44.3 \text{ m/s}$$

4. Loan interest – payment

Let $B(t)$ = remaining balance (\$) after t years.

Interest: balance grows continuously at $0.04B$ \$/yr.

Repayment: constant 1800 \$/yr.

Initial: $B(0) = 8000$.

$$B' = 0.04 B - 1800, \quad B(0) = 8000$$

Interval: valid until the loan is paid off. If T is the payoff time, $B(T) = 0$, so

$$0 \leq t \leq T, \quad B(T) = 0$$