1.1H

Covariance of Relativistic Newton Eqn

$$x' = \Gamma(x - Vt) \qquad \Gamma^2 = \frac{1}{1-V^2}$$

$$t' = \Gamma(t - Vx)$$

$$x'(t),\ x(t),\ t'(t)$$

$$\frac{dx'}{dt} = \Gamma(u - V)$$

$$\frac{dt'}{dt} = \Gamma(1 - Vu)$$

$$\Rightarrow \boxed{u' = \frac{u-V}{1-Vu}} \quad \text{— } 1 \qquad 1 - Vu \neq 0$$

$$\frac{du'}{dt} = \frac{a}{1-Vu} + \frac{(u-V)Va}{(1-Vu)^2}$$

$$= \frac{a(1-V^2)}{(1-Vu)^2} = \frac{a}{\Gamma^2(1-Vu)^2}$$

$$a' = \frac{du'}{dt'} \qquad \boxed{a' = \frac{a}{\Gamma^3(1-Vu)^3}} \quad \text{— } 2$$

$$\boxed{\frac{du'}{dt'} = \frac{F'}{m'}} \Rightarrow \boxed{\frac{du}{dt} = \frac{F'}{m'}\Gamma^3(1-Vu)^2}$$

This is not covariant.

Can redefine $\frac{F'}{m'} \equiv \frac{F}{m}\Gamma^3$

$$\Rightarrow \boxed{\frac{du}{dt} = \frac{F}{m}(1-Vu)^3}$$ still not covariant

Now consider $\boxed{\frac{d}{dt'}(\gamma' u') = \frac{F'}{m'}}$

$$\boxed{\gamma'^2 \equiv (1-u'^2)^{-2}}$$

Is this covariant?

First, note that $(\gamma u)^{\circ} = \gamma \mathring{u} + \mathring{\gamma} u$

$\gamma^2 \equiv \frac{1}{1-u^2} \Rightarrow \cancel{2}\gamma\mathring{\gamma} = \frac{\cancel{2}u\mathring{u}}{\cancel{(1-u^2)^2}}\gamma^4$

$\Rightarrow (\gamma u)^{\circ} = \gamma\mathring{u} + u^2\mathring{u}\gamma^3$

$= \gamma\mathring{u}(1 + u^2\gamma^2) = \gamma^3\mathring{u}$

So, equivalently, is $\boxed{\gamma'^3\frac{du'}{dt'} = \frac{F'}{m'}}$ Covariant? (3)

Using (2) in (3) $\Rightarrow$

$$\gamma'^3 \frac{du}{dt} \frac{1}{\Gamma^3(1-Vu)^3} = \frac{F'}{m'} \qquad (4)$$

Convert γ' to γ, using (1)

$$1 - u'^2 = \frac{1}{\gamma'^2} = 1 - \frac{(u-V)^2}{(1-Vu)^2}$$

$$= \frac{1 + (Vu)^2 - \cancel{2uV} - (u^2 + V^2 - \cancel{2uV})}{(1-Vu)^2}$$

$$= \frac{(1-u^2) + (Vu)^2 - V^2}{(1-Vu)^2} = \frac{(1-u^2)(1-V^2)}{(1-Vu)^2}$$

$$\therefore \frac{1}{\gamma'^2} = \frac{1}{\gamma^2 \Gamma^2 (1-Vu)^2}, \quad \text{Use this in}$$

$$\Rightarrow \frac{\gamma^3 \Gamma^3 (1-Vu)^3}{\Gamma^3 (1-Vu)^3} \frac{du}{dt} = F'/m'$$

$$\Rightarrow \boxed{\gamma^3 \frac{du}{dt} = \frac{F}{m}} \qquad \frac{F}{m} \equiv \frac{F'}{m'},$$

Covariant with (3)

Hassan

1.2H $\qquad \ddot{\vec{r}} = \vec{r}/x, \quad x \neq 0$

(a) $\Rightarrow \ddot{x} = 1, \ \ddot{y} = y/x, \ \ddot{z} = z/x. \quad z(1)=0, \ \dot{z}(1)=0 \Rightarrow \boxed{z=0}$

$\dot{x} = t, \ \dot{x}(1)=1. \ \boxed{x = t^2/2}, \ x(1) = 1/2$

$\ddot{y} = y/(t^2/2) \Rightarrow \ddot{y} = \frac{2y}{t^2}$. let $y = t^{\alpha} \Leftrightarrow \alpha(\alpha-1) = 2$

$\Rightarrow \alpha = 2$ or $\alpha = -1 \Rightarrow y = At^2 - B/t$; $\dot{y} = 2At + B/t^2$

$y(1) = 0 \Rightarrow 0 = A - B$; $\dot{y}(1) = 3 \Rightarrow 3 = 2A + B = 3A$

$\Rightarrow A = 1, \ B = 1 \Rightarrow \boxed{y(t) = t^2 - 1/t}$

(b) $\vec{L} = \vec{r} \times \vec{v}$ = const, central force

at $t=1$, $\vec{r}(1) = [1/2, 0, 0]$, $\vec{v}(1) = [1, 3, 0]$

$\Rightarrow \vec{L}(1) = \frac{3}{2}\hat{z} \Rightarrow \boxed{\frac{3}{2} = x\dot{y} - y\dot{x}}$ const of motion

$\Rightarrow x^2 (y/x)^{\cdot} = 3/2 \Rightarrow \left(\frac{y}{x}\right)^{\cdot} = \frac{3}{2x^2}$.

But $\ddot{x} = 1 \Rightarrow \boxed{x = t^2/2} \Rightarrow \left(\frac{y}{x}\right)^{\cdot} = \frac{6}{t^4} \Rightarrow \frac{y}{x} = \frac{-2}{t^3} + C$

$\Leftrightarrow 0 = -2 + C \Rightarrow \frac{y}{x} = 2(1 - 1/t^3) \Rightarrow \boxed{y = t^2 - 1/t}$

1.3H $\ddot{\vec{r}} = -\vec{\nabla}(xy) \Rightarrow \boxed{\ddot{x} = -y,\ \ddot{y} = -x},\ \ddot{z} = 0$

(a) $\vec{r}(0) = [1, 1, 0],\ \vec{v}(0) = 0 \Rightarrow \boxed{z(t) = 0}$

$\ddddot{x} = -\ddot{y} = x \Rightarrow \ddddot{x} = x$. Try $x = e^{\alpha t}$

$\Rightarrow \alpha^4 = 1 \Rightarrow \alpha^2 = \pm 1 \Rightarrow \alpha = \pm 1, \pm i$

$\Rightarrow x = \{\cos t, \sin t, \cosh t, \sinh t\}$

$\Rightarrow \boxed{x(t) = \cos t}$ satisfies $x(0) = 1,\ \dot{x}(0) = 0$ & $\ddot{x}(0) = -1$, $\dddot{x}(0) = 0$

$\therefore\ y = -\ddot{x} \Rightarrow \boxed{y(t) = \cos t}$ satisfies $y(0) = 1,\ \dot{y}(0) = 0$ & $\ddot{y}(0) = -1$, $\dddot{y}(0) = 0$

(b) $E = \frac{1}{2}v^2 + xy = \text{const}$

$\Rightarrow E = 1 \Rightarrow \boxed{\frac{1}{2}(\dot{x}^2 + \dot{y}^2) + xy = 1}$ ①

(c) $M = \dot{x}\dot{y} + \frac{1}{2}(x^2 + y^2)$

$\dot{M} = \ddot{x}\dot{y} + \dot{x}\ddot{y} + x\dot{x} + y\dot{y} = 0$

$= -y\dot{y} - x\dot{x} + x\dot{x} + y\dot{y}$

$\Rightarrow M = \text{const} = \frac{1}{2}(1+1) = 1$

$\Rightarrow \boxed{\frac{1}{2}(x^2 + y^2) + \dot{x}\dot{y} = 1}$ ②

(d) ① $\Rightarrow \frac{1}{2}(\dot{x} + \dot{y})^2 + xy - \dot{x}\dot{y} = 1$

② $\Rightarrow \frac{1}{2}(x+y)^2 - xy + \dot{x}\dot{y} = 1$

Add $\Rightarrow \boxed{\frac{1}{2}\dot{p}^2 + \frac{1}{2}p^2 = 2}$

Harmonic oscillator

$p(0) = 2,\ \dot{p}(0) = 0 \Rightarrow \boxed{p = 2\cos t}$

Likewise, using $(\dot{x} - \dot{y})^2$, etc, $\boxed{\frac{1}{2}\dot{q}^2 = \frac{1}{2}q^2} \Rightarrow \dot{q} = \pm q$

$q(0) = 0,\ \dot{q}(0) = 0 \Rightarrow \boxed{q = 0} \Rightarrow \boxed{x = y}$ $\boxed{x = \cos t}$

1.4H $\quad \dot{\vec{V}} = \vec{V} \times \hat{z}, \quad \dot{\vec{r}} = \vec{V}$

(a) $\Rightarrow \dot{V}_z = 0 \Rightarrow V_z = 0$

(b) $\vec{r} = r\hat{r} \Rightarrow \dot{\vec{r}} = \vec{v} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}$

$\Rightarrow \dot{\vec{v}} = \ddot{r}\hat{r} + 2\dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} - r\dot{\phi}^2\hat{r}$

$\vec{v} \times \hat{z} = -\dot{r}\hat{\phi} + r\dot{\phi}\hat{r}$

$\Rightarrow \boxed{\ddot{r} - r\dot{\phi}^2 = r\dot{\phi}} \quad \boxed{r\ddot{\phi} + 2\dot{r}\dot{\phi} = -\dot{r}}$

(c) $r = C, \dot{\phi} = D \Rightarrow -CD^2 = CD \Rightarrow CD(D+1) = 0$

$C \neq 0, D \neq 0 \Rightarrow D = -1 \Rightarrow \boxed{r = C, \ \dot{\phi} = -1, \ \phi = -t}$