Monopole – Part I

$$m\dot{\vec{v}} = q\vec{v}\times\vec{B}, \quad \vec{B} = b\vec{r}/r^3$$

$$\dot{\vec{v}} = \frac{\vec{v}\times\vec{r}}{r^3}\,\frac{qb}{m} \qquad \boxed{\frac{qb}{m}=1} \qquad [b]=[B]L^2$$

$$\boxed{\dot{\vec{v}} = \vec{v}\times\vec{r}/r^3}$$

$\Rightarrow$ Dimensions of qb/m

$\rightarrow \frac{qB}{m}L^2 = \frac{L^2}{T}$ = ang mom per mass

$T = \text{const}$

(1) $\vec{v}\cdot\dot{\vec{v}} = 0 = \frac{d}{dt}\left(\frac{1}{2}v^2\right)$

$$\Rightarrow \boxed{T = \tfrac{1}{2}v^2 = \text{const}}$$

$\vec{D} = \text{const}$

(2) $\vec{D} = \vec{L} - qb\vec{r}/r \Rightarrow \frac{\vec{D}}{m} = \vec{r}\times\vec{v} - \frac{\vec{r}}{r}$

$$\boxed{m=1} \qquad \boxed{\vec{D} = \vec{r}\times\vec{v} - \vec{r}/r}$$

$$\dot{\vec{D}} = \vec{r}\times\dot{\vec{v}} - \vec{v}/r + \frac{\vec{r}}{r}\,\frac{\dot{r}}{r}$$

$$= \vec{r}\times\frac{(\vec{v}\times\vec{r})}{r^3} - \frac{\vec{v}}{r} + \frac{\vec{r}\dot{r}}{r^3} = \cancel{\frac{\vec{v}}{r}} - \frac{\vec{r}\cdot\vec{v}\,\vec{r}}{r^3} + \frac{\vec{r}\dot{r}}{r^2}$$

$$= \frac{-r\dot{r}\vec{r}}{r^3} + \frac{\vec{r}\dot{r}}{r^2} = 0$$

$$\boxed{\vec{D} = \vec{\text{const}}}$$

③ Let $\vec{D} = D_0 \hat{z}$. Let $r = r_0, \theta = \theta_0$

$\vec{v}_0 = -\hat{\phi} v_0$, let $r_0 v_0 = \frac{qb}{m} \tan\theta_0 \to \tan\theta_0$

Components in $\{r, \theta, \phi\}$

$\hat{r}$: $\boxed{D_0 \cos\theta = -1}$ (A)

$\hat{\theta}$: $-D_0 \sin\theta = \hat{\theta} \cdot \vec{r} \times \vec{v} = -r^2 \sin\theta \dot{\phi}$

$\Rightarrow \boxed{D_0 = r^2 \dot{\phi}}$ (B)

$\hat{\phi}$: $0 = \hat{\phi} \cdot \vec{r} \times \vec{v} = \hat{\theta} \cdot \vec{v} r = r^2 \dot{\theta}$

$\Rightarrow \boxed{0 = r^2 \dot{\theta}}$ (C)

(A) + (C) are redundant.

using initial conditions, $D_0 \cos\theta_0 = -1$,

$$D_0 = r_0^2 \dot{\phi}_0 = -\frac{r_0 v_0}{\sin\theta_0} = \frac{-1}{\cos\theta_0}$$

Consistent

3 constants of motion, with $r_0 v_0 = \tan\theta_0$

$\boxed{r^2 \dot{\phi} = -\frac{1}{\cos\theta_0}}$ $\boxed{\theta = \theta_0}$ ④

Also $T = \text{const}$ $\Rightarrow$ $\boxed{v_0^2 = \dot{r}^2 + r^2 \sin^2\theta \dot{\phi}^2}$

⑤ Equivalent 1-D for r

$$v_0^2 = \dot{r}^2 + \frac{r^2 \sin^2\theta_0}{\cos^2\theta_0}\frac{1}{r^4}$$

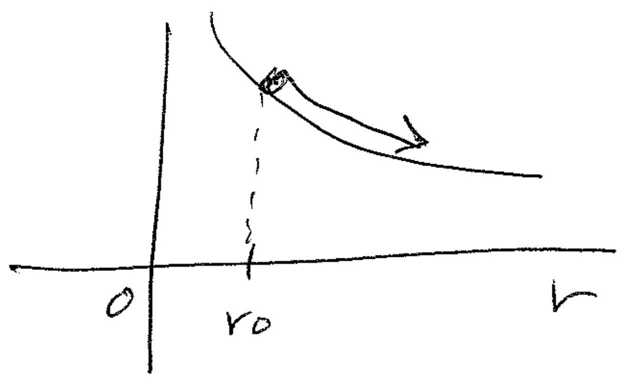
$$\Rightarrow \quad 1 = \frac{\dot{r}^2}{v_0^2} + \frac{1}{r^2}\frac{\sin^2\theta_0}{\cos^2\theta_0\, v_0^2}$$

$$\Rightarrow \quad \boxed{\frac{\dot{r}^2}{v_0^2} + \frac{r_0^2}{r^2} = 1}$$ equivalent 1-D eqn

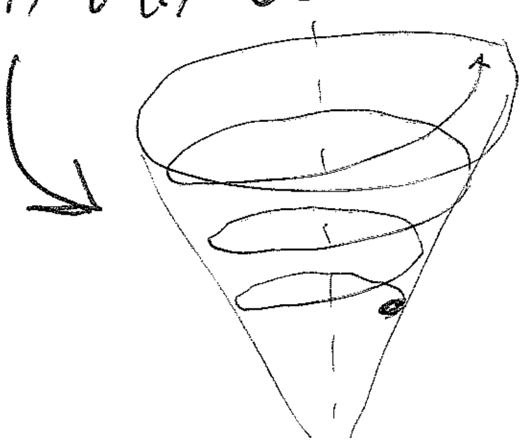
$$\frac{1}{2}\dot{r}^2 + V_{eff}(r) = E \quad \Rightarrow E = \frac{1}{2}v_0^2$$

$$V_{eff}(r) = \frac{1}{2}\frac{r_0^2}{r^2}v_0^2$$

rolls downhill from r_0, and $\theta = \theta_0 \Rightarrow$



(4) $\theta(t) = \theta_0$



$r \to \infty$, $\dot{r} \to v_0$

stays on cone of θ_0, spirals out

3a

6 Solve

$$\frac{\dot r}{v_0} = \sqrt{1 - \frac{r_0^2}{r^2}}$$

$$\int_{r_0}^{r} \frac{dr}{\sqrt{}} = \int_0^t dt \Rightarrow \boxed{r^2 = r_0^2 + v_0^2 t^2}$$

Check $\cancel{2} r \dot r = \cancel{2} v_0^2 t$

$$\cancel{r^2} \dot r^2 = v_0^2 \left(\cancel{r^2} - \frac{r_0^2}{r^2}\right)$$

$$\Rightarrow \frac{\dot r^2}{v_0^2} = \left(1 - r_0^2/r^2\right) \checkmark$$

Part II Vec Potential

1 let $\vec{A} = A\hat{\phi} = \rho A \vec{\nabla}\phi$, $\rho = r\sin\theta$

$$\Rightarrow \vec{B} = \frac{b\vec{r}}{r^3} = \vec{\nabla}\times\vec{A} = \vec{\nabla}(\rho A)\times\vec{\nabla}\phi$$

$$\Rightarrow b/r = \vec{\nabla}(\rho A)\cdot\vec{\nabla}\phi\times\vec{r} = \frac{r\hat{\theta}}{\rho}\cdot\vec{\nabla}(\rho A)$$

$$\Rightarrow \frac{b}{r} = \frac{1}{\rho}\frac{\partial}{\partial\theta}(\rho A) \Rightarrow \frac{\partial(\rho A)}{\partial\theta} = b\sin\theta$$

$$\rho A = -b\cos\theta$$

$$\boxed{A = \frac{-b}{r\tan\theta}} \quad \vec{A} = \hat{\phi}A$$

$\vec{r} \neq 0$
$\Rightarrow \vec{\nabla}\cdot\vec{B} = 0$

$\vec{r} \neq 0 \Rightarrow \vec{\nabla}\cdot\vec{B} = 0$

2 Constants of Motion

$$L = T - U = T + q\vec{v}\cdot\vec{A}$$

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + \rho^2\dot{\phi}^2\right)$$

$$\vec{v}\cdot\vec{A} = \vec{v}\cdot\hat{\phi}A = \rho\dot{\phi}A$$

$$L = L(\dot{r}, \dot{\theta}, \dot{\phi}; r, \theta) \quad \partial L/\partial\phi = 0$$
$$\partial L/\partial t = 0$$

$\therefore P_\phi = \text{const}; \quad H = \text{constant}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) = 0 \Rightarrow \frac{\partial L}{\partial\dot{\phi}} = \text{const}$$

$$\Rightarrow \quad m\rho^2\dot{\phi} - \frac{q\rho b}{r\tan\theta} = \text{const}$$

$$\rho^2\dot{\phi} - \frac{\rho}{r\tan\theta} = \text{const}, \quad \rho^2\dot{\phi} - \cos\theta = \text{const}$$

$$t=0, \quad \rho_0 = r_0\sin\theta_0, \quad \rho_0\dot{\phi}_0 = -v_0, \quad \theta=\theta_0$$

$$\Rightarrow \boxed{\rho^2\dot{\phi} - \cos\theta = -\rho_0 v_0 - \cos\theta_0} \; *$$

$$h = \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L$$

$$= \dot{r}(m\dot{r}) + \dot{\theta}(mr^2\dot{\theta}) + \dot{\phi}\left(m\rho^2\dot{\phi} - \frac{qb\rho}{r\tan\theta}\right)$$

$$-(T + q\vec{v}\cdot\vec{A}) = \text{const}$$

$$\Rightarrow \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 = \text{const}$$

$$\boxed{\dot{r}^2 + r^2\dot{\theta}^2 + \rho^2\dot{\phi}^2 = v_0^2} \; *$$

* 2 constants of motion (obvious constants)

θ eqn

(9) $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \left(\frac{\partial L}{\partial \theta}\right)$$

$$\rho\dot{\phi}A = -b\dot{\phi}\cos\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = mr^2\sin\theta\cos\theta\,\dot{\phi}^2$$

$$+\, q b \dot{\phi}\sin\theta$$

$$\boxed{\frac{d}{dt}(r^2\dot{\theta}) = r^2\sin\theta\cos\theta\,\dot{\phi}^2 + \dot{\phi}\sin\theta}$$

$$\frac{qb}{m} = 1$$

$$RHS = (r^2\cos\theta\,\dot{\phi} + 1)\,\dot{\phi}\sin\theta$$

Compute RHS

$$\dot{\phi}\sin\theta = -\frac{\sin\theta}{\rho^2}\left[\rho_0 v_0 + \cos\theta_0 - \cos\theta\right]$$

$$= -\frac{\sin\theta}{\rho^2}\left[\frac{\sin^2\theta_0 \cancel{\tan}\theta_0}{\cos\theta_0} + \cos\theta_0 - \cos\theta\right]$$

$$\dot{\phi}\sin\theta = \frac{-\sin\theta}{\rho^2\cos\theta_0}\left(1 - \cos\theta_0\cos\theta\right)$$

$$r^2\cos\theta\,\dot{\phi} + 1 = \frac{\cos\theta}{\sin^2\theta}\left[-\rho_0 v_0 - \cos\theta_0 + \cos\theta\right] + 1$$

$$= \frac{\cos\theta}{\sin^2\theta}\left[-\frac{\sin\theta_0}{\cos\theta_0} - \cos\theta_0 + \cos\theta\right] + 1$$

$$= \frac{\cos\theta}{\sin^2\theta}\,\frac{1}{\cos\theta_0}\left(-1 + \cos\theta\cos\theta_0\right) + 1$$

$$= \frac{-\cos\theta}{\sin^2\theta\cos\theta_0} + \frac{\cos^2\theta}{\sin^2\theta} + 1$$

$$= \frac{1}{\sin^2\theta\cos\theta_0}\left(-\cos\theta + \cos\theta_0\right)$$

$$\therefore \boxed{RHS = \frac{1}{\rho^2\sin\theta\cos^2\theta_0}\left(\cos\theta - \cos\theta_0\right)\left(1 - \cos\theta\cos\theta_0\right)}$$

10 Stability

$$\frac{d}{dt}\left(r^2\dot\theta\right) = \frac{(\cos\theta-\cos\theta_0)(1-\cos\theta\cos\theta_0)}{r^2\sin\theta\cos^2\theta_0}$$

$$r^2 = v_0^2t^2+r_0^2, \quad \theta=\theta_0$$

$$r\to r+\breve{r}, \quad \theta\to\theta_0+\breve{\theta}$$

$$\Rightarrow \frac{d}{dt}\left(r^2\dot{\breve{\theta}}\right) = \frac{\cancel{\sin^2\theta_0}\,(-\cancel{\sin\theta_0})\,\breve{\theta}}{\cancel{r^2\sin^2\theta_0}\,\cancel{\sin\theta_0}\cos^2\theta_0}$$

$$\Rightarrow \boxed{r^2\frac{d}{dt}\left(r^2\frac{d\breve{\theta}}{dt}\right) = \frac{-\breve{\theta}}{\cos^2\theta_0}}$$

Let $\frac{dt}{r^2}\equiv du \Rightarrow \boxed{\frac{d^2\breve{\theta}}{du^2} = \frac{-\breve{\theta}}{\cos^2\theta_0}}$

$$\frac{1}{\cos^2\theta_0} = \tan^2\theta_0+1 = r_0^2v_0^2+1$$

$$\Rightarrow \boxed{\omega^2 = 1+r_0^2v_0^2} \quad \text{stable}$$

where $\boxed{\tan(r_0v_0u)\equiv v_0t/r_0}$

$t\to\infty$, $\breve{\theta}\to$ const, stable