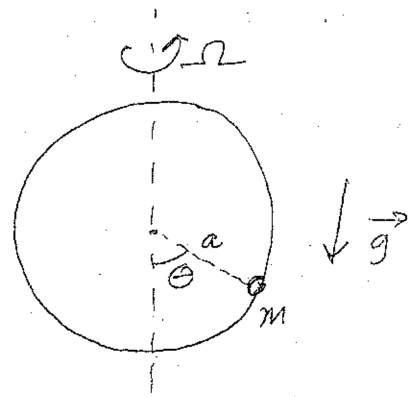
Phys601/F08/Midterm – in class

50 min, 2 pages of notes



Useful formulae:

$\cos 2t = 2\cos^2 t - 1$
$2 \sin t \cos t = \sin 2t$
$\sin t = t - t^3/6 + \ldots$
$\cos t = 1 - t^2/2 + \ldots$

Hoop (4 points per part except 6 for #7)

A point mass m is constrained to move on a massless hoop of radius a fixed in a vertical plane that rotates about its vertical symmetry axis with constant angular speed Ω. The only external force is gravity, $\boldsymbol{g}$.

1. Find the Lagrangian and obtain the Euler-Lagrange equation of motion.

2. Identify the effective potential V_{eff}.

3. Show that $\theta = 0$ is an equilibrium (ie for $d\theta/dt=0$ and $d^2\theta/dt^2=0$). Under what conditions on $\{g, m, a, \Omega\}$ is this a stable equilibrium. Demonstrate this from V_{eff}.

4. Find the frequency of small oscillations about this equilibrium assuming stability.

5. Show that there is another $0 < \theta < \pi$ which could also be an equilibrium. Under what conditions on $\{g, m, a, \Omega\}$ is such an equilibrium possible? Under what conditions is this equilibrium stable? Demonstrate all this from V_{eff}.

6. Find the frequency of small oscillations about the equilibrium in 5. assuming stability.

7. Suppose $\Omega^2 = g/a$. Check that $\theta = 0$ is the only equilibrium. By assuming that $\theta \ll 1$, ie, we are only interested in small oscillations, expand the E-L equation about $\theta = 0$ and find the ODE for $\theta(t)$ for small oscillations in this case (keep small terms only up to lowest non-vanishing order). Suppose $(d\theta/dt)(0) = 0$ and $\theta(0) = \varepsilon \ll 1$. Find the period of oscillation. You may leave your answer in terms of a definite integral but, by making the integral dimensionless, make sure you obtain a dimensional scaling for the period.

Hoop

$$T = \frac{1}{2} m a^2 \dot{\theta}^2 + \frac{1}{2} m a^2 \sin^2\theta \, \Omega^2$$

$$V = -mga\cos\theta$$

① $L = T - V \qquad ma^2 = 1$

$\qquad\qquad\qquad\quad g/a = 1$

$$\Rightarrow \boxed{L = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\sin^2\theta\,\Omega^2 + \cos\theta}$$

$$\partial L/\partial\dot{\theta} = \dot{\theta}, \quad \partial L/\partial\theta = -\sin\theta + \Omega^2\sin\theta\cos\theta$$

$$\Rightarrow \boxed{\ddot{\theta} = -\sin\theta\left(1 - \Omega^2\cos\theta\right)} \quad ①$$

② $$\boxed{V_{eff} = -\cos\theta - \frac{1}{2}\sin^2\theta\,\Omega^2}$$

③ From ①, $\theta = 0 \Rightarrow$ equil

$$V_{eff}'' = \cos\theta - \Omega^2(\cos^2\theta - \sin^2\theta)$$

$$= \cos\theta - \Omega^2\cos 2\theta$$

$$V_{eff}''\big|_{\theta=0} = 1 - \Omega^2 > 0$$

$$\Leftrightarrow \boxed{\Omega^2 < 1} \quad \Omega^2 < g/a$$

4. $$\ddot{\tilde{\theta}} = -\tilde{\theta} + \Omega^2 \tilde{\theta}$$
$$= -(1-\Omega^2)\tilde{\theta}$$
$$\Rightarrow \boxed{\omega^2 = (1-\Omega^2)}$$

5. RHS ① $= 0 \Rightarrow \cos\theta = \frac{1}{\Omega^2}$

$\Rightarrow$ solution for if $\boxed{\Omega^2 > 1}$

$\cancel{\cos\theta}$ $\boxed{\theta = \cos^{-1}\frac{1}{\Omega^2}}$

$$V''_{eff} = \frac{1}{\Omega^2} - \Omega^2\left(\frac{\cancel{2}}{\Omega^4} - 1\right)$$
$$= -\frac{1}{\Omega^2} + \Omega^2 = \Omega^2\left(1 - \frac{1}{\Omega^4}\right) > 0$$

$\boxed{\text{stable for } \Omega^2 > 1}$

6. $$\ddot{\tilde{\theta}} = -V''_{eff}(\theta - \theta_0)$$
$$\ddot{\tilde{\theta}} = -\Omega^2\left(1 - \frac{1}{\Omega^4}\right)\tilde{\theta}$$
$$\boxed{\omega^2 = \Omega^2\left(1 - \frac{1}{\Omega^4}\right)}$$

7 $\underline{\Omega^2 = 1}$

$$\ddot{\theta} = -\sin\theta\,(1-\cos\theta)$$

$\theta = 0$ is an equil, only one

$$\sin\theta(1-\cos\theta) \simeq (\theta - \theta^3/6)(\theta^2/2)$$

$$\boxed{\ddot{\theta} \simeq -\theta^3/2}$$

$$\times\dot{\theta} \quad \frac{d}{dt}\left(\frac{\dot{\theta}^2}{2}\right) = -\frac{d}{dt}\left(\frac{\theta^4}{8}\right)$$

$$\dot{\theta}^2 = -\frac{\theta^4}{4} + \frac{\epsilon^4}{4}$$

$$\Rightarrow 2\dot{\theta} = -\sqrt{\epsilon^4 - \theta^4}$$

$$-\int_{\epsilon}^{-\epsilon} \frac{2\,d\theta}{\epsilon^2(1-\theta^4/\epsilon^4)^{1/2}} = \int_0^{T/2} dt$$

$$\boxed{\frac{T}{4} = \frac{1}{\epsilon}\int_{-1}^{1} \frac{ds}{\sqrt{1-s^4}}} \Rightarrow \boxed{T = (\text{Const})\,\frac{4}{\epsilon}\sqrt{\frac{a}{g}}}$$

$$\int_{-1}^{1} \frac{ds}{\sqrt{1-s^4}}$$