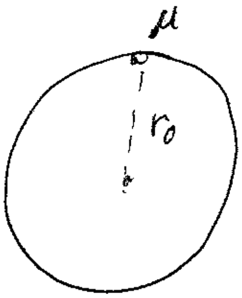
G3.11 Circular orbit $\Rightarrow \mu r_0 \omega^2 = k / r_0^2$

$$\Rightarrow \boxed{\omega^2 = \frac{k}{r_0^3 \mu}} \qquad \tau = \frac{2\pi}{\omega}$$



$$\mu \ddot{r} = \frac{-k}{r^2} \qquad l = 0$$

$$\times \dot{r} \Rightarrow \frac{1}{2}\mu \frac{\dot{r}^2}{2} = +\frac{k}{r} + E$$

$$\dot{r}(0) = 0 \Rightarrow \boxed{\frac{1}{2}\mu \dot{r}^2 = k\left(\frac{1}{r} - \frac{1}{r_0}\right)}$$

$$\dot{r} = -\sqrt{\frac{2k}{\mu}}\sqrt{\frac{1}{r} - \frac{1}{r_0}} \qquad \boxed{\dot{s} = -\alpha\sqrt{\frac{1}{s} - 1}} \qquad s = \frac{r}{r_0}$$

$$\alpha = \sqrt{\frac{2k}{\mu}}\frac{1}{r_0^{3/2}}$$

$$\int_1^0 \frac{ds}{\sqrt{\frac{1}{s} - 1}} = -\alpha T, \quad T = \text{time to drop}$$

LHS = dimensionless

$$\Rightarrow T = \frac{C}{\alpha} \Rightarrow \frac{T}{\tau} = \frac{C}{\tau\alpha} = \frac{C\omega}{2\pi\alpha}$$

$$\frac{\omega^2}{\alpha^2} = \frac{k}{r_0^3 \mu}\frac{\mu r_0^3}{2k} = \frac{1}{2} \Rightarrow \boxed{\frac{T}{\tau} = \frac{C}{2\pi}\frac{1}{\sqrt{2}}}$$

where $\int_0^1 \frac{ds}{\sqrt{1/s - 1}} = C \Rightarrow C = \frac{\pi}{2} \Rightarrow \boxed{\frac{T}{\tau} = \frac{1}{4\sqrt{2}}}$

G3020

$$\vec{F} = -mC\vec{r}, \Rightarrow \vec{F} = -mC\vec{\nabla}\frac{r^2}{2}$$

$$V = mCr^2/2$$

$$\text{Total } V = -\frac{k}{r} + \frac{mCr^2}{2}, \quad C \to 0.$$

$$F = -\frac{dV}{dr} = -\frac{k}{r^2} - mCr$$

(a) Circular equilibrium: $mr\omega^2 = \frac{k}{r^2} + mCr$

$$\boxed{l = mr^2\Omega} \quad ① \Rightarrow \boxed{\frac{l^2}{mr^3} = \frac{k}{r^2} + mCr} \quad ②$$

$$\Rightarrow r \Rightarrow \Omega$$

(b) $m\ddot{r} = mr\dot{\phi}^2 - \frac{k}{r^2} - mCr$

$$m\ddot{r} = \frac{l^2}{mr^3} - \frac{k}{r^2} - mCr \Rightarrow m\ddot{r} = \frac{-3l^2 r}{m^2 r^4} + \frac{2k}{mr^3} r - mCr$$

$$\Rightarrow \boxed{\omega^2 = \frac{3l^2}{m^2r^4} - \frac{2k}{mr^3} + C} \quad ③$$

where r is from ②

Solve for r from ②, insert into ① $\Rightarrow \Omega$

insert into ③ $\Rightarrow \omega$

Need $(\omega - \Omega)$.

expansion in C

$$\frac{l^2}{mr_0^3} = \frac{k}{r_0^2} \Rightarrow \boxed{r_0 = \frac{l^2}{km}}$$

$$\delta r \; \frac{-3\cancel{l^2}k}{r^3\cancel{m}\cancel{r^4}} = \frac{-2k\delta r + mCr}{r^3}$$

$$\Rightarrow \frac{-3k\delta r}{r^3} = mCr \Rightarrow \boxed{\delta r = -\frac{mCr_0^4}{3k}} \quad (4)$$

$$\boxed{r = r_0 + \delta r}$$

$$\Omega = \frac{l}{mr^2} \simeq \frac{l}{mr_0^2} - \frac{2l}{mr^3}\delta r = \frac{l}{mr_0^2}\left(1 - \frac{2\delta r}{r_0}\right)$$

$$\boxed{\Omega \simeq \frac{l}{mr_0^2}\left(1 - \frac{2\delta r}{r_0}\right)} \quad (5)$$

From (3)

$$\omega_0^2 = \frac{3l^2}{m^2r_0^4} - \frac{2\cancel{k}}{mr_0^3} = \frac{3l^2}{m^2r_0^4} - \frac{2l^2}{m^2r_0^{\cancel{3}4}}$$

$$\omega_0^2 = \frac{l^2}{m^2r_0^4} \qquad \boxed{\omega_0 = \frac{l}{mr_0^2} = \Omega_0}$$

$$2\omega\omega_1 = \frac{-12l^2}{m^2r^5}\delta r + \frac{6l^2}{m^2r^{\cancel{4}5}}\delta r + C$$

$$\boxed{2\omega\omega_1 = \frac{-6l^2}{m^2r_0^5}\delta r + C} \quad (6)$$

From (5) $\quad \Omega_1 = \frac{-l}{mr_0^2} \cdot \frac{2\delta r}{r_0} = -\omega_0 \frac{2\delta r}{r_0}$

From (6) $\quad \omega_1 = -\frac{3l^{\cancel{2}}\delta r}{m^{\cancel{2}} r_0^{\cancel{5}} \cancel{\omega}} + \frac{c}{2\omega}$

$$\omega_1 = \frac{-3l}{mr_0^2}\frac{\delta r}{r_0} + \frac{c}{2\omega_0}$$

$$\omega_1 = -3\omega_0 \frac{\delta r}{r_0} + \frac{c}{2\omega_0}$$

$$\cancel{\Omega} - \omega = \omega_{precession} = \Omega_1 - \omega_1$$

$$\frac{\Omega - \omega}{\omega_0} = -\cancel{\omega_0}\frac{2\delta r}{r_0} + 3\cancel{\omega_0}\frac{\delta r}{r_0} - \frac{c}{2\omega_0^2}$$

$$\frac{\delta r}{r_0} = -\frac{mCr_0^3}{3k} = -\frac{\cancel{m^2} C \cancel{r_0} \cancel{r_0^3}}{3\cancel{l^2}\omega_0^2} = -\frac{c}{3\omega_0^2}$$

$$\therefore \frac{\Omega - \omega}{\omega_0} = \frac{-3c}{3\omega_0^2} - \frac{c}{2\omega_0^2} = -\frac{3}{2}\frac{c}{\omega_0^2}$$

$$\boxed{\frac{\Omega - \omega}{\omega_0} \simeq -\frac{3}{2}\frac{C}{\omega_0^2}}$$

$$\omega_0 = \frac{l}{mr_0^2} = \frac{l\, k^2 m^2}{m\, l^4}$$

$$\boxed{\omega_0 = \frac{k^2 m}{l^3}}$$

$\omega = \Omega + |\text{correction}|$

so $\omega > \Omega \Rightarrow$ precess opposite to orbit direction

5.1Q

$$m\ddot{r} = mr\dot{\phi}^2 + F(r)$$

$$mr^2\dot{\phi} = l \qquad F(r) = -\frac{k}{r^2} e^{-r/a}$$

$$\ddot{r} = \frac{l^2}{mr^3} - \frac{k}{r^2} e^{-r/a}$$

$$\boxed{-V'_{eff} = \frac{l^2}{mr^3} - \frac{k}{r^2} e^{-r/a}}$$

$$\boxed{+V_{eff} = +\frac{l^2}{2mr^2} + \int_\infty^r dr \frac{k}{r^2} e^{-r/a}} , \quad V(r\to\infty) \to 0$$

Circular motion $\Rightarrow V'_{eff} = 0$

$$\Rightarrow \boxed{\frac{l^2}{mr_0} = k e^{-r_0/a}}$$

stable for $V''_{eff} > 0$

$$-V''_{eff} = \frac{-3l^2}{mr^4} + \frac{2k}{r^3} e^{-r/a} + \frac{k}{ar^2} e^{-r/a}$$

$$\cancel{-V''_{eff}\Big|_{r_0} = \frac{-3k e^{-r_0/a}}{r_0^3} + \frac{2k e^{-r_0/a}}{r_0}}$$

$$-V''_{eff} = \frac{-3l^2}{mr_0^4} + \frac{k}{r_0^3} e^{-r_0/a}\left[2 + \frac{r_0}{a}\right]$$

$$= -\frac{3l^2}{\ } + \frac{l^2}{mr_0^4}\left(2 + r_0/a\right) < 0 \text{ for stable}$$

$$\Rightarrow \quad 3 > 2 + \frac{r_0}{a}$$

$$\Rightarrow \quad \boxed{r_0 < a} \quad \text{Stable} \qquad r_0 = R$$

$$\ddot{r} = F_{eff} = -V'_{eff}(r)$$

$$V'_{eff}(r) \simeq \underbrace{V'_{eff}(r_0)}_{=0} + V''_{eff}(r_0)\,\tilde{r}$$

$$\Rightarrow m\ddot{\tilde{r}} = -V''_{eff}(r_0)\,\tilde{r}$$

$$\Rightarrow \boxed{\omega^2 = \frac{l^2}{m^2 r_0^4}\left(1 - \frac{r_0}{a}\right)} > 0 \quad \text{for } r_0 < a$$

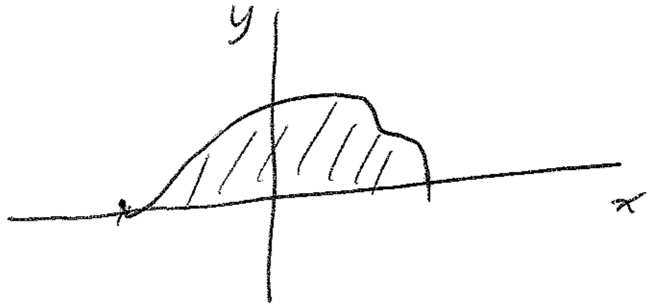
frequency of small osc

5.1H

Maximize Area holding length fixed

$$A = \int_{-1}^{1} dx\, y(x)$$

$$L = \int_{-1}^{1} dx \sqrt{1+y'^2}$$



let $\delta(A - \lambda L) = 0$

$$\Rightarrow \int_{-1}^{1} dx \left[\delta y - \frac{\lambda y'}{\sqrt{1+y'^2}} \delta y' \right] = 0$$

$$\Rightarrow \int_{-1}^{1} dx \left[1 + \frac{d}{dx} \frac{\lambda y'}{\sqrt{1+y'^2}} \right] \delta y = 0$$

$$\Rightarrow \boxed{\frac{d}{dx}\left(\frac{\lambda y'}{\sqrt{1+y'^2}} \right) = -1}$$

$$\frac{\lambda y'}{\sqrt{1+y'^2}} = c - x \xrightarrow{y'(0)=0} c = 0$$

$$\Rightarrow \lambda^2 y'^2 = x^2(1+y'^2)$$

$$(\lambda^2 - x^2) y'^2 = x^2, \quad y' = \frac{-x}{\sqrt{\lambda^2-x^2}} = \frac{d}{dx}\sqrt{\lambda^2-x^2}$$

$$y = \sqrt{\lambda^2 - x^2} \cancel{+} - c \quad \Rightarrow (y+c)^2 = \lambda^2 - x^2$$

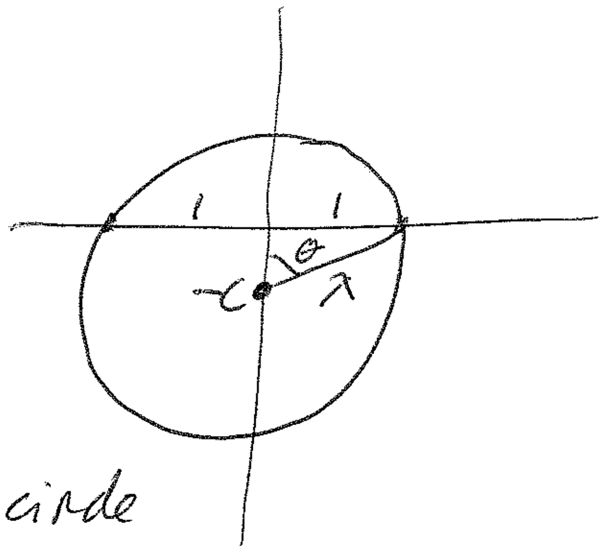
$\boxed{(y+c)^2 + x^2 = \lambda^2}$ circle @ $(x,y) = (0,-c)$

~~$y(1) = 0 \Rightarrow (1-c)^2 = \lambda^2, \quad y(-1) = 0 \Rightarrow (1+c)^2 = \lambda^2$~~

Thus,

$$\boxed{(y+c)^2 + x^2 = \lambda^2}$$

$$\boxed{c = \sqrt{\lambda^2 - 1}}$$

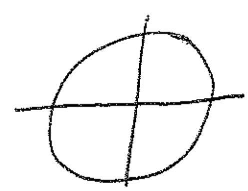


Shape is part of a circle defined as above.

<u>Note</u> L = arc length $= 2\lambda\theta$, and ~~tan θ~~ $\sin\theta = \frac{1}{\lambda}$

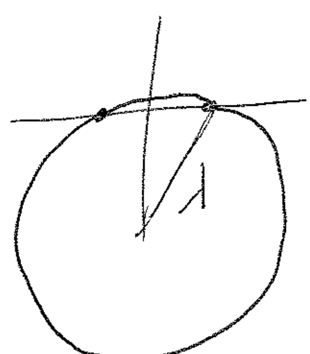
$$\Rightarrow \boxed{\sin\left(\frac{L}{2\lambda}\right) = \frac{1}{\lambda}} \leftarrow \text{determines } \lambda$$

$$L = \pi \Rightarrow \sin\left(\frac{\pi}{2\lambda}\right) = \frac{1}{\lambda} \Rightarrow \boxed{\lambda = 1}$$



$$L \to 2 \Rightarrow \sin\left(\frac{1}{\lambda}\right) \to \frac{1}{\lambda}$$

$$\Rightarrow \lambda \to \infty$$



$L \to \infty \Rightarrow$ other way, by symmetry