

12.1

In this problem we study two initial value in time problems, namely the diffusion equation and the wave equation.

1. Consider the 1-D heat conduction equation $\partial_t T = D\partial_x^2 T$. The temperature $T(x,t)$ between two flat surfaces from $x=0$ to $x=a$ is given initially by $T(x,0) = f(x)$. There are heat reservoirs so $T(0,t)=0$ and $T(a,t)=0$. Find $T(x,t)$ for general $f(x)$.
2. Now consider the 1-D wave equation $\partial_t^2 \xi = c^2 \partial_x^2 \xi$, where $\xi(x,t)$ is the displacement (of a string, eg). The string is fixed at two ends from $x=0$ to $x=a$. The displacement is given initially by $\xi(x,0) = f(x)$. Also, the velocity of the string is given initially by $\partial_t \xi(x,0) = g(x)$. Find $\xi(x,t)$ for general $f(x)$ and $g(x)$. Take note of the overall multiplicity of possible solutions in this problem compared to the diffusion problem in #1 and, thus, why two initial conditions are needed.

Now extend the above problems to 2D as follows:

3. As an extension of #1, there is a temperature distribution $T(x,y,t)$ inside a square of sides a by a . At $t=0$, $T = f(x,y)$. Find $T(x,y,t)$.
4. Likewise for #2, consider a square cavity. For each eigenmode, there corresponds an eigenfrequency, ie, the normal mode frequencies of the “cavity modes”. Identify these and sketch the lowest 4 spatial eigenfunctions (ie, the “drum” modes). You are not asked to do the initial value problem, which would require two initial conditions and a sum over all possible eigenfrequencies.

12.2

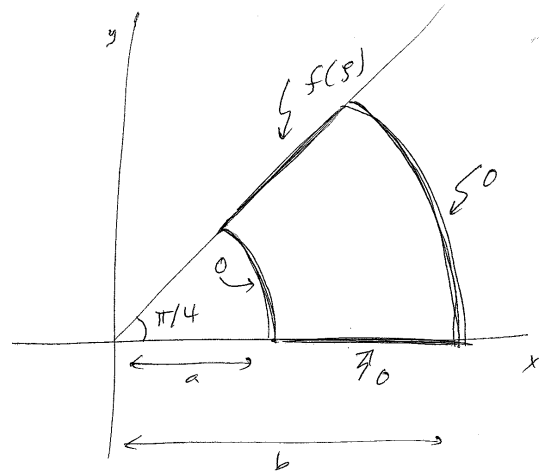
Consider the operator $L[f] = x(d/dx)[x(df/dx)]$ in the space of functions $\{f(x)$ in the domain $x=[1,a]$ such that $f(1)=0$ and $f(a)=0$, $a > 1$. *Note: the domain starts at $x=1$.*

1. Show that the vector space of the $\{f(x)\}$ with the operator L forms a Sturm-Liouville system. Define the inner product (f,g) , including any weight function. Show explicitly that L is Hermitean.
2. Find the eigenfunctions φ_n and the associated eigenvalues. Explicitly calculate (φ_m, φ_m) .
3. Expand any $f(x)$ in the set as a series in the $\varphi_n(x)$. Formally, obtain an expression for the coefficients C_n in the expansion.
4. Find C_n for $f(x) = \ln(x) \cdot [\ln(a/x)]$ inside $x=[1,a]$.

Note: the identity $u^w = e^{w \ln(u)}$, u real, may be useful.

12.3

Consider the 2D geometry shown in the Fig. A scalar function $\psi(x,y)$ satisfies $\nabla^2\psi = 0$ inside the boundary. Boundary conditions on ψ are shown in the Figure. Find ψ inside. Make use of the fact that the function is zero at two radii and, thus, one expects oscillatory solutions in the radial coordinate. The orthogonality condition may be unfamiliar but can be checked by using an appropriate weight function.



12.4

Two concentric spheres have radii a and $b > a$. The outer sphere is divided into two hemispheres. A function $\psi(\mathbf{x})$ satisfies $\nabla^2\psi = 0$ in between the two spheres. The inner sphere is maintained at $\psi = 0$. The top hemisphere of the outer sphere is maintained at $\psi = +V_0$ and the bottom hemisphere is maintained at $\psi = -V_0$. Find ψ in between the spheres.

12.5

A long cylinder of radius a is filled with a material of heat conduction coefficient κ . Thus, the diffusion of heat in the material is described by the diffusion equation $\partial T/\partial t = \kappa \nabla^2 T$. In the case that the material is also heated by a given external heater, $H(\mathbf{x})$, the governing equation is $\partial T/\partial t = \kappa \nabla^2 T + H(\mathbf{x})$, where $T = T(\mathbf{x}, t)$. The entire cylinder is placed in a heat bath of temperature $= 0$.

- (a) Suppose H is highly localized so that $H = A\delta(\rho - \rho_0)$, where $\rho_0 < a$, $A = \text{constant}$, and we are using cylindrical coordinates. The temperature inside the material builds up until it gets to a steady state as a balance between heating and conduction. Find $T(\mathbf{x})$ in steady state, using the symmetries of the problem. [Note: inhomogeneous equations are in general solved by Green's functions, except in this case the source is particularly simple.]
- (b) Suppose, after steady state is reached, we turn off H . In this case, T will decay to zero. Find $T(\mathbf{x}, t)$, again using the symmetries of the initial condition. Leave your final answer in terms of integrals over known functions. What does the approximate shape of the T function look like for large times (ie, the leading order term)?