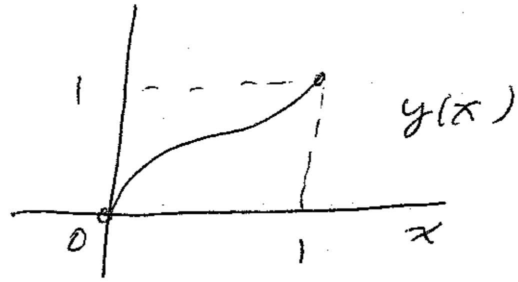
G23



$$l = \int_{x=0}^{x=1} ds$$

$$ds^2 = dx^2 + dy^2 = dx^2(1+y'^2)$$

$$\boxed{l = \int_0^1 dx\sqrt{1+y'^2}} \qquad \begin{matrix} y(0)=0 \\ y(1)=0 \end{matrix}$$

$$\delta l = 0 = \int_0^1 dx \frac{\frac{1}{2}\, 2y'\,\delta y'}{\sqrt{1+y'^2}} = 0$$

$$\Rightarrow -\int_0^1 dx\, \delta y \frac{d}{dx}\left[\frac{y'}{\sqrt{1+y'^2}}\right] = 0 \quad \forall\, \delta y$$

$$\Rightarrow \frac{y'}{\sqrt{1+y'^2}} = \text{const} = C$$

$\Rightarrow y' = D = \text{const}$ is a solution

$$\Rightarrow \boxed{y = x} \qquad \begin{matrix} y(0)=0 \\ y(1)=1 \end{matrix} \quad \Rightarrow \text{straight line}$$

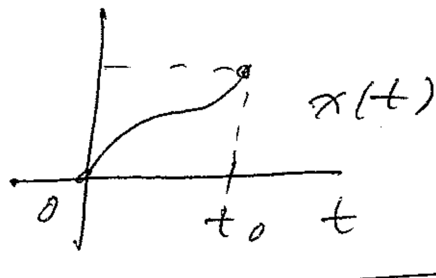
G2.5 $\qquad V(x) = -Fx\,,\ F = \text{const}$

$$L = \frac{1}{2} m \dot{x}^2 + Fx$$

$$x(t)\ ;\ x(0) = 0\,,\ x(t_0) = a$$

$x(t)$ must be such that

$$\delta S = \int_0^{t_0} dt\, L = 0$$



We are asked to try $\boxed{x(t) = A + Bt + Ct^2}$ and minimize S over all possible $\{A, B, C\}$.

Note: $\quad x(0) = 0 \Rightarrow A = 0$

$$x(t_0) = a \Rightarrow Bt_0 + Ct_0^2 = a$$

Thus, $B + C$ are not independent,

ie, $\boxed{\delta B + \delta C t_0 = 0}$ (*)

Now $\quad S = \int_0^{t_0} dt \left[\frac{1}{2}(B + 2Ct)^2 + \frac{F}{m}(Bt + Ct^2) \right]$

$$\Rightarrow \delta S = \int_0^{t_0} dt \left[(B + 2Ct)(\delta B + 2t\delta C) + \frac{F}{m}(\delta B t + \delta C t^2) \right]$$

$$= \int_0^{t_0} dt \left[(B + 2Ct)(-t_0 + 2t)\delta C + \frac{F}{m}(-t_0 + t)t\,\delta C \right]$$

$$= \int_0^{t_0} dt \left[-Bt_0 + 2Bt - 2Ctt_0 + 4Ct^2 - \frac{F}{m}t_0 t + \frac{F}{m}t^2 \right] \delta C$$

$$\Rightarrow 0 = \int_0^{t_0} dt \left[-\cancel{Bt_0^2} + \cancel{Bt_0^2} - 2Ct_0^3 + \frac{4}{3}Ct_0^3 - \frac{Ft_0^3}{m2} + \frac{Ft_0^3}{m3} \right] \delta C$$

$\forall \delta C \quad \Rightarrow C = \frac{F}{2m}\,,\ B = (a - \frac{F}{m}t_0^2/2)/t_0\,.$ Checks E-L soln.

2.12

let $L = L(q_k, \dot{q}_k, \ddot{q}_k, t)$

let $\delta L = 0 = \int_{t_1}^{t_2} dt\, L$

$$\Rightarrow \quad 0 = \int dt \left[\frac{\partial L}{\partial q_k} \delta q_k + \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k + \frac{\partial L}{\partial \ddot{q}_k} \delta \ddot{q}_k \right]$$

$$= \int dt \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) + \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_k} \right) \right] \delta q_k$$

$$\Rightarrow \boxed{\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}_k} \right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k} \right) + \left(\frac{\partial L}{\partial q_k} \right) = 0}$$

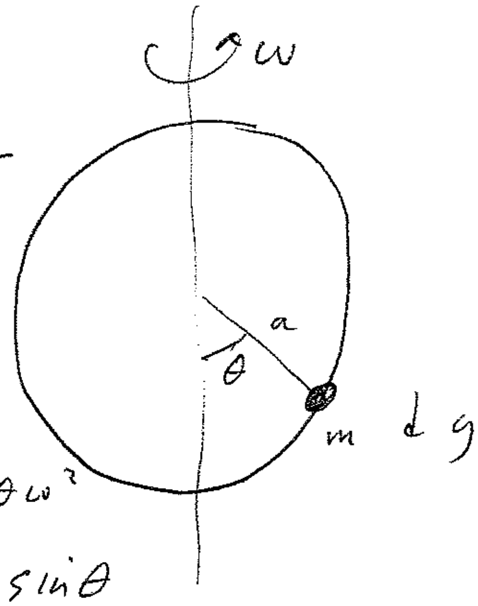
Suppose $L = -\frac{m}{2} q \ddot{q} - \frac{k}{2} q^2$

$$\partial L / \partial \ddot{q} = -\frac{mq}{2}, \quad \frac{\partial L}{\partial q} = -\frac{m}{2} \ddot{q} - kq$$

$$\Rightarrow -\frac{m\ddot{q}}{2} - \frac{m}{2} \ddot{q} - kq = 0$$

$$\Rightarrow \boxed{m\ddot{q} = -kq} \quad \text{H.O.}$$

G2.18



$$T = \frac{1}{2} m a^2 \dot\theta^2 + \frac{1}{2} m a^2 \sin^2\theta\, \omega^2$$

$$V = -mg\cos\theta\, a$$

$$\frac{\partial L}{\partial \dot\theta} = m a^2 \dot\theta, \quad \frac{\partial L}{\partial \theta} = m a^2 \sin\theta\cos\theta\,\omega^2 - mga\sin\theta$$

$$\Rightarrow \; m a^2 \ddot\theta = m a^2 \omega^2 \sin\theta\cos\theta - mga\sin\theta$$

$$\Rightarrow \; \boxed{\ddot\theta = \left(\omega^2\cos\theta - \frac{g}{a}\right)\sin\theta}$$

equilibria $(\ddot\theta = 0) \Rightarrow \theta = 0$, or $\cos\theta = \frac{g}{a\omega^2}$

2^{nd} are possible $\Leftrightarrow \boxed{\frac{g}{a} < \omega^2}$

Energy function $h = \dot\theta \frac{\partial L}{\partial \dot\theta} - L$ = const since $\partial_t L = 0$

$$\Rightarrow \; h = \dot\theta^2 m a^2 - \frac{1}{2} m a^2 \dot\theta^2 - \frac{1}{2} m a^2 \sin^2\theta\,\omega^2 - mga\cos\theta$$

$$\boxed{\frac{h}{ma^2} = \frac{1}{2}\dot\theta^2 - \frac{\omega^2 \sin^2\theta}{2} - \frac{g}{a}\cos\theta}$$

$$V_{eff} = -\frac{\omega^2}{2}\sin^2\theta - \frac{g}{a}\cos\theta$$

V'_{eff} = as above

$$V''_{eff} = -\omega^2\cos 2\theta + \frac{g}{a}\cos\theta$$

$\theta = 0 \Rightarrow V''_{eff} = -\omega^2 + \frac{g}{a}$, stable only if $\omega^2 < \frac{g}{a}$

G 2.24   H.O. $L = \frac{m}{2}\dot{x}^2 - \frac{kx^2}{2}$

Assume solns periodic in $\frac{2\pi}{\omega}$

$$\Rightarrow \boxed{x(t) = \sum_{l=0} a_l \cos(l\omega t)}$$

ω not known, all a_l

$t \to t + 2\pi/\omega \Rightarrow l\omega t \to l\omega t + 2\pi l$
$\Rightarrow$ same $x(t)$

If $x(t)$ is a solution, then S must be an extremum. $S = S(a_l, \omega, k/m)$

$\Rightarrow \delta S = 0 \quad \forall \delta a_l$

$$\frac{S}{m} = \int_0^{2\pi/\omega} dt \left[\frac{1}{2}\dot{x}^2 - \frac{1}{2}\frac{k}{m}x^2 \right]$$

$$\dot{x} = -\sum (+a_l l\omega)\cos(l\omega t)$$

$$\dot{x}^2 - \frac{k}{m}x^2 = \sum_l (a_l l\omega)\cos(l\omega t) \sum_{l'} (a_{l'} l'\omega)\cos(l'\omega t)$$
$$- \frac{k}{m}\sum_l a_l \cos(l\omega t) \sum_{l'} a_{l'} \cos(l'\omega t)$$

only $l = l'$ survives and $\cos^2(l\omega t)$ averages to $\frac{1}{2}\frac{2\pi}{\omega}$.

$\Rightarrow \frac{S}{m} = \sum_l \frac{1}{2} a_l^2 [l^2\omega^2 - k/m]\frac{2\pi}{\omega}$. $\delta S = 0 \ \forall \delta a_l$

$\Rightarrow a_l(l^2\omega^2 - k/m) = 0$. If $l\omega \neq \sqrt{k/m} \Rightarrow$ all $a_l = 0$

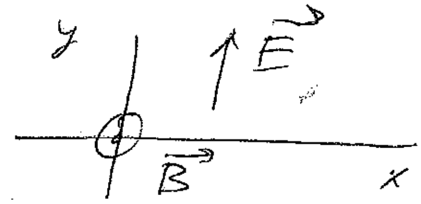
$\Rightarrow$ trivial. If $l_0\omega = \sqrt{k/m} \Rightarrow a_{l_0} \neq 0$, other $a_{l \neq l_0} = 0$.

$$\Rightarrow \boxed{x(t) = a_{l_0}\cos\left(\sqrt{\tfrac{k}{m}}\, t\right)}$$

Q1/H

$\vec{r}(0) = 0, \dot{\vec{r}}(0) = 0$

$\vec{E} = E\hat{y}, \quad \vec{B} = B\hat{z}.$



① $\vec{B} = \vec{\partial} \times \vec{A}$

$\vec{A} = \hat{y} A_y \Rightarrow B = \partial_x A_y \Rightarrow \boxed{A_y = Bx}$

$\vec{E} = -\vec{\partial}\varphi - \partial_t \vec{A} = -\vec{\partial}\varphi = \hat{y} E$

$\Rightarrow \boxed{\varphi = -Ey}$

② $L = T - U, \quad U = q(\varphi - \vec{v} \cdot \vec{A})$

$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

$U = -q(Ey + \dot{y} B x)$

③ $\partial L / \partial \dot{x} = m\dot{x}, \quad \partial L/\partial x = +q \dot{y} B$

$\partial L / \partial \dot{y} = m\dot{y} + qBx, \quad \partial L/\partial y = qE$

$\Rightarrow m\ddot{x} = qB\dot{y}, \quad m\ddot{y} + qB\dot{x} = qE$

$\Rightarrow \ddot{x} = \Omega \dot{y}, \quad \ddot{y} + \Omega \dot{x} = \Omega (E/B), \quad \Omega = qB/m$

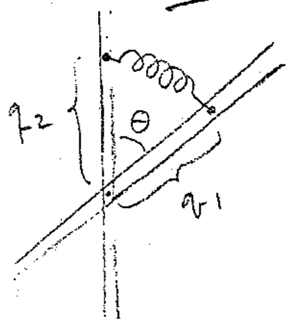
④ $\dot{x} = \Omega y$, using $t=0$ conditions.

$\Rightarrow \ddot{y} + \Omega^2 y = \Omega(E/B) \Rightarrow y_p = (E/B), \; y_h \propto \begin{Bmatrix} \cos\Omega t \\ \sin\Omega t \end{Bmatrix}$

$\Rightarrow \boxed{y(t) = (E/\Omega B)[1 - \cos\Omega t]} \quad \dot{x} = \Omega y \Rightarrow x \overset{\Omega}{=} (E/B)\left[t - \frac{\sin\Omega t}{\Omega}\right]$

$\boxed{x(t) = (E/\Omega B)[\Omega t - \sin\Omega t]}$

4.10



$$T = \frac{1}{2} m (\dot{q}_1^2 + \dot{q}_2^2)$$

$$V = \frac{1}{2} k (q_1 - q_2)^2 = \frac{1}{2} k (q_1^2 + q_2^2 - 2 q_1 q_2 \cos\theta)$$

$$L = T - V$$

$$\left.\begin{aligned} m\ddot{q}_1 &= -k(q_1 - q_2\cos\theta) \\ m\ddot{q}_2 &= -k(q_2 - q_1\cos\theta) \end{aligned}\right\} \text{ Eq. O' Motion}$$

c) $\theta = 0 \rightarrow$

$$m\ddot{q}_1 = -k(q_1 - q_2)$$
$$m\ddot{q}_2 = -k(q_2 - q_1)$$

Eq's for 2 masses joined by a spring in 1D.

$\theta = \frac{\pi}{2} \rightarrow$

$$m\ddot{q}_1 = -kq_1$$
$$m\ddot{q}_2 = -kq_2$$

motion decouples in orthogonal directions

d) $q_1 = Ae^{-i\omega t}$, $q_2 = Be^{-i\omega t}$, $\omega_0^2 \equiv \frac{k}{m}$

$$\rightarrow \begin{aligned} -\omega^2 A + \omega_0^2 (A - B\cos\theta) &= 0 \\ -\omega^2 B + \omega_0^2 (B - A\cos\theta) &= 0 \end{aligned}$$

$$\det \begin{vmatrix} \omega_0^2 - \omega^2 & -\omega_0^2 \cos\theta \\ -\omega_0^2\cos\theta & \omega_0^2 - \omega^2 \end{vmatrix} = [(\omega_0^2 - \omega^2) - \omega_0^2\cos\theta] \times [(\omega_0^2 - \omega^2) + \omega_0^2\cos\theta] = 0$$

$$\rightarrow \omega_\pm^2 = \omega_0^2 (1 \pm \cos\theta) \qquad \frac{A}{B} = \mp 1$$

$$\rightarrow \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left(a e^{i\omega_+ t} + b e^{-i\omega_+ t} \right) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \left(c e^{i\omega_- t} + d e^{-i\omega_- t} \right)$$