

WHAT IS A LEAKY ATTRACTOR? FROM THE PAPER "LAMINAR CHAOTIC SADDLE WITHIN A TURBULENT ATTRACTOR"

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#intermittency #synchronization



Paper link



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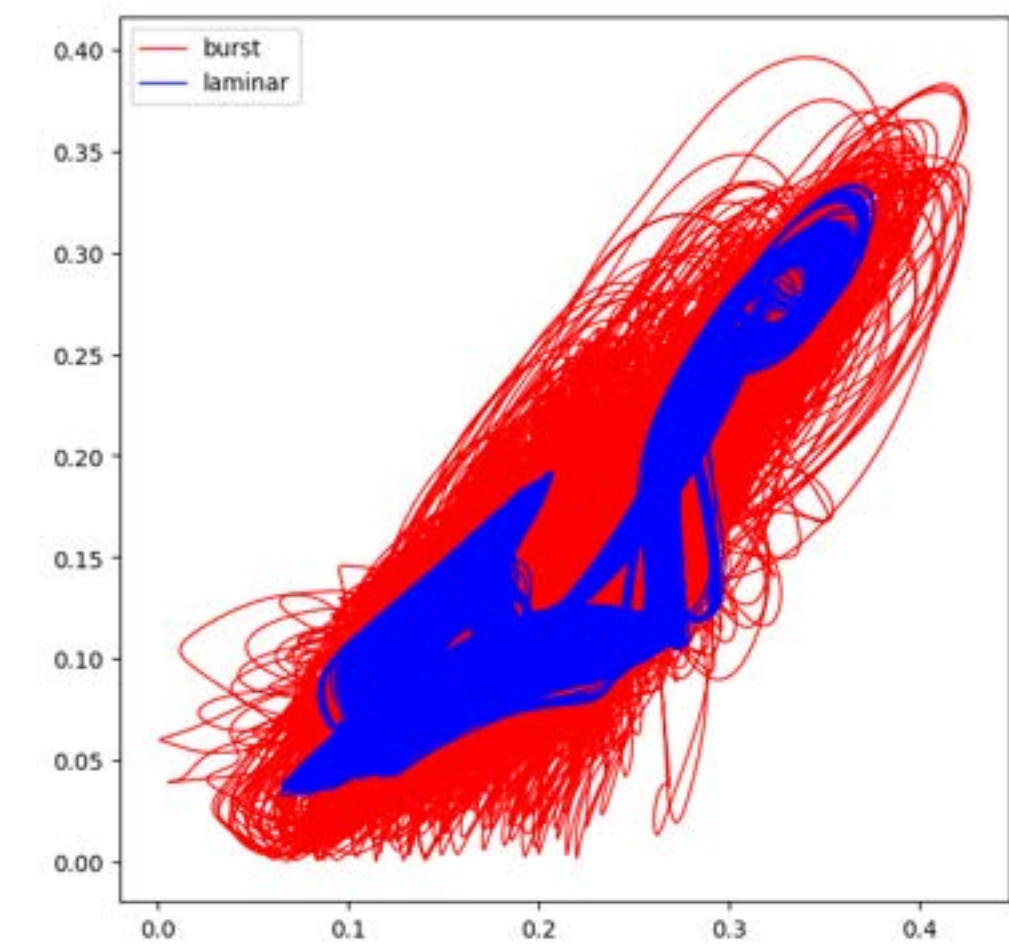


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Abstract

A leaky attractor is an invariant set with non-attracting directions. When the set of points with non-attracting neighborhoods has zero measure, it is called a Milnor attractor; when positive, it corresponds to what is known as a chaotic saddle.

In this study, we investigate leaky attractors in a turbulent model exhibiting intermittency with alternating bursty and laminar phases, as well as in a coupled dynamical system of two chaotic oscillators. We numerically confirm that leaky attractors characterize physical phenomena such as laminar in intermittency and synchronization. Furthermore, we demonstrate that synchronization phenomena also occur in the turbulent model. This poster is based on Kato, Kobayashi, Saiki, and Yorke, *Phys. Rev. E*, 2024.^[1]



Leaky attractor embedded in chaotic attractor

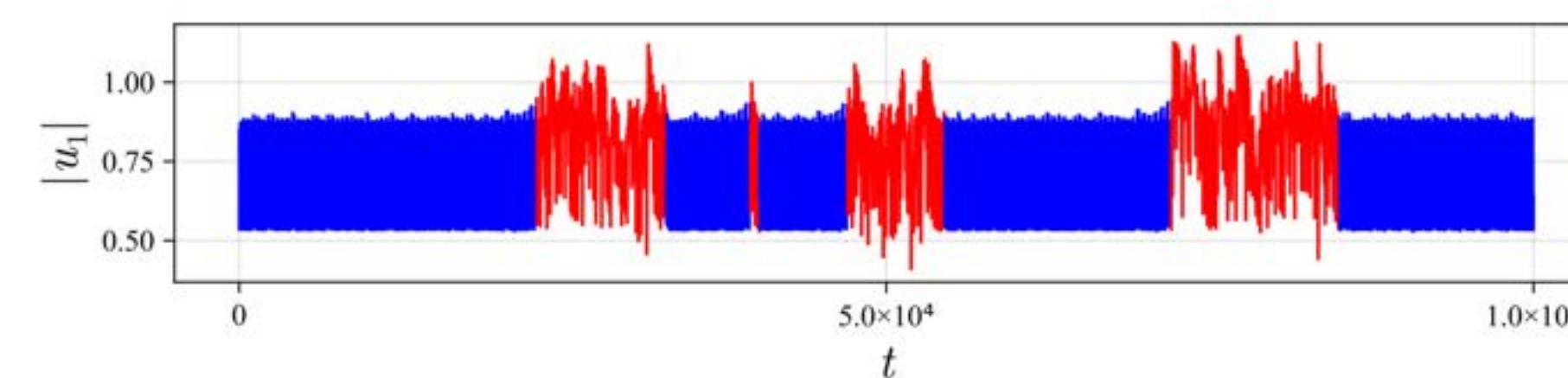
Mathematical Models

GOY Shell Model (GOY=Gledzer-Ohkitani-Yamada)

Shell models of fluid turbulence are ODEs introduced to understand the chaotic properties of fully developed turbulence. They model nonlinear interactions between neighboring wavenumber shells in Fourier space. We employ the so-called GOY shell model^[3], which is a system of ordinary differential equations of complex variables:

$$\left(\frac{d}{dt} + \nu k_j^2\right) u_j = i[A_j u_{j+1} u_{j+2} + B_j u_{j-1} u_{j+1} + C_j u_{j-1} u_{j-2}]^* + f \delta_{j,1}$$

where $k_j = 10^{-4} \times 2^j$ and $f = 5(1+i)10^{-3}$. At $\beta = 0.4162$, we can observe intermittent states as shown in the figure below:



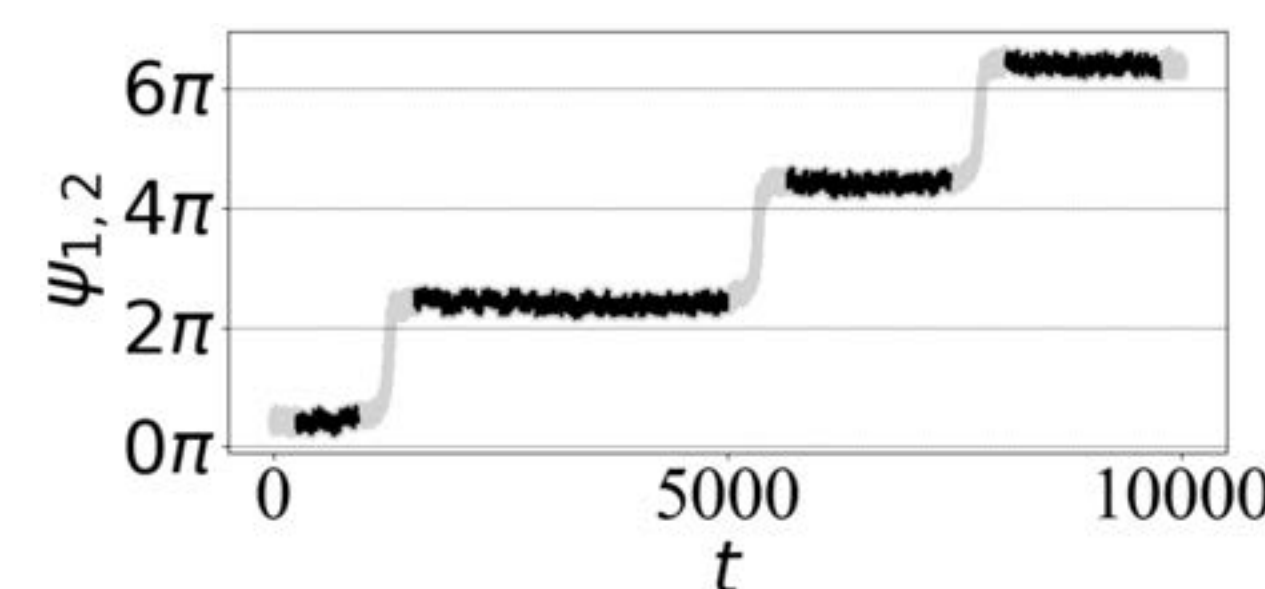
Time evolution of $|u_1|$. Intermittency between laminar and bursty states occurs.

Coupled Rössler Systems

To analyze leaky attractors in chaotic synchronization models, we study a system of coupled Rössler oscillators:

$$\begin{cases} \dot{x}_{1,2} = -\omega_{1,2} y_{1,2} - z_{1,2} + \varepsilon(x_{2,1} - x_{1,2}), \\ \dot{y}_{1,2} = \omega_{1,2} x_{1,2} - a y_{1,2}, \\ \dot{z}_{1,2} = f + z_{1,2}(x_{1,2} - c), \end{cases}$$

where a, f, c are fixed to the classical value for Rössler model, and $\omega_{1,2} = \omega_0 \pm \delta$ ($\omega_0 = 0.97$, $\delta = 0.02$), and various coupling strengths ε . **The difference of ω_1 and ω_2 makes two Rössler oscillator asymmetric.** Due to this asymmetry, we need to introduce a specific definition of synchronicity. We define the phase $\phi_j(t)$ of each oscillator (via unwrapped phase), and call it **phase synchronization** when the phase difference stays locked (mod 2π) within a small threshold ε . See the paper for details.

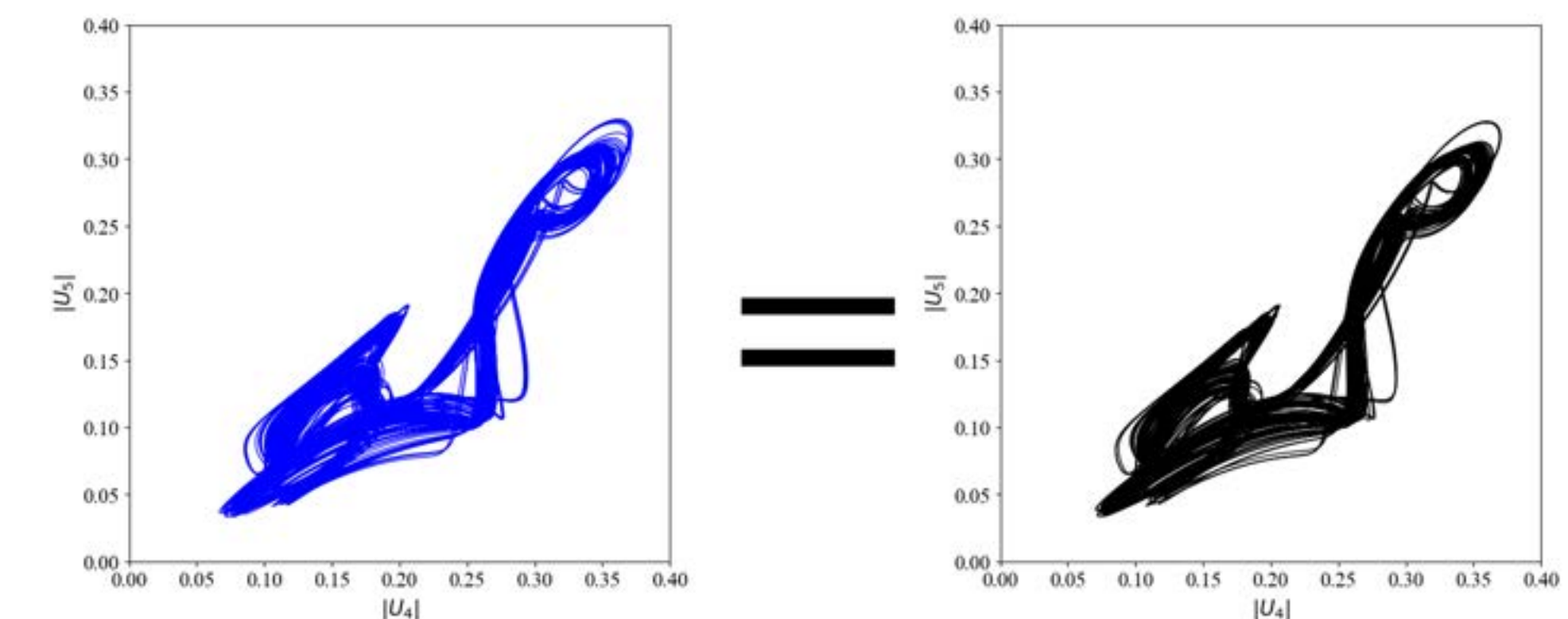


Phase difference (y-axis) is locked during black.

Take-home message

Leaky attractor in turbulence = Phase-synchronized state

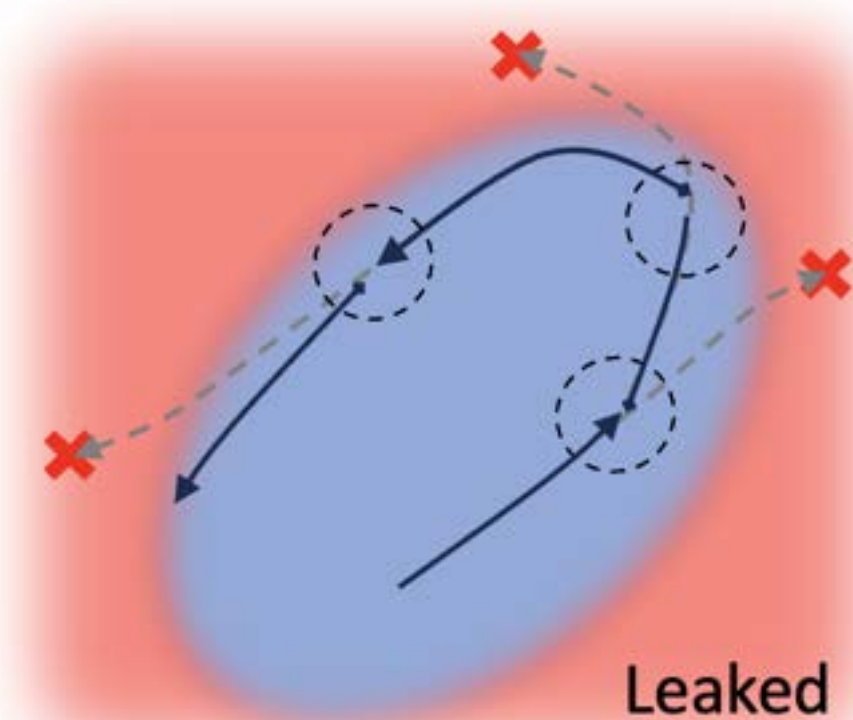
Laminar states in intermittency and phase-synchronized states in coupled chaotic systems correspond to the same leaky attractor, a non-attracting invariant set embedded in a chaotic attractor. In this case, the structure is a genuine dynamical object, not an artifact of trajectory selection, and robustly underlies coherent behavior in turbulent and high-dimensional chaotic systems.



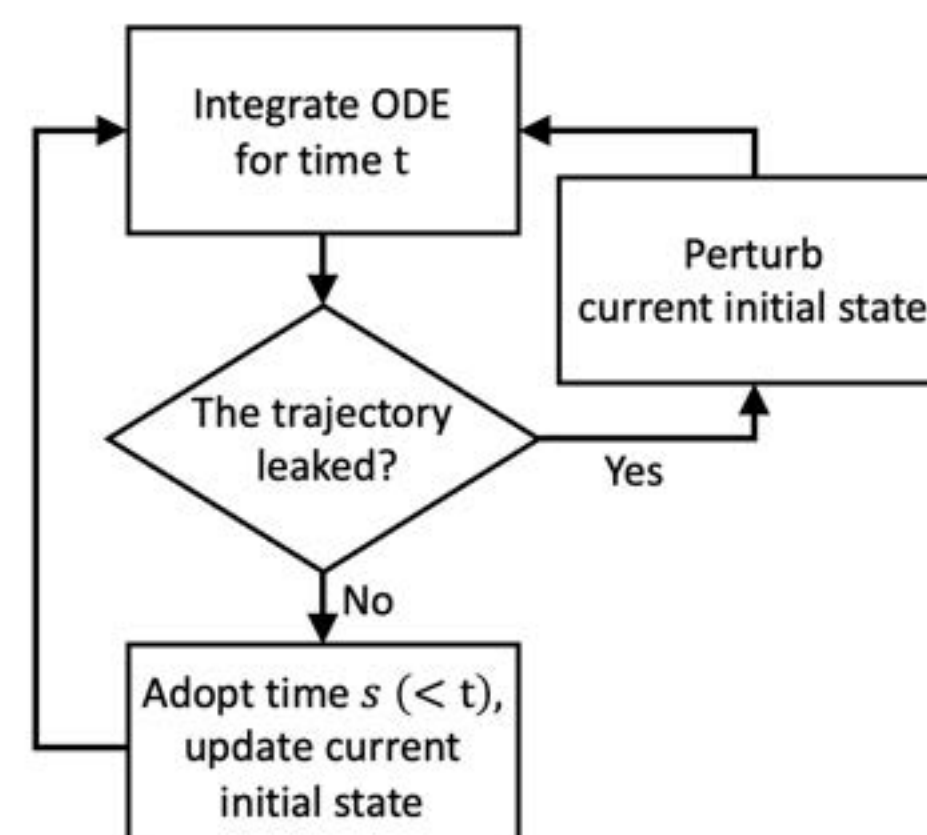
Left: Laminar state in intermittency. Right: Phase-synchronized state.

How to find the leaky attractor: stagger-and-step method

To identify the leaky attractor within a chaotic attractor, we employ the stagger-and-step method, introduced by Sweet, Nusse, and Yorke (2001)^[2]. In essence, the Stagger and Step method applies small perturbations to trajectories as needed to keep them confined within the leaky attractor, thereby constructing long trajectories that remain trapped there and robustly capture it in our case.



Leaked



Process of Stagger-and-Step method

Skematic Figure of Stagger-and-Step method

Conclusion

- Laminar states in intermittent dynamics are supported by a leaky attractor, a non-attracting chaotic invariant set embedded in a turbulent attractor.
- In coupled chaotic oscillators, the phase-synchronized state corresponds to the same type of leaky attractor.
- In the GOY shell model, laminar states exhibit partial inter-shell phase-synchronized states, which a leaky attractor also characterizes.
- These results indicate that the leaky attractor provides a unified framework to characterize coherence and intermittency in high-dimensional chaotic systems: coherent dynamics and phase synchronization.

References

- Kato, H., Kobayashi, M. U., Saiki, Y. & Yorke, J. A. *Physical Review E* **110**, L052202 (2024).
- Sweet, D., Nusse, H. E. & Yorke, J. A. en. *Physical Review Letters* **86**, 2261–2264 (2001).
- Yamada, M. & Ohkitani, K. *Journal of the Physical Society of Japan* **56**, 4210–4213 (1987).