8.1

a) $H = \dot{q}_i p_i - L$

$\rightarrow \quad L = \dot{q}_i p_i - H$

$$dL = d\dot{q}_i p_i + \dot{q}_i dp_i - \frac{\partial H}{\partial q_i} dq_i - \frac{\partial H}{\partial p_i} dp_i - \frac{\partial H}{\partial t} dt$$

$$= d\dot{q}_i p_i + \dot{p}_i dq_i - \frac{\partial H}{\partial t} dt$$

$$= \frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt$$

$$\rightarrow \quad \frac{\partial L}{\partial q_i} = \dot{p}_i \, , \quad \frac{\partial L}{\partial \dot{q}_i} = p_i \, , \quad \frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t}$$

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b) Sim.

$$dL' = -d\dot{p}_i q_i - \cancel{\dot{p}_i dq_i} + \cancel{\dot{p}_i dq_i} - \dot{q}_i dp_i - \frac{\partial H}{\partial t} dt$$

$$= \frac{\partial L'}{\partial p_i} dp_i + \frac{\partial L'}{\partial \dot{p}_i} d\dot{p}_i + \frac{\partial L'}{\partial t} dt$$

$$\rightarrow \quad \frac{\partial L'}{\partial p_i} = -\dot{q}_i \, , \quad \frac{\partial L'}{\partial \dot{p}_i} = -q_i \, , \quad \frac{\partial L'}{\partial t} = -\frac{\partial H}{\partial t}$$

8.27

a) $p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}\sin^2\omega t + \frac{m\omega}{2} q \sin 2\omega t$

$$H = p\dot{q} - L = \frac{m}{2}\left[\dot{q}^2 \sin^2\omega t - \omega^2 q^2\right]$$

$$= \frac{1}{2m\sin^2(\omega t)}\left(p - m\frac{\omega}{2} q \sin(2\omega t)\right)^2 - \frac{m\omega^2}{2} q^2$$

$\frac{\partial H}{\partial t} \neq 0 \Rightarrow$ H is not conserved.

$$\dot{Q} = \dot{q}\sin\omega t + \omega q \cos\omega t \; ; \; Q = q\sin\omega t$$

$$L = \frac{m}{2}\left(\frac{\dot{Q}}{\sin(\omega t)} - \omega Q \frac{\cos(\omega t)}{\sin^2(\omega t)}\right)^2 \sin^2(\omega t)$$

$$+ m\omega\left(\frac{\dot{Q}}{\sin(\omega t)} - \omega Q \frac{\cos(\omega t)}{\sin^2(\omega t)}\right) Q\cos(\omega t)$$

$$+ \frac{m\omega^2 Q^2}{2\sin^2(\omega t)}$$

$$= \frac{m}{2}\left\{\dot{Q}^2 + \omega^2 Q^2\right\} \Rightarrow p' = m\dot{Q}$$

$$\Rightarrow H = \frac{1}{2m}\left\{p'^2 - m^2\omega^2 Q^2\right\}$$

$\frac{\partial H}{\partial t} = 0 \Rightarrow$ H is conserved

Van der Pol Oscillator

$$\ddot{x} + x = 2\nu\dot{x}(1-x^2) \; ; \quad x(t)$$

$\nu \ll 1$ let $x(t) \sim 1$

Lowest order $\quad \ddot{x}_0 + x_0 = 0$

$$\Rightarrow \boxed{x_0 = A(\tau)\sin\xi} \, , \quad \xi = t + \phi(\tau)$$

1st order $\quad \ddot{x}_1 + x_1 + 2\dfrac{\partial^2 x_0}{\partial t\,\partial\tau} = 2\nu\dfrac{\partial x_0}{\partial t}\left(1 - x_0^2\right)$

$$\frac{\partial^2 x_0}{\partial t\,\partial\tau} = +\frac{\partial}{\partial\tau}(A\cos\xi) = +A_\tau\cos\xi - A\sin\xi\,\phi_\tau$$

$$\frac{\partial x_0}{\partial t} = A\cos\xi$$

$$\frac{\partial x_0}{\partial t}x_0^2 = A\cos\xi\,A^2\sin^2\xi$$
$$= \frac{1}{2}A^3\sin 2\xi\,\sin\xi$$

$$2\sin 2\xi\,\sin\xi = \cos\xi - \cos 3\xi$$

$$\ddot{x}_1 + x_1 + 2A_\tau\cos\xi - 2\phi_\tau A\sin\xi = +2\nu A\cos\xi$$
$$- \frac{2\nu}{2}\frac{A^3}{2}\left(\cos\xi - \cos 3\xi\right)$$

If we had not allowed $A(\tau), \phi(\tau) \Rightarrow$

$$\ddot{x}_1 + x_1 = 2\nu A\cos\xi - \frac{\varepsilon\nu A^3}{2}\cos\xi + \frac{\nu A^3}{2}\cos 3\xi$$

at resonance $\Rightarrow$ secular

With $A(\tau), \phi(\tau)$, we eliminate secularity by setting coeff of $\sin\xi$ + $\cos\xi = 0 \Rightarrow$

$$\frac{d\phi}{d\tau} = 0, \quad \frac{2dA}{d\tau} = 2\nu A - \frac{\nu A^3}{2}$$

$$\Rightarrow \boxed{\frac{dA}{d\tau} = \nu A\left(1 - \frac{A^2}{4}\right)} \qquad \frac{1}{2}\ln\left(\frac{x^2}{4-x^2}\right)$$

$$\frac{dA}{A(1-A^2/4)} = \nu\, d\tau \Rightarrow \int_{A_0}^{A} \frac{dA}{A(1-A^2/4)} = \nu\tau$$

$$\Rightarrow \frac{A^2}{A_0^2}\left(\frac{1-A_0^2/4}{1-A^2/4}\right) = e^{2\nu t} \qquad A^2<4,\ A_0^2<4$$

$$\boxed{A^2 = \frac{A_0^2 e^{2\nu t}}{\left[1 - \frac{A_0^2}{4}\left(1-e^{2\nu t}\right)\right]}} \qquad \xrightarrow{\nu t \to 0} A_0^2 e^{2\nu t} \qquad \xrightarrow{\nu t \to \infty} 4$$