Problem 20.1H Kepler orbits

$$\ell = m r^2 \dot\phi$$

$$E = \frac{1}{2} m(\dot r^2 + r^2 \dot\phi^2) - k/r$$

$$\Rightarrow \frac{1}{2} m \dot r^2 = E + \frac{k}{r} - \frac{1}{2}\frac{\ell^2}{m r^2}$$

$$\frac{dr}{d\phi} = \frac{\dot r}{\dot\phi} \Rightarrow \frac{1}{2}\frac{\cancel{m}\ell^2}{m r^4}\left(\frac{dr}{d\phi}\right)^2 = E + \frac{k}{r} - \frac{1}{2}\frac{\ell^2}{m r^2}$$

$$\left(\frac{dr}{d\phi}\right) = \frac{dr}{du}\left(\frac{du}{d\phi}\right) = -\frac{1}{u^2}\left(\frac{du}{d\phi}\right) \quad , \; u \equiv \frac{1}{r}$$

$$\Rightarrow \frac{1}{2}\frac{\ell^2}{m}\left(\frac{du}{d\phi}\right)^2 = E + ku - \frac{1}{2}\frac{\ell^2}{m}u^2$$

$$\left(\frac{du}{d\phi}\right)^2 = \frac{2mE}{\ell^2} + \frac{2mk}{\ell^2}u - u^2$$

$$\int \frac{du}{\sqrt{\frac{2mE}{\ell^2} + \frac{2mk}{\ell^2}u - u^2}} = \int d\phi$$

agrees with Eq (3.50), Goldstein

$$\Rightarrow \boxed{\frac{1}{r} = \frac{mk}{\ell^2}\left(1 + \sqrt{1 + \frac{2E\ell^2}{mk^2}}\cos(\phi - \phi')\right)} \quad \text{Eq (3.55)}$$

2.2H L-R-L Vector

(a) $\vec{A} = \vec{p} \times \vec{L} - km\vec{r}/r$, for $V = -\frac{GMm}{r}$

$$\dot{\vec{A}} = \dot{\vec{p}} \times \vec{L} - km\frac{\vec{v}}{r} + \frac{km}{r^2}\frac{\vec{r}}{r}\dot{r}$$

use $\dot{\vec{p}} = -k\vec{r}/r^3$

$$\Rightarrow \dot{\vec{A}} = -\frac{k}{r^3}\underbrace{\vec{r}\times(\vec{r}\times\vec{p})}_{\vec{r}\cdot\vec{p}\,\vec{r} - r^2\vec{p}} - \frac{k\vec{p}}{r} + \frac{km\vec{r}\dot{r}}{r^3}$$

cancel

$$= -k\frac{\vec{r}\,\vec{r}\cdot\vec{p}}{r^3} + \frac{km\vec{r}}{r^3}\dot{r}$$

$$= 0$$

using $\vec{r}\cdot\vec{v} = r\dot{r}$

(b) $\dot{\vec{p}} = -k\vec{r}/r^3$, $\vec{r} \times \dot{\vec{p}} = (\vec{r}\times\vec{p})^{\circ} = 0$

$$\Rightarrow \vec{L} \times \dot{\vec{p}} = (\vec{L}\times\vec{p})^{\circ} = -\frac{k\vec{L}\times\vec{r}}{r^3}$$

$$RHS = -k\frac{(\vec{r}\times\vec{p})\times\vec{r}}{r^3} = -\frac{k\vec{p}}{r} + k\frac{\vec{r}\,\vec{r}\cdot\vec{p}}{r^3}$$

$$= -k\frac{\dot{\vec{r}}m}{r} + \frac{k\vec{r}\dot{r}m}{r^2} = -km\left(\frac{\vec{r}}{r}\right)^{\circ}$$

$$\Rightarrow d\vec{A}'/dt = 0 , \vec{A}' = \vec{L}\times\vec{p} + km\vec{r}/r$$

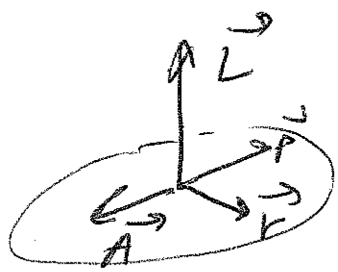
(c) Show that $\vec{A} \cdot \vec{L} = 0$ identically

$$\vec{A} = \vec{p} \times \vec{L} - km\vec{r}/r = \text{constant vector}$$
$$\vec{A} \cdot \vec{L} = -km\vec{r} \cdot \vec{L}/r = 0$$

So this component yields $0 = 0$.

Show that A^2 is made up from L and $\mathcal{E}$

$$A^2 = |\vec{p} \times \vec{L}|^2 + (km)^2 - 2km\frac{\vec{r}}{r} \cdot \vec{p} \times \vec{L}$$



$$|\vec{p} \times \vec{L}|^2 = p^2 L^2$$
$$\vec{r} \cdot \vec{p} \times \vec{L} = L^2$$

$$\therefore A^2 = p^2 L^2 + k^2 m^2 - 2km\frac{L^2}{r}$$

$$\frac{A^2}{2mL^2} = \frac{p^2}{2m} + \frac{k^2 m}{2L^2} - \frac{k}{r}$$

But $\mathcal{E} = \frac{p^2}{2m} - \frac{k}{r}$

$$\Rightarrow \boxed{\frac{A^2}{2mL^2} = \mathcal{E} + \frac{k^2 m}{2L^2}} \quad A = A(L, \mathcal{E})$$

(d) Take $\vec{r}$ at $\phi = 0$ to be parallel to $\vec{A}$

$\Rightarrow$ fix $\phi = 0$

$$\Rightarrow \quad \vec{A} \cdot \vec{r} = A r \cos\phi \;, \quad A \geq 0$$

$$= \vec{r} \cdot (\vec{p} \times \vec{L}) - kmr$$

$$= L^2 - kmr$$

$$\Rightarrow \quad \boxed{\frac{L^2}{r} = km + A\cos\phi} \qquad \begin{array}{l} A \geq 0 \\ r > 0 \end{array}$$

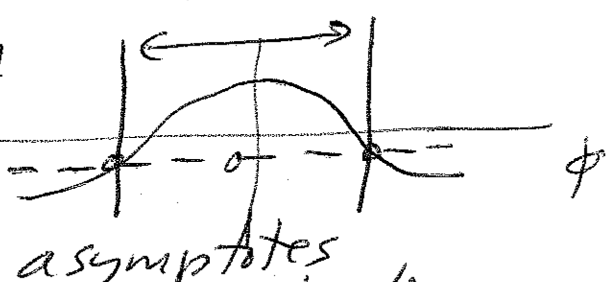
$\Rightarrow r(\phi)$

- $A = 0 \Rightarrow r = \text{const} \Rightarrow$ circle

- $A < km \Rightarrow$ ellipse since RHS always > 0 but wobbles

- $km < A \Rightarrow$ only certain ϕ allowed, ie, $\cos\phi \geq -km/A$

Note: as $\cos\phi \to -km/A$
$\Rightarrow$ RHS $\to 0 \Rightarrow r \to \infty$



$\Rightarrow$ hyperbola with asymptotes $\cos\phi_0 \to -km/A$

- $A \to \infty \Rightarrow \phi_0 = \pm\pi/2$

(2.3H) EM energy density

$$m_i \dot{\vec{v}}_i = q_i (\vec{E}_i + \vec{v}_i \times \vec{B}_i)$$

$$\vec{E}_i \to \vec{E}(\vec{r}_i, t), \text{ etc}$$

$$\vec{v}_i \cdot m_i \dot{\vec{v}}_i = \frac{d}{dt}(T_i), \quad T = \sum_i T_i$$

$$\Rightarrow \frac{dT}{dt} = \sum_i q_i \vec{v}_i \cdot \vec{E}(\vec{r}_i, t)$$

$$\text{RHS} \to \int d^3r\, \vec{J} \cdot \vec{E}$$

From ME, $\vec{\nabla} \times \vec{B} = \vec{J} + \partial_t \vec{E}$

$$\Rightarrow \text{RHS} = \int d^3r\, \vec{E} \cdot [\underset{①}{\vec{\nabla} \times \vec{B}} - \underset{②}{\partial_t \vec{E}}]$$

$$① = \int d^3r\, [-\vec{\nabla} \cdot (\vec{E} \times \vec{B}) + \vec{B} \cdot \vec{\nabla} \times \vec{E}]$$

$$= -\int d\vec{S} \cdot \vec{E} \times \vec{B} - \int d^3r\, \vec{B} \cdot \partial \vec{B} / \partial t$$

using $\vec{\nabla} \times \vec{E} = -\partial_t \vec{B}$

$$= -\int d\vec{S} \cdot \vec{E} \times \vec{B} - \int d^3r\, \partial_t (B^2/2)$$

$$= -\int d\vec{S} \cdot \vec{E} \times \vec{B} - \frac{d}{dt} \int d^3r\, \frac{1}{2} B^2$$

$$② = \int d^3r\, \vec{E}\cdot(-\partial_t \vec{E}) = -\frac{d}{dt}\int d^3r\, \frac{1}{2}E^2$$

$$\Rightarrow \boxed{\frac{d}{dt}\left[T + \int d^3r\, \frac{1}{2}(E^2+B^2)\right] = -\int d\vec{S}\cdot\vec{E}\times\vec{B}}$$

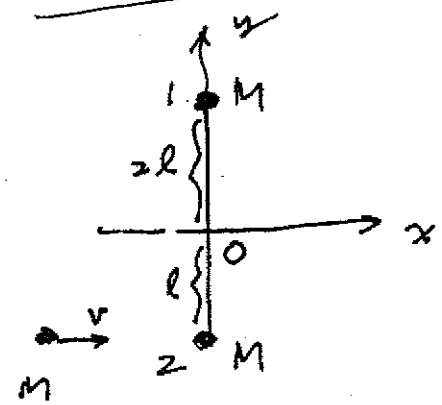
"Poynting Flux"

For closed box,

$$T + \left[\frac{1}{2}(E^2+B^2)\right] = \text{const}$$

assuming $\int_{box} d\vec{S}\cdot\vec{E}\times\vec{B} = 0$

2.1Q



a) $x_{cm} = (vt + 0 + 0)M/3M = \frac{vt}{3}$

$y_{cm} = \frac{2l - l - l}{3M} = 0$

x_{cm}, y_{cm} is the same before & after the collision

b) $\vec{L} = \vec{r} \times \vec{p} = -l\hat{j} \times m\vec{v}$ --- before

$= mlv\,\hat{k}$

same for after collision.

c) In CMS, we have pure rotation about CM.

$L = I\omega$, $I = M(2l)^2 + 2M \cdot l^2$

$= 6Ml^2$

$L = mlv$

$\therefore \omega = \frac{v}{6l}$

d) In lab frame. the motion is the CM motion plus rotation about CM.

$$\therefore \begin{cases} x_1 = \frac{vt}{3} - 2l \sin \omega t, & y_1 = 2l \cos(t\omega) \\ x_2 = \frac{vt}{3} + l \sin \omega t, & y_2 = -l \cos \omega t \end{cases}$$

$(\omega = v/6l)$