

1.3 Separable

$$y' = f(t)g(y)$$

$$\frac{dy}{g(y)} = f(t) dt \implies \int \frac{dy}{g(y)} = \int f(t) dt$$

Stationary first. $g(c) = 0 \implies y \equiv c$, lost when dividing.

$$y' = t y^2, \quad y(0) = 1$$

Can you integrate it?

- 1) stationary $y \equiv 0$; $y(0) = 1 \neq 0$, so divide by y^2 .
- 2) separate, integrate: $-\frac{1}{y} = \frac{t^2}{2} + C$ 3) $y(0) = 1 \implies C = -1$
- 4) solve: $y = \frac{2}{2-t^2}$, three pieces — keep the one holding $t = 0$
- 5) interval $(-\sqrt{2}, \sqrt{2})$: blow-up as $t \rightarrow \pm\sqrt{2}$ $y(0) = a$: at $\pm\sqrt{2/a}$

1.4 Existence & Uniqueness

We can know a solution exists and is unique without ever solving the ODE.

$$y' = 3y^{2/3}, \quad y(0) = 0$$

Unique without solving?

- 1) f continuous everywhere 2) $\partial_y f = 2y^{-1/3}$ not continuous at $y = 0$
- 3) theorem says nothing 4) $y \equiv 0$, $y = t^3$ both solve it $\implies$ not unique.

Thm 4.1. $y' = f(t, y)$, $y(t_0) = y_0$, with f over a set S :

$f, \partial_y f$ continuous over S , $(t_0, y_0) \in \text{int}(S) \implies$ **unique** $y = Y(t)$
on the largest $(a, b) \ni t_0$ whose graph stays in $\text{int}(S)$.

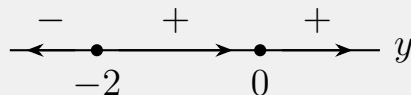
1.5 Graphical method:

autonomous
 $\rightarrow$ phase line
 $y' = g(y)$

$$y' = 3y^2(y + 2)$$

What do solutions do, without solving?

- 1) Where are the equilibria? $g(y) = 0 \implies y = -2, 0$
- 2) Sign of g in between? $-, +, +$
- 3) Then the behavior is already fixed: -2 unstable, 0 semistable.



0211 Review for Quiz 1

1. $t y' = y + 2$

3. $y' + 2y^2 \cos t = 0$, $y(0) = \frac{1}{3}$

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2. $y' + \frac{2t}{t^2+4} y = \frac{t^2}{t^2+4}$

4. $y' = \frac{t}{t^2+1} (y + 1)$, $y(0) = 1$

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1. $t y' = y + 2$ separable & linear

Normal form needs $t \neq 0$: $y' = \frac{y+2}{t}$.

Stationary: $y + 2 = 0 \implies y \equiv -2$.

Otherwise separate:

$$\int \frac{dy}{y+2} = \int \frac{dt}{t}$$

$$\implies \ln|y+2| = \ln|t| + c$$

Exponentiate: $|y+2| = e^c|t|$, so $y+2 = At$ with $A \in \mathbb{R}$ ($A = 0$ gives back the stationary solution).

$$\boxed{y = At - 2} \quad \text{on } t > 0 \text{ or } t < 0.$$

$t = 0$ is excluded: there $f = \frac{y+2}{t}$ is undefined, so Thm 4.1 says nothing. Every branch runs into $(0, -2)$, so $y(0) = -2$ has infinitely many solutions and $y(0) = b \neq -2$ has none.

3. $y' + 2y^2 \cos t = 0$, $y(0) = \frac{1}{3}$ separable
 $y' = -2y^2 \cos t$, so $f(t) = -2 \cos t$, $g(y) = y^2$.

Stationary: $y \equiv 0$; $y(0) = \frac{1}{3} \neq 0$, so divide by y^2 .

$$\int \frac{dy}{y^2} = -2 \int \cos t \, dt$$

$$\implies -\frac{1}{y} = -2 \sin t + C$$

So $\frac{1}{y} = 2 \sin t + c$; $t = 0$, $y = \frac{1}{3} \implies c = 3$.

$$\boxed{y = \frac{1}{3 + 2 \sin t}}$$

$3 + 2 \sin t \in [1, 5]$, never 0, so the solution lives on $(-\infty, \infty)$.

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2. $y' + \frac{2t}{t^2+4} y = \frac{t^2}{t^2+4}$ linear

$$a(t) = \frac{2t}{t^2+4},$$

$$A(t) = \int \frac{2t}{t^2+4} dt = \ln(t^2+4).$$

Integrating factor $\mu = e^{A(t)} = t^2 + 4$:

$$((t^2+4)y)' = (t^2+4) \cdot \frac{t^2}{t^2+4} = t^2$$

Integrate: $(t^2+4)y = \frac{t^3}{3} + c$.

$$\boxed{y = \frac{t^3/3 + c}{t^2+4}}$$

a, f are continuous on all of $\mathbb{R}$ and $t^2+4 > 0$, so every solution lives on $(-\infty, \infty)$ — the linear case: interval known before solving.

4. $y' = \frac{t}{t^2+1} (y+1)$, $y(0) = 1$
 separable & linear

$$f(t) = \frac{t}{t^2+1}, \quad g(y) = y+1.$$

Stationary: $y \equiv -1$; $y(0) = 1 \neq -1$, so divide by $y+1$.

$$\int \frac{dy}{y+1} = \int \frac{t \, dt}{t^2+1}$$

$$\implies \ln|y+1| = \frac{1}{2} \ln(t^2+1) + c$$

Exponentiate: $y+1 = A\sqrt{t^2+1}$;

$y(0) = 1 \implies A = 2$.

$$\boxed{y = 2\sqrt{t^2+1} - 1}$$

Check: $y' = \frac{2t}{\sqrt{t^2+1}} = \frac{t}{t^2+1} \cdot 2\sqrt{t^2+1}$.

$t^2+1 > 0$, so the interval is $(-\infty, \infty)$.