GPS 1.8 Lagrange equations unchanged under $L \to L + \frac{dF}{dt}$

$$L(q_j, \dot{q}_j, t) \quad , \quad F(q_j, t)$$

Consider $L' = L + dF/dt$

$$\Rightarrow L' = L + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q_k}\dot{q}_k$$

$$\frac{\partial L'}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{q}_j} + \frac{\partial F}{\partial q_j}$$

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{d}{dt}\left(\frac{\partial F}{\partial q_j}\right) \quad A$$

$$\frac{\partial L'}{\partial q_j} = \frac{\partial L}{\partial q_j} + \frac{\partial}{\partial q_j}\left(\frac{dF}{dt}\right) \quad B$$

$$A = \frac{\partial^2 F}{\partial t \partial q_j} + \frac{\partial^2 F}{\partial q_j \partial q_k}\dot{q}_k$$

$$\therefore A = B$$

$$B = \frac{\partial^2 F}{\partial t \partial q_j} + \frac{\partial^2 F}{\partial q_k \partial q_j}\dot{q}_k$$

SUBTRACTING $\Rightarrow$ $\boxed{\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) - \left(\frac{\partial L'}{\partial q_j}\right) = 0}$ if $L =$ Lagrangian

GPS 1.9 $\quad$ c = 1 units $\quad$ Eqns of Motion unchanged under EM gauge transfmn

$$\vec{A} \to \vec{A} + \partial\psi/\partial\vec{r}$$

$$\varphi \to \varphi - \partial\psi/\partial t$$

$$L = \frac{1}{2} m v^2 - q(\varphi - \vec{v}\cdot\vec{A})$$

$$L' = \frac{1}{2} m v^2 - q(\varphi - \vec{v}\cdot\vec{A}) + q\left(\frac{\partial\psi}{\partial t} + \vec{v}\cdot\frac{\partial\psi}{\partial\vec{r}}\right)$$

But $\displaystyle \frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \vec{v}\cdot\frac{\partial\psi}{\partial\vec{r}}$

$$\Rightarrow L' = L + q\frac{d\psi}{dt}$$

From GPS 1.8, equations of motion are unchanged

GPS 1.10 L equs "covariant" under point transfmns

$$q_k = q_k(s_\ell, t) \;,\quad L = L(q_k, \dot{q}_k, t)$$

point transfmn

$$\dot{q}_k = \frac{\partial q_k}{\partial s_\ell}\dot{s}_\ell + \frac{\partial q_k}{\partial t}$$

$$\text{(B)} = \left.\frac{\partial L}{\partial s_\ell}\right)_{\dot{s}_\ell, t} = \frac{\partial q_k}{\partial s_\ell}\frac{\partial L}{\partial q_k} + \frac{\partial \dot{q}_k}{\partial s_\ell}\frac{\partial L}{\partial \dot{q}_k}$$

$$\left.\frac{\partial L}{\partial \dot{s}_\ell}\right)_{s_\ell, t} = \cancelto{0}{\frac{\partial q_k}{\partial \dot{s}_\ell}}\frac{\partial L}{\partial q_k} + \frac{\partial \dot{q}_k}{\partial \dot{s}_\ell}\frac{\partial L}{\partial \dot{q}_k} = \frac{\partial q_k}{\partial s_\ell}\frac{\partial L}{\partial \dot{q}_k}$$

$$\text{(A)} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_\ell}\right) = \frac{\partial q_k}{\partial s_\ell}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\partial L}{\partial \dot{q}_k}\frac{d}{dt}\left(\frac{\partial q_k}{\partial s_\ell}\right)$$

$$\text{(A)} - \text{(B)} = \frac{\partial q_k}{\partial s_\ell}\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k}\right] + \frac{\partial L}{\partial \dot{q}_k}\underbrace{\left[\frac{d}{dt}\left(\frac{\partial q_k}{\partial s_\ell}\right) - \frac{\partial \dot{q}_k}{\partial s_\ell}\right]}_{\text{(C)}},$$

But $\frac{d}{dt}\left(\frac{\partial q_k}{\partial s_\ell}\right) = \frac{\partial^2 q_k}{\partial s_\ell \partial s_m}\dot{s}_m + \frac{\partial^2 q_k}{\partial t \partial s_\ell}$; $\frac{\partial \dot{q}_k}{\partial s_\ell} = \frac{\partial^2 q_k}{\partial s_\ell \partial s_m}\dot{s}_m + \frac{\partial^2 q_k}{\partial s_\ell \partial t}$

$$\Rightarrow \text{(C)} = 0 \Rightarrow \text{(A)} - \text{(B)} = 0$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}_\ell}\right) - \left(\frac{\partial L}{\partial s_\ell}\right) = 0}$$

GPS 1.20

$$L = \frac{m^2 \dot{x}^4}{12} + m\dot{x}^2 V(x) - V^2(x)$$

$$\frac{\partial L}{\partial \dot{x}} = \frac{1}{3} m^2 \dot{x}^3 + 2m\dot{x}\, V(x)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m^2 \dot{x}^2 \ddot{x} + 2m\ddot{x} V + 2m\dot{x}^2 \frac{\partial V}{\partial x}$$

$$\frac{\partial L}{\partial x} = m\dot{x}^2 \frac{\partial V}{\partial x} - 2V\frac{\partial V}{\partial x}$$

Euler - Lagrange

$$\rightarrow \quad m^2 \dot{x}^2 \ddot{x} + 2m\ddot{x} V + m\dot{x}^2 \frac{\partial V}{\partial x} + 2V\frac{\partial V}{\partial x} = 0$$

$$\rightarrow \left(\frac{1}{2} m\dot{x}^2 + V\right)\left(m\ddot{x} + \frac{\partial V}{\partial x}\right) = 0$$

$$\frac{\partial L}{\partial t} = 0 \rightarrow \frac{1}{2} m\dot{x}^2 + V = \text{const.}$$

$$\rightarrow m\ddot{x} = -\frac{\partial V}{\partial x}$$

I'm not sure what adjective the G-man wants, but clearly different Lagrangians can lead to the same Eq's o' Motion.

3.1H Identities

(a) $$\frac{\partial}{\partial \vec{r}}(\vec{r}\cdot\vec{A})\Big|_k = \frac{\partial}{\partial x_k}(x_\ell A_\ell) = \delta_{\ell k} A_\ell = A_k$$
$$= \vec{A}\big|_k$$

more directly,
$$\frac{\partial}{\partial \vec{r}}(\vec{r}\cdot\vec{A}) = \left(\frac{\partial \vec{r}}{\partial \vec{r}}\right)\cdot\vec{A} = \overleftrightarrow{\mathbb{1}}\cdot\vec{A} = \vec{A}$$

(b) $$\frac{\partial}{\partial \vec{r}}\left(\frac{1}{2}r^2\right) = \frac{\partial}{\partial \vec{r}}\left(\frac{1}{2}\vec{r}\cdot\vec{r}\right) = \overleftrightarrow{\mathbb{1}}\cdot\vec{r} = \vec{r}$$

(c) $$\frac{d}{dt}\left(\frac{1}{2}r^2\right) = \frac{d}{dt}\left(\frac{1}{2}\vec{r}\cdot\vec{r}\right) = \frac{d\vec{r}}{dt}\cdot\vec{r} = \vec{v}\cdot\vec{r}$$

(d) $$\frac{d}{dt} f[\vec{r}(t), t] = \frac{d}{dt} f[x_k(t), t]$$
$$= \frac{dx_k}{dt}\frac{\partial f}{\partial x_k} + \frac{\partial f}{\partial t} = \vec{v}\cdot\frac{\partial f}{\partial \vec{r}} + \frac{\partial f}{\partial t}$$

3.2 H "Fit" force into L formulation

$$\vec{F} = Q\vec{v}\cdot[\vec{r}\vec{D} - \vec{D}\vec{r}]$$

No constraints $\Rightarrow \frac{d}{dt}(m\vec{v}) = \vec{F}$

$LHS = \frac{d}{dt}\left(\frac{\partial T}{\partial \vec{v}}\right)$, since $T = \frac{1}{2}mv^2$

$\frac{\partial T}{\partial \vec{v}} = m\vec{v}$

(see 3.1H)

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \vec{v}}\right) - \left(\frac{\partial T}{\partial \vec{r}}\right) = \vec{F}$$

Can we write $\vec{F}$ so it looks like LHS?

<u>Try each one</u> $\quad \vec{v}\cdot\vec{r} = \frac{d}{dt}\left(\frac{1}{2}\vec{r}\cdot\vec{r}\right)$?

What about $\vec{v}\cdot\vec{D}\vec{r} = \frac{\partial}{\partial \vec{r}}\left[\vec{v}\cdot\vec{D}\frac{r^2}{2}\right]$ acceptable form

What about $\frac{\partial}{\partial \vec{v}}[\quad]$

$$= \frac{\partial}{\partial \vec{v}}\left[\vec{v}\cdot\vec{D}\frac{r^2}{2}\right] = \vec{D}\frac{r^2}{2}$$

So, then, $\frac{d}{dt}\frac{\partial}{\partial \vec{v}}\left[\vec{v}\cdot\vec{D}\frac{r^2}{2}\right] = \vec{D}\frac{d}{dt}\frac{r^2}{2}$

$= \vec{D}\,\vec{v}\cdot\vec{r}$

So, $\vec{v}\cdot\vec{r}\vec{D} - \vec{v}\cdot\vec{D}\vec{r}$

$= \frac{d}{dt}\frac{\partial}{\partial \vec{v}}\left[\frac{1}{2}r^2\vec{v}\cdot\vec{D}\right] - \frac{\partial}{\partial \vec{r}}\left[\frac{1}{2}r^2\vec{v}\cdot\vec{D}\right]$

$\Rightarrow \boxed{\begin{aligned} L &= T - U \\ U &= Q\frac{r^2}{2}\vec{v}\cdot\vec{D}\end{aligned}}$ works

Systematically

$\vec{F} = Q\vec{v}\cdot[\vec{r}\vec{D} - \vec{D}\vec{r}] = -Q\vec{v}\times(\vec{r}\times\vec{D})$

Suppose $\vec{D}\times\vec{r} \equiv \vec{B}(\vec{r})$

$\Rightarrow \vec{F} = Q\vec{v}\times\vec{B}(\vec{r})$. But we know that Lorentz force "fits", provided we let $\vec{B} = \vec{\nabla}\times\vec{A}$. Try $\vec{D}\times\vec{r} = \vec{\nabla}\times\vec{A}$, let $\vec{A} \stackrel{?}{=} \vec{D}r^2/2 \Rightarrow \vec{\nabla}\times\vec{A} \stackrel{?}{=} \vec{\nabla}\frac{r^2}{2}\times\vec{D} = \vec{r}\times\vec{D}$. $\Rightarrow \boxed{\vec{A} = \vec{D}\frac{r^2}{2}}$