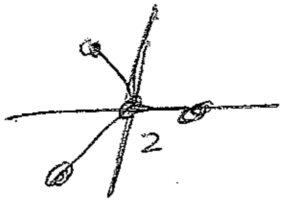
$\boxed{9.1H}$

$$\epsilon^2 x^6 - \epsilon x^4 - x^3 + 8 = 0 \ , \quad \epsilon \ll 1$$

<u>Naively</u> $\quad -x_0^3 + 8 = 0 \Rightarrow x_0 =$



$$\boxed{x_0 = 2, 2e^{i2\pi/3}, 2e^{-i2\pi/3}}$$

$$\cancel{\epsilon^2 x_0^6} - \epsilon x_0^4 - 3x_0^2 x_1 = 0$$

smaller

$$\Rightarrow \quad 3x_1 = -\epsilon x_0^2$$

$$\Rightarrow \quad \boxed{x_1 = -\frac{4}{3}\epsilon \left[1, e^{4\pi i/3}, e^{-4\pi i/3} \right]}$$

$$\Leftrightarrow \quad |x_1| \ll |x_0| \quad \Rightarrow \quad \boxed{\frac{\epsilon}{3} \ll 1} \quad \text{OK}$$

<u>lost 3 roots</u> $\quad$ Try $|x| \to \infty$

- Clearly x^6 term dominates for $|x| \ggg 1$.
- Clearly "8" term is always small since $x^3 \gg 8$.
- Question is which of the x^4 or x^3 terms must be retained for $|x|$ large but not ∞.

if we retain x^4 term, then we are saying

$$|\epsilon^2 x^6| \sim |\epsilon x^4| \gg |x^3| \Rightarrow x^2 \sim \frac{1}{\epsilon} \text{ and } |\epsilon x| \gg 1$$

$$\rightarrow \epsilon \gg 1 \Rightarrow \text{inconsistent}$$

$\therefore$ lets retain x^3 term, ie, lets try

$|\epsilon^2 x^6| \sim |x^3| >> |\epsilon x^4|$

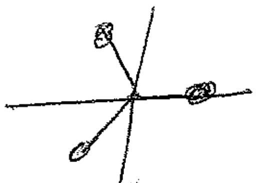
$\Rightarrow x^3 \epsilon^2 \sim 1$ and $|\epsilon x| << 1$

$\Rightarrow \epsilon^3 \frac{1}{\epsilon^2} << 1 \Rightarrow$ consistent

$\therefore$ Try x^6 & x^3 terms to lowest

$$\Rightarrow \quad \epsilon^2 x_0^6 - x_0^3 = 0$$

$$\Rightarrow x_0^3 = \frac{1}{\epsilon^2} \Rightarrow x_0 = \frac{1}{\epsilon^{2/3}}$$



$$\boxed{x_0 = \frac{1}{\epsilon^{2/3}} \left[1, e^{2\pi i/3}, e^{-2\pi i/3}\right]}$$

$$\Rightarrow \epsilon^2 6 x_0^5 x_1 - \underset{\uparrow}{\epsilon x_0^4} - 3x_0^2 x_1 + \underset{\uparrow}{8} = 0$$

$\frac{\epsilon}{\epsilon^{8/3}} : 1 \quad \therefore$ 8 is smaller

$$\Rightarrow (\epsilon^2 6 x_0^5 - 3x_0^2) x_1 = \epsilon x_0^4 \Rightarrow 3(2x_0^3\epsilon^2 - 1) x_1 = \epsilon x_0$$

$$\Rightarrow 3x_1 = \epsilon x_0^2 \Rightarrow \boxed{x_1 = \frac{1}{3\epsilon^{1/3}}\left[1, e^{4\pi i/3}, e^{-4\pi i/3}\right]}$$

$|x_1| << |x_0| \Leftrightarrow \boxed{\epsilon^{1/3} << 1}$ OK

9.2H)

$$\epsilon y'' + (1+x)y' + y = 0 \;, \epsilon \ll 1, x \geq 0$$

Naively $(1+x)y_0' + y_0 = 0$

$$\Rightarrow [(1+x)y_0]' = 0$$

$$\Rightarrow \boxed{y_0 = \frac{C}{1+x}}$$

$$[(1+x)y_1]' + \epsilon y_0'' = 0$$

$$\Rightarrow [(1+x)y_1]' + \epsilon y_0' = D$$

$$\Rightarrow [(1+x)y_1] = D + \frac{\epsilon C}{(1+x)^2}$$

homogeneous soln

$$y_1 = \frac{D}{1+x} + \frac{\epsilon C}{(1+x)^3}$$

$$\Rightarrow \boxed{y_1 \simeq \frac{\epsilon C}{(1+x)^3}}$$

$|y_1| \ll |y_0|$

$\Leftarrow$ $\boxed{\epsilon \ll (1+x)^2}$

Always ok for $x \geq 0$

$$\boxed{y(1) = \frac{C}{1+x} + \frac{\epsilon C}{(1+x^3)} + \cdots}$$

Lost a solution. Try WKB, $y = e^S, |S| \gg 1$

$$\Rightarrow \boxed{\epsilon S'^2 + \epsilon S'' + (1+x)S' + 1 = 0} \quad ①$$

$\uparrow$ small $\uparrow$ small

$$\Rightarrow \quad \epsilon S_0'^2 + (1+x)S_0' = 0$$

$$\epsilon S_0' = -(1+x), \quad \epsilon S_0 = -\left(x+\frac{x^2}{2}\right) + \text{const}$$

$$\Rightarrow \boxed{y_0 = Ce^{-\frac{1}{\epsilon}(x + x^2/2)}}$$

$$2\epsilon S_0'S_1' + \epsilon S_0'' + (1+x)S_1' + 1 = 0$$

~~⟹ Me~~ $\Rightarrow -2(1+x)S_1' \cancel{-1} + (1+x)S_1' \cancel{+1} = 0$

$$\Rightarrow \quad -(1+x)S_1' = 0 \Rightarrow S_1' = 0.$$

What about S_2'?

$$2\epsilon S_0'S_2' + \overset{0}{\cancel{\epsilon S_1'^2}} + \overset{0}{\cancel{\epsilon S_1''}} + (1+x)S_2' = 0$$

$$\Rightarrow \quad -2(1+x)S_2' + (1+x)S_2' = 0$$

$\Rightarrow S_2' = 0.$ ∴ $S' = S_0'$ only (all h.o. terms $= 0$). So it must be an <u>exact</u> solution. Try $\epsilon S' = -(1+x)$ in ①. $\Rightarrow \quad \epsilon S'^2 \cancel{=} 1 + (1+x)S' + 1 = 0$

$\Rightarrow -(1+x)S' + (1+x)S' = 0$ ∴ <u>Exact</u>

$$\Rightarrow \boxed{y^{(2)} = Ce^{-\frac{1}{\epsilon}(x+x^2/2)}}$$ 2nd solution

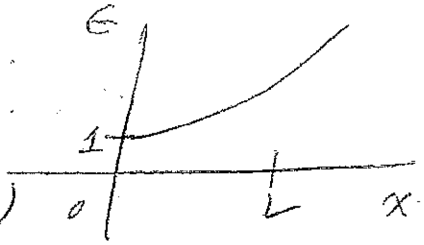
9.3H

W1

Validity of WKB

Suppose $\partial_t^2 \psi = \frac{c^2}{\epsilon(x)} \partial_x^2 \psi$, $\psi(x,t)$

Suppose $\epsilon(x) = (1+x^2/2L^2)^2$



Find WKB solution for $\psi(x,t)$ for $x > 0$ assuming $\psi \to e^{-i\omega t}$, and $\left(\frac{\omega}{c}\right)L >> 1$. Find ψ to first order and demand $|S_1'| \ll |S_0'|$ to check on self-consistent validity of solution. (Assume propagation to the right)

$$\partial_t^2\psi = \frac{c^2}{\epsilon(x)}\psi^{2}_{x^2} \Rightarrow \frac{d^2\psi}{dx^2} = -\frac{\epsilon(x)}{c^2}\omega^2\psi$$

$$\psi(x) = e^{iS(x)} \Rightarrow +S'^2 \underline{\cancel{+}}\, iS'' = +\epsilon(x)\left(\frac{\omega}{c}\right)^2$$

$$k = S' \Rightarrow k^2 - ik' = \epsilon(x)\,(\omega/c)^2$$

$$\Rightarrow \boxed{k_0 = \left(\frac{\omega}{c}\right)(1+x^2/2L^2)} \quad 2k_0k_1 = ik_0'$$

rightward propgn.

$$\Rightarrow \quad k_1 = \frac{i}{2}\frac{k_0'}{k_0} = \frac{i}{2}\frac{x/L^2}{(1+x^2/2L^2)}$$

Demand $|k_1| \ll |k_0| \Rightarrow \frac{(x/L\cancel{^2})}{(1+x^2/2L^2)} \ll \left(\frac{\omega L}{c}\right)(1+x^2/2L^2)$

$$\Rightarrow \boxed{\frac{(x/L)}{(1+x^2/2L^2)^2} \ll \left(\frac{\omega L}{c}\right)}$$

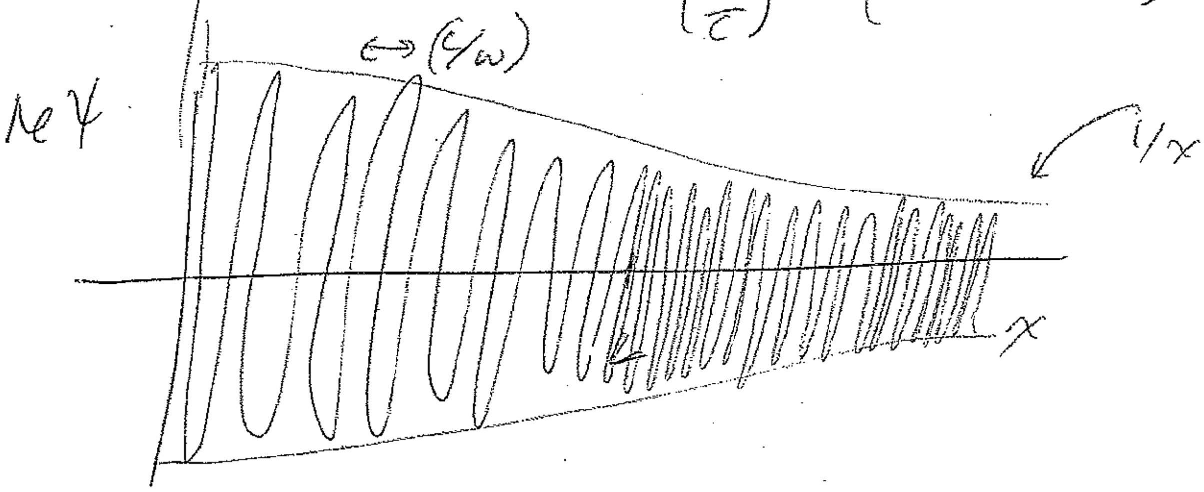
This is good $\forall x$ provided $(LHS)_{max} \ll RHS$

$$\Rightarrow \boxed{1 \ll (\omega L/c)}$$

$$S_0 = \int dx\, k_0 = \left(\frac{\omega}{c}\right)\left(x + \frac{x^3}{6L^2}\right)$$

$$e^{iS_1} = e^{i\int k_1\, dx} = e^{-\frac{1}{2}\ln|k_0|} = \frac{1}{k_0^{1/2}}$$

$$\Rightarrow \psi(x,t) \simeq \frac{e^{i\frac{\omega}{c}(x-ct)}\; e^{i\left(\frac{\omega}{c}\right)\frac{x^3}{6L^2}}}{\left(\frac{\omega}{c}\right)^{1/2}\left(1+x^2/2L^2\right)^{1/2}}$$



N3

Geom optics

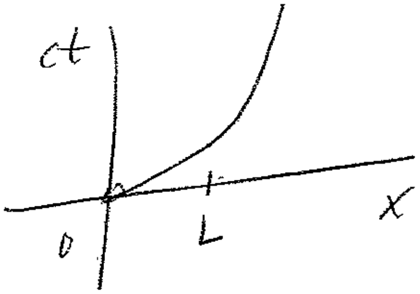
$$k = \frac{\omega}{c}\left(1 + x^2/2L^2\right)$$

$$\frac{\partial}{\partial k} \Rightarrow \quad c = \frac{\partial \omega}{\partial k}\left(1 + x^2/2L^2\right)$$

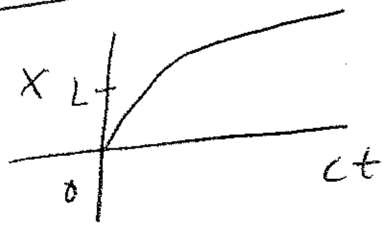
$$\frac{\partial}{\partial x} \Rightarrow \quad 0 = \frac{\partial \omega}{\partial x}\left(1 + x^2/2L^2\right) + \frac{\omega x}{L^2}$$

$$\Rightarrow \frac{\overset{\circ}{x}}{c} = \frac{1}{\left(1 + x^2/2L^2\right)} \quad \Rightarrow \overset{\circ}{x} + \frac{(x^3)^{\circ}}{6L^2} = c$$

$$\Rightarrow \boxed{x + \frac{x^3}{6L^2} = ct}$$



$\Rightarrow$



$$\overset{\circ}{k} = \frac{-\,\omega x/L^2}{\left(1 + x^2/2L^2\right)} \quad \text{or} \quad \boxed{\frac{k\epsilon}{\omega} = 1 + \frac{x^2}{2L^2}}$$

$\Downarrow$

$k(t)$ in terms of $x(t)$

$t \to 0 \Rightarrow x \to 0 \Rightarrow k \to \text{const}$

$t \to \infty \Rightarrow \frac{kc}{\omega} \to \frac{x^2}{2L^2} \Rightarrow k \sim x^2$