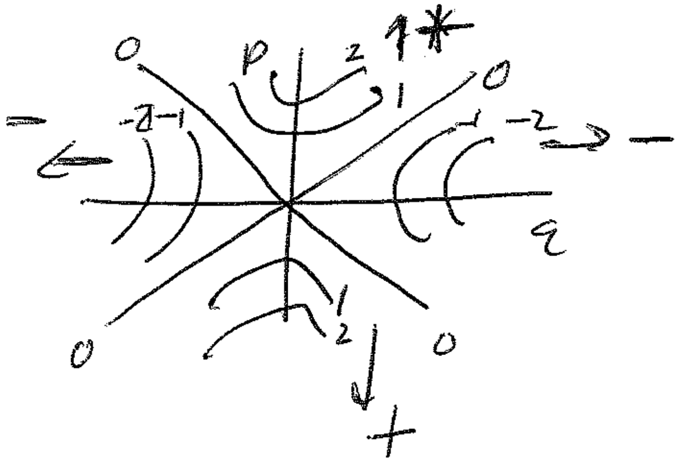
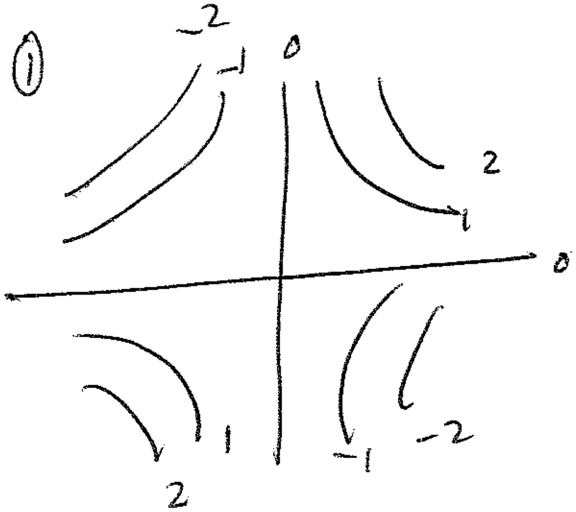
(8.1H) $H = \frac{p^2}{2} - \frac{q^2}{2}$



①



$Q' = pq$

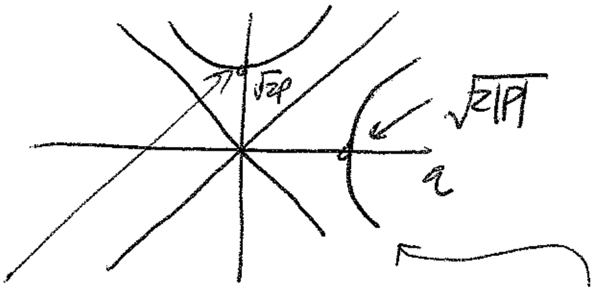
$\vec{D}H \cdot \vec{D}Q' \to (\partial_q, \partial_p)H \cdot (\partial_q, \partial_p)Q$

$= (-q, p) \cdot (p, q) = -pq + pq = 0$

② $\{P, Q\}$, CT $P = H$ $P = \frac{p^2}{2} - \frac{q^2}{2}$ ← links p, q, P

$p = \pm\sqrt{2P + q^2}$

$2P + q^2 > 0$



$P > 0 \Rightarrow 2P + q^2 > 0$, $P < 0 \Rightarrow q^2 > 2|P|$

Use F_2 $p = \partial F_2 / \partial q$, $Q = \partial F_2 / \partial P$

$\Rightarrow F_2 = \int^q dq' \, (2P + q^2)^{1/2}$ ← for all lines $p > 0$

$\Rightarrow Q = \int^q dq' \frac{1}{\sqrt{2P + q^2}}$

For $P>0$ $\quad Q = \int^{q/\sqrt{2P}} \frac{ds}{\sqrt{1+s^2}}$

$s = \sinh t$, $\quad 1+s^2 = \cosh^2 t$, $\quad ds = \cosh t\, dt$

$$\int^x \frac{ds}{\sqrt{1+s^2}} = \int^x dt = \sinh^{-1} x$$

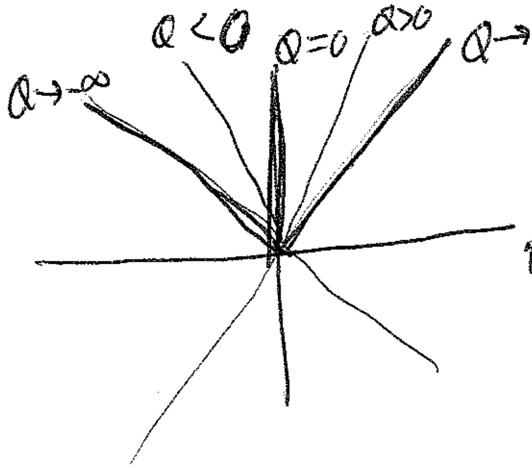
$$\Rightarrow \sinh Q = \frac{q}{\sqrt{2P}}, \quad p^2 = q^2 + 2P$$

$$= 2P\sinh^2 Q + 2P$$

$$= 2P\cosh^2 Q$$

$$\Rightarrow \boxed{\tanh Q = \frac{q}{p}}$$

$$\boxed{\begin{aligned} q &= \sqrt{2P}\,\sinh Q \\ p &= \sqrt{2P}\,\cosh Q \end{aligned}}$$



$p<0$, $P>0$ is similar

For $P<0$ $\quad Q = \int_{\sqrt{2|P|}}^{q} \frac{dq'}{\sqrt{q'^2 - 2|P|}}$

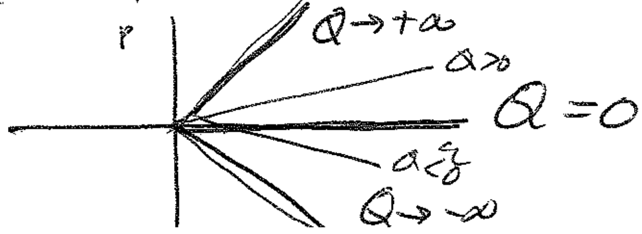
$$Q = \int_1^{q/\sqrt{2|P|}} \frac{ds}{\sqrt{s^2-1}}$$

$s = \cosh t$

$$\Rightarrow Q = \cosh^{-1}\left(\frac{q}{\sqrt{2|P|}}\right) - \cosh^{-1} 1$$

$$\Rightarrow \cosh Q = q/\sqrt{2|P|}, \quad \sinh Q = p/\sqrt{2|P|}$$

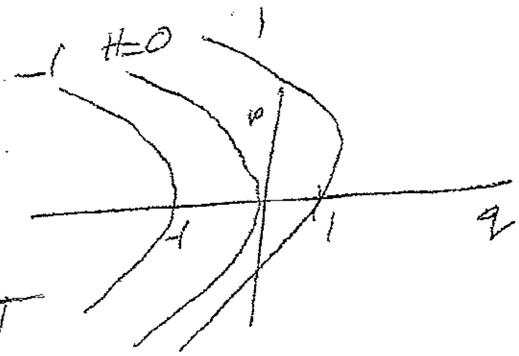
$$\boxed{\tanh Q = \frac{p}{q}}$$



similar on $q<0$ side

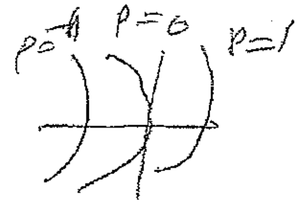
8.2H 1)

① $H = \frac{p^2}{2m} + mgz$



$m=1,\ g=1$

$\Rightarrow \boxed{H = \frac{1}{2}p^2 + q}$



② let $\boxed{P = H = \frac{1}{2}p^2 + q}$

Try $F_2(q,P)$. $p = \frac{\partial F_2}{\partial q}$

$\Rightarrow \frac{\partial F_2}{\partial q} = \sqrt{2(P-q)}\ ,\ p>0$

$\Rightarrow F_2 = \int^q dq' \sqrt{2(P-q')}\ ,\ p>0$

$\Rightarrow Q = \frac{\partial F_2}{\partial P} = \int^q \frac{dq'}{\sqrt{2(P-q')}}\ ,\ p>0$

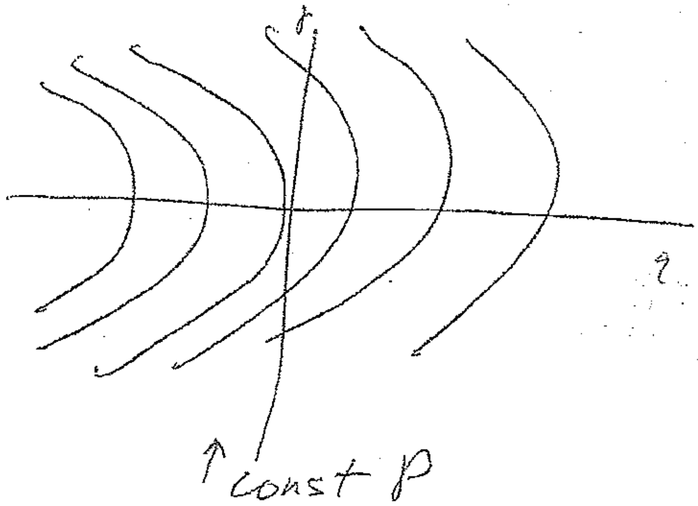
$\Rightarrow Q = -\sqrt{2}(P-q)^{1/2}\ ,\ p>0$

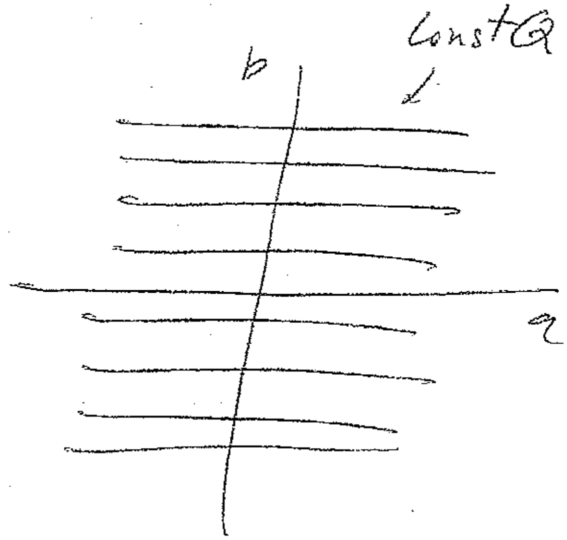
$\Rightarrow Q = -\sqrt{2}\left[\frac{1}{2}p^2 + \cancel{q} - \cancel{q}\right]^{1/2} = -\sqrt{2}\frac{p}{\sqrt{2}}$

$\Rightarrow \boxed{Q = -p}\ \ p>0$

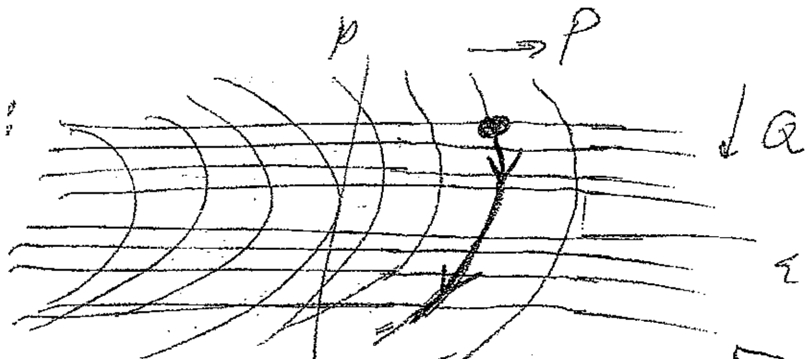
Thus $P=\frac{1}{2}p^2+q$, $Q=-p$

or $q=P-\frac{1}{2}Q^2$, $p=-Q$





$\Rightarrow$ Grid:



③ $H=P \Rightarrow \dot{P}=0, \dot{Q}=+1$

$\Rightarrow \boxed{Q=Q_0+t}$ and $\boxed{P=\text{const}}$

Thus, as t evolves, mass moves as shown

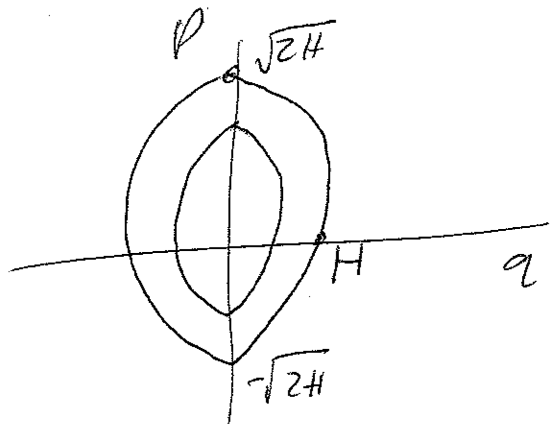
$p=-Q_0-t$; $q=P_0-\frac{1}{2}p^2$

8.3H

$$H = \frac{1}{2} m \dot{x}^2 + V'|x|$$

$V'/m = 1,\ m = 1\quad x \to q,\ \dot{x} \to p$

① $$\boxed{H = \frac{1}{2} p^2 + |q|}$$



② In class, we showed that for a periodic Q and P = P(H), we must have

$$P = \frac{1}{2\pi} \oint p\,dq = \frac{1}{2\pi} (\text{Area in } H)$$

$$\therefore P = \frac{1}{2\pi} 4 \int_0^{\sqrt{2H}} \left(H - \frac{1}{2} p^2\right) dp$$

$$= \frac{2}{\pi} \left[H\sqrt{2H} - \frac{1}{\cancel{2}} \frac{1}{3} \sqrt{2H}\, \cancel{2}H \right] = \frac{2}{\pi} H \sqrt{2H}\, \frac{2}{3}$$

$$\boxed{P = \frac{4\sqrt{2}}{3\pi} H^{3/2}}$$

$$\dot{Q} = \partial H / \partial P = \frac{1}{dP/dH}$$

$$= \frac{2\pi}{4\sqrt{2H}} = \omega$$

$T = 2\pi/\omega$

≠ const.

Find Q using F_2

$$p = \frac{\partial F_2}{\partial q}$$

$$\left.\frac{\partial F}{\partial q}\right)_P = \sqrt{2(H-q)} \qquad q>0$$

$$F = \int^q dq \sqrt{2(H-q)}$$

$$Q = \frac{\partial F}{\partial P} = \frac{dH}{dP} \int^q dq \frac{1}{\sqrt{2(H-q)}}$$

$$Q(q=0) = 0$$

$$\Rightarrow Q = \frac{dH}{dP} \int_0^q dq \frac{1}{\sqrt{2(H-q)}}$$

$$= \frac{dH}{dP} \left[\sqrt{2(H-q)}\right]_q^0$$

$$Q = \frac{dH}{dP} \left[\sqrt{2H} - \sqrt{2(H-q)}\right]$$

We want $Q(p=0) = \pi/2$

$$\Rightarrow \frac{\pi}{2} = \frac{dH}{dP} \left| \sqrt{2H} \right|_{p=0}$$

$$\Rightarrow \quad \frac{dP}{dH} = \frac{2}{\pi}\sqrt{2}\, H^{1/2}$$

$$P = \frac{2\sqrt{2}}{\pi} H^{3/2} \frac{2}{3}$$

$$\boxed{P = \frac{4\sqrt{2}}{3\pi} H^{3/2}} \quad P(H) \qquad \text{checks}$$

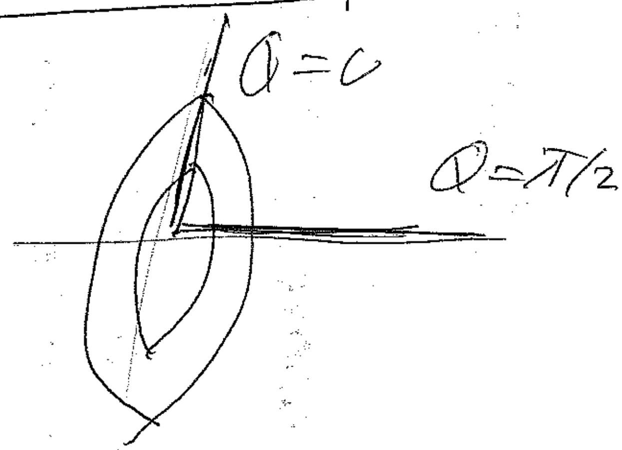
AA vars

$$Q = \frac{\left[\sqrt{2H} - \sqrt{2(H-q)}\right]}{\frac{2}{\pi}\sqrt{2H}}$$

$$\boxed{Q = \frac{\pi}{2}\left[1 - \sqrt{1 - q/H}\right]} \quad H(p,q)$$

$$Q(q=0) = 0$$

$$Q(p=0) = \frac{\pi}{2}$$



Periodic

Use $F_4(p,P)$

$$q = -\frac{\partial F}{\partial p} \qquad Q = \frac{\partial F}{\partial P} \quad , P(H)$$

$$H = \frac{1}{2}p^2 + q \,, \quad q > 0$$

$$\Rightarrow +\left.\frac{\partial F}{\partial p}\right)_P = \frac{1}{2}p^2 - H(P) \Rightarrow F = \frac{p^3}{6} - Hp + f(P)$$

$$\Rightarrow Q = \frac{\partial F}{\partial P} = -H'p + \frac{df}{dP} = H'\left(-p + \frac{df}{dH}\right)$$

we want $Q(q=0) = 0$

$$\Rightarrow 0 = H'\big|_{q=0}\left(\frac{df}{dH}\bigg|_{q=0} - p\right)$$

for this to work, the parenthesis must vanish for all p. Thus, $\left.\frac{df}{dH}\right|_{q=0} = p$ must be true. Study the dimensions of this condition: we have $\left.\frac{f}{H}\right|_{q=0} \sim p$

$$\Rightarrow \frac{f|_{q=0}}{H|_{q=0}} = \frac{f|_{q=0}}{p^2} \sim p \Rightarrow f|_{q=0} \sim p^3$$

But $p^3 \sim H^{3/2}\Big|_{q=0}$

$\therefore$ Try $\quad f = C H^{3/2}$

Then parentheisis $= \left(C \frac{3}{2} H^{1/2}\Big|_{q=0} - p \right) = 0$

$$\Rightarrow \frac{3}{2} C \left(\frac{1}{2}\right)^{1/2} p - p = 0$$

$$\Rightarrow \boxed{C = \sqrt{2}\,\frac{2}{3}} \quad \boxed{f = \frac{2\sqrt{2}}{3} H^{3/2}}$$

But we must also demand that

$$Q(p=0) = \pi/2$$

$$\Rightarrow H'\left(\frac{df}{dH} - p\right)_{p=0} = \pi/2$$

$$\Rightarrow \frac{\sqrt{2H}}{dP/dH}\Bigg|_{p=0} = \frac{\pi}{2} \Rightarrow \frac{dP}{dH}\Bigg|_{p=0} = \frac{2}{\pi}\sqrt{2H}\Bigg|_{p=0}$$

$$\Rightarrow P = \frac{2}{\pi}\sqrt{2}\, H^{3/2}\, \frac{2}{3} \Rightarrow \boxed{P(H) = \frac{4}{3\pi}\sqrt{2}\, H^{3/2}}$$

agrees with previ

$$\Rightarrow Q = \frac{\left(\frac{df}{dH} - p\right)}{dP/dH} = \frac{\left(\sqrt{2H} - p\right)}{\frac{2}{\pi}\sqrt{2H}}$$

$$Q = \frac{\pi}{2}\left(1 - \frac{p}{\sqrt{2H}}\right)$$

$$\boxed{Q = \frac{\pi}{2}\left(1 - \sqrt{1 - q/H}\right)}$$

$H(p,q)$

checks