Phys601/F11/Problem Set 11 **Due Dec 5, 2011**

~~To be upgraded~~

11.1H Normal modes

A particle of mass m and charge q is connected to the origin by a spring of constant $m\omega_0^2$. Crossed **E** and **B** fields are externally applied. $\mathbf{E} = E_0\hat{\mathbf{x}}$ and $\mathbf{B} = B_0\hat{\mathbf{z}}$, where E_0 and B_0 are constants.

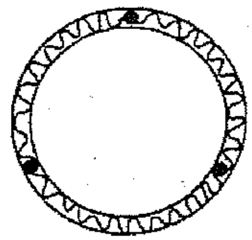
1. Write down the Lagrangian for the system assuming the motion is in the 2D x-y plane only.
2. Find a static equilibrium. What are the balancing forces for this?
3. Obtain the equations for small oscillations about this equilibrium.
4. Find the normal mode eigenfrequencies. Is the system stable?
5. What is the effect of E_0 on your normal modes.
6. What are the approximate eigenfrequencies for $B_0 \to 0$ and $B_0 \to \infty$. Keep corrections up to the first non-vanishing order. Comment.

11.1G Rotating Frames

Goldstein Ch4 Problem 4.24

11.1Q Normal modes

I.1



Three small balls of equal mass m and negligible radius a move without friction in a circular tunnel of radius $R \gg a$. There is no gravity. The balls are connected by springs, each of unstretched length length $\frac{2}{3}\pi R$ and spring constant k. At equilibrium the balls are in the positions shown in the figure above.

The total potential energy of the balls is

$$V = \frac{1}{2}kR^2\left((\phi_1 - \phi_2)^2 + (\phi_2 - \phi_3)^2 + (\phi_3 - \phi_1)^2\right),$$

where ϕ_i is the angular displacement of ball i from its equilibrium position. The balls are now perturbed from their equilibrium position.

(a) What are the eigenfrequencies of oscillation, and their degeneracies?

[9 points]

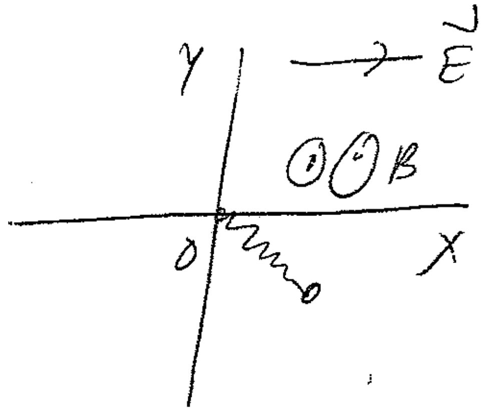
(Hint: The equation for eigenvalues, although cubic in principle, does not have the constant term. Thus the roots are easy to find.)

(b) Find and describe the eigenmodes corresponding to the frequencies you found in (a)?

[9 points]

(c) Note that the Lagrangian does not change if the same angle is added to the three coordinates ϕ_i. This suggests that in this problem, in addition to energy, there is another conserved quantity. What is this constant of the motion? [2 points]

1101H



$$\vec{E} = (E_0, 0, 0)$$

$$\vec{B} = (0, 0, B_0)$$

$$\vec{A} = [0, B_0 x, 0]$$

$$\partial_x A_y = B_z = B_0$$

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$V_E = -qE_0 x, \quad V_k = \frac{1}{2} m \omega_0^2 (x^2 + y^2 + z^2)$$

$$V_B = -qB_0 \dot{y} x$$

$$L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2} m \omega_0^2 (x^2 + y^2 + z^2) + qE_0 x + qB_0 \dot{y} x$$

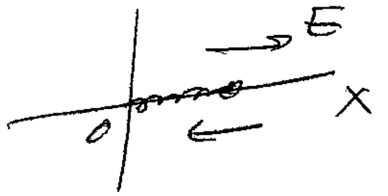
$\omega_0 = 1$ defines time; let $\frac{qB_0}{m} \equiv \Omega$;
let $qE_0/m = 1$ (defines length); $m = 1 \Rightarrow$ energy

$$\Rightarrow \boxed{L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}(x^2 + y^2 + z^2) + Ex + \Omega \dot{y} x}$$

where $E \to \frac{qE_0}{m}$

$$\Rightarrow \boxed{\ddot{x} = -x + E + \Omega \dot{y}, \quad \ddot{y} = -y - \Omega \dot{x}, \quad \ddot{z} = -z}$$

<u>Equilibrium</u> $\boxed{x_0 = E,\ y_0 = 0,\ z_0 = 0}$



stretched spring

let $x = E + \tilde{x},\ y = \tilde{y},\ z = \tilde{z}$

z decouples $\Rightarrow$ $\boxed{\omega^2 = 1 \text{ for } \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$

else, $-(\omega^2 - 1)\tilde{x} = -i\omega\Omega\tilde{y}$
$-(\omega^2 - 1)\tilde{y} = i\omega\Omega\tilde{x}$

$\Leftrightarrow$ $\boxed{(\omega^2 - 1)^2 = (\omega\Omega)^2}$

Note*: E has dropped out. So E_0 not relevant to eigenfrequencies

$\Rightarrow$ $\omega^2 - 1 = \pm\omega\Omega$

$\boxed{\omega^2 \mp \omega\Omega - 1 = 0}$ $\Rightarrow$ $\boxed{\omega = \dfrac{\pm\Omega \pm \sqrt{\Omega^2 + 4}}{2}}$

stable $\forall\,\Omega$

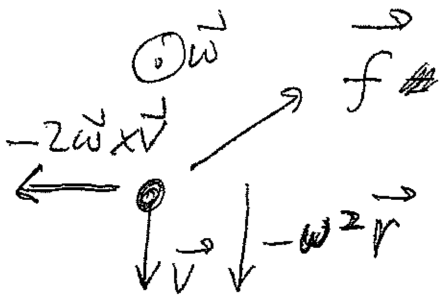
- * E_0 does not affect eigenfrequencies
- $B_0 \to 0 \Rightarrow \boxed{\omega \simeq \pm 1}$ just spring oscillations

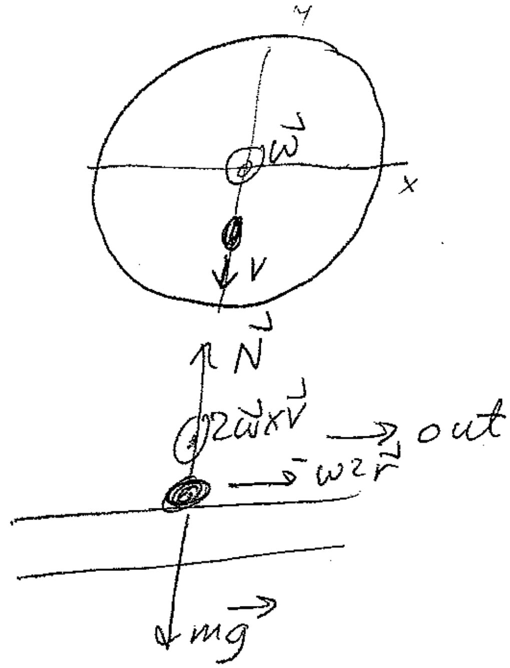
$B_0 \to \infty \Rightarrow \dfrac{\omega}{\Omega} \simeq \dfrac{\pm 1 \pm 1}{2} \longrightarrow 1, 0, -1, 0$

Circular motion

$0 \to \dfrac{1}{\Omega^2}$

GPS 4.24





$$\vec N = m\vec g$$

$$|\vec f| \le \mu|\vec N|$$

Need $|\vec f| \le \mu|\vec N| = \mu m g$

Force balance $\Rightarrow$ $\vec f = m\omega^2\vec r + m2\vec\omega\times\vec v$

$$\Rightarrow |\vec f|^2 = m^2(\omega^4 r^2 + 4\omega^2 v^2)$$

$$\Rightarrow \cancel{m^2}\omega^2(r^2\omega^2 + 4v^2) \le \mu^2\cancel{m^2}g^2$$

$\omega = 3$, $\mu = 0.3$, $v = 0.5\ \frac{cm}{s}$;

$$9(9r^2 + 4\cdot\tfrac{1}{4}) \le (0.3)^2(980)^2$$

Note

$|F_c| \ll \mu m g$

$$9r^2+1 \lesssim \frac{\cancel{9}}{10^2}\,\frac{10^6}{\cancel{9}} \simeq 10^4$$

$$9r^2 \lesssim 10^4,\ \boxed{r \lesssim \frac{10^2}{3} \simeq 33\,cm}$$

$\underline{\underline{11 \cdot 1\, a}}$

$$V = \frac{1}{2} k R^2 \left[(\phi_1 - \phi_2)^2 + (\phi_2 - \phi_3)^2 + (\phi_3 - \phi_1)^2 \right]$$

$$T = \frac{1}{2} m R^2 \left[\dot{\phi}_1^2 + \dot{\phi}_2^2 + \dot{\phi}_3^2 \right]$$

$$L = T - V \ ; \quad R = 1, \ k/m = 1$$

$$\partial L / \partial \dot{\phi}_1 = \dot{\phi}_1 \ , \quad \partial L / \partial \phi_1 = -(\phi_1 - \phi_2) + (\phi_3 - \phi_1)$$

$$\Longrightarrow \ \ddot{\phi}_1 = -(\phi_1 - \phi_2) + (\phi_3 - \phi_1)$$

$\ddot{\phi}_2 =$ cyclic permutation

$\ddot{\phi}_3 =$

$$\phi_1 \to \phi_1 e^{-i\omega t} \ \text{etc} \ \Longrightarrow$$

a)

$$\begin{bmatrix} (\omega^2 - 2) & 1 & 1 \\ 1 & (\omega^2 - 2) & 1 \\ 1 & 1 & (\omega^2 - 2) \end{bmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = 0$$

$$\omega^2 - 2 \equiv D$$

$$\Rightarrow D[D^2-1] + [1-D] + [1-D] = 0$$

$$\Rightarrow D(D-1)(D+1) = 2(D-1)$$

$$\Rightarrow D = 1 \quad \text{or} \quad D(D+1) = 2$$

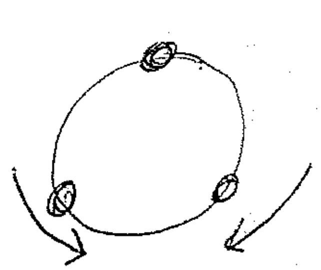
$$\Rightarrow D=1, \quad D=1, \quad D=-2$$

$$\Rightarrow \boxed{\omega^2 = 3, \quad \omega^2 = 3, \quad \omega^2 = 0}$$

degenerate — zero freq.

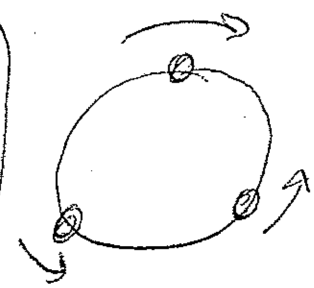
(b) Eigenmodes $D = 1 \Rightarrow \phi_1 + \phi_2 + \phi_3 = 0$

$$\Rightarrow \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$



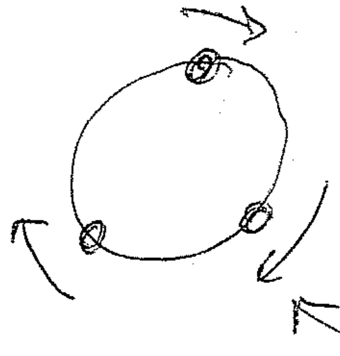
OR

$$\begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$



$$D = -2 \Rightarrow \phi_2 + \phi_3 = 2\phi_1 \Rightarrow \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\Rightarrow$



zero frequency

(c) Note that the CM angular momentum = const

$\Rightarrow (\phi_1 + \phi_2 + \phi_3)^{\cdot\cdot} = 0$

Check this with the original eqns.