<u>Phys604/F15/Hassam/Take Home</u>

- **Due by Wednesday Dec15 12noon, in my office (slip it under door if I am not there)**
- **Posted Tuesday Dec14 12noon**
- **Please BOX all your important answers to each subsection**
- **Must show all relevant work but be concise**
- **You may consult any online materials or texts**
- **You may not consult other persons**

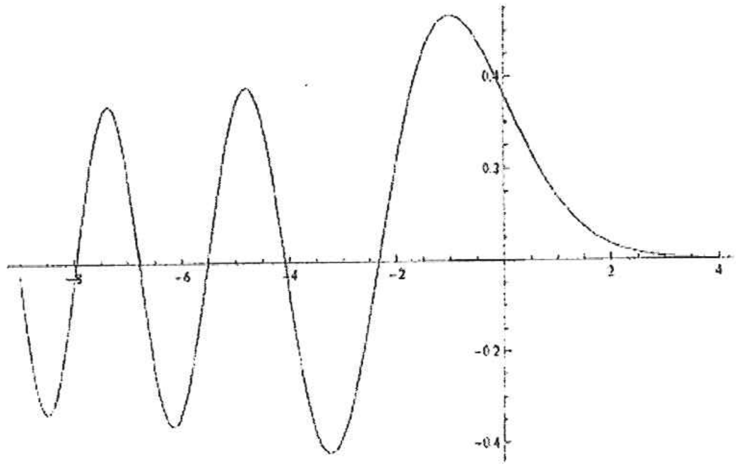
You will need the below information:

<u>Information on Airy functions:</u> The Ai(s) and Bi(s) functions satisfy the equation $d^2y/ ds^2 = sy$, $y = y(s)$. These functions are completely defined by Frobenius' series in powers of s; the series converge for all $|s| < \infty$. We have also obtained integral representations for both Airy functions in class: the Ai(s) function is picked to be the linear combination such that Ai(s) decays as $s \to +\infty$. The Bi(s) function blows up as $s \to +\infty$. The asymptotic behaviors here for positive s are $Ai(s) \sim \exp[-(2/3)s^{3/2}]$, while $Bi(s) \sim \exp[(2/3)s^{3/2}]$. For $s < 0$, both functions oscillate, asymptoting as $\exp[\pm i(2/3)(-s)^{3/2}]$. Roots are defined by $Ai(-s_{An}) = 0$ and $Bi(-s_{Bn}) = 0$, where the s_n are positive and n = 1, 2, 3,….. See the attached sheets for specific roots and more info.

Ai(x)

```
In[14]:= Plot[AiryAi[x], {x, -9, 4}]
```

Out[14]=



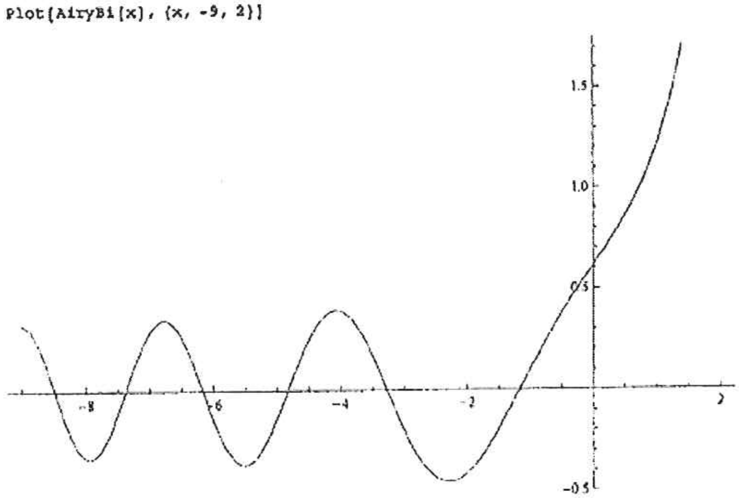

```
In[16]:= FindRoot[AiryAi[x], {x, -3}]
         FindRoot[AiryAi[x], {x, -5}]
         FindRoot[AiryAi[x], {x, -6}]
         FindRoot[AiryAi[x], {x, -7}]
Out[16]= {x → -2.33811}
Out[17]= {x → -4.08795}
Out[18]= {x → -5.52056}
Out[19]= {x → -6.78671}
```

Roots

Bi(x)

```
In[21]:= Plot[AiryBi[x], {x, -9, 2}]
```

Out[21]=




```
In[30]:= FindRoot[AiryBi[x], {x, -2}]
         FindRoot[AiryBi[x], {x, -3.5}]
         FindRoot[AiryBi[x], {x, -5}]
         FindRoot[AiryBi[x], {x, -6.5}]
Out[30]= {x → -1.17371}
Out[31]= {x → -3.27109}
Out[32]= {x → -4.83074}
Out[33]= {x → -6.16985}
```

Roots

TH.1 Sturm-Liouville (20 points)

(a) Consider the space of all f(x) in the domain x=[0,∞], with f(0)=0 and f(∞)=0. The inner product is defined (f,g)=∫dxf*(x)g(x). Consider the operator A = $d^2/dx^2 - x$. Show that the allowed f's and the operator A form a Sturm-Liouville system.

(b) Find the eigenfunctions of A. (You will need to use what you know about Airy functions. Needed facts are provided.) Define the orthogonality condition, leaving the norm in terms of an integral expression. Sketch the first 3 eigenfunctions. Write a formal expansion for an f(x) in this space in terms of these eigenfunctions. Obtain a formal expression for the coefficients of the expansion. Assume you have all the information on Airy functions as given online.

TH.2 Diffusion (20 points)

A long cylinder of radius a is filled with a material of constant heat conduction coefficient κ. Thus, the diffusion of heat in the material is described by the diffusion equation $\partial T/\partial t = \kappa\nabla^2 T$. In the case that the material is also heated by a given external heater, H(**x**), the governing equation is $\partial T/\partial t = \kappa\nabla^2 T + H(\mathbf{x})$, where T = T(**x**,t). The entire cylinder is placed in a heat bath of temperature = 0.

(a) Suppose H is highly localized so that $H = A\delta(\rho-\rho_0)$, where $\rho_0 < a$, A = constant, and we are using cylindrical coordinates. The temperature inside the material builds up until it gets to a steady state as a balance between heating and conduction. Find T(**x**) *in steady state*, using the symmetries of the problem. [Note: inhomogeneous equations are in general solved by Green's functions, except in this case the source is particularly simple.]

(b) Suppose, after steady state is reached, we turn off H. In this case, T will decay to zero. Find T(**x**,t), again using the symmetries of the initial condition. Leave your final answer in terms of integrals over known functions. What does the approximate shape of the T function look like for large times (ie, the leading order term)? [this part can be done independently of part (a); assume that you know the steady state T(**x**,0).]

TH.3 Green (40 points)

(a) A function $\psi(\mathbf{x})$ satisfies $\nabla^2\psi = f(\mathbf{x})$ inside a sphere of radius a. The surface of the sphere, S, is maintained at $\psi(S) = h(r,\theta)$, where h is given in spherical coordinates. $f(\mathbf{x})$ is given in between the spheres to be azimuthally symmetric, ie, $f=f(r,\theta)$. Find the solution to ψ for general f and h using Green functions.

(b) Suppose $f=0$ and $h=\cos(\theta)$. Find $\psi(\mathbf{x})$ explicitly using G from (a).

(c) Find $\psi(\mathbf{x})$ for $f=0$ and $h=\cos(\theta)$ by *directly solving* Laplace's equation.