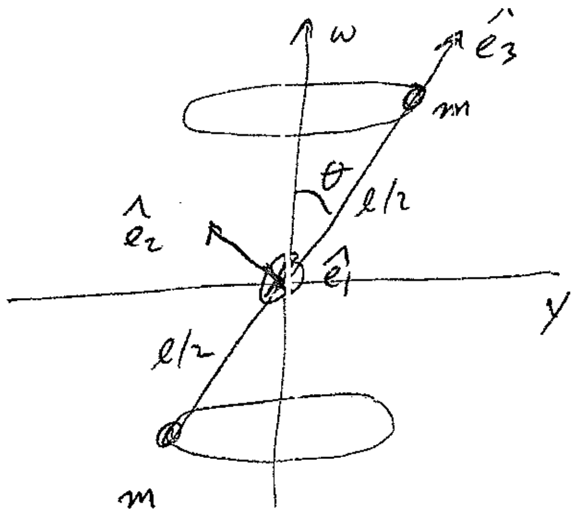
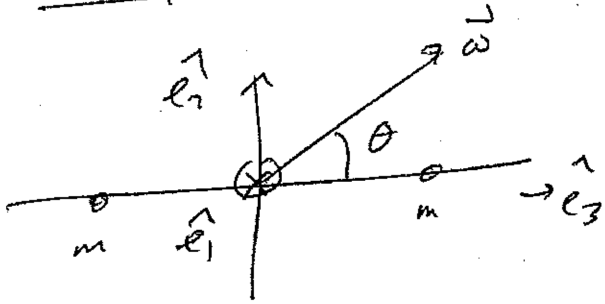
$\vec{N}$ from fixed $\vec{\omega}$

N1



① Body based



$$I_{xx} = \sum m z^2 = ml^2/2$$

$$I_{yy} = ml^2/2$$

$$I_{zz} = 0$$

$$I = \frac{ml^2}{2}\begin{pmatrix} 1 & & \\ & 1 & \\ & & 0 \end{pmatrix}$$

$$\omega_1 = 0$$

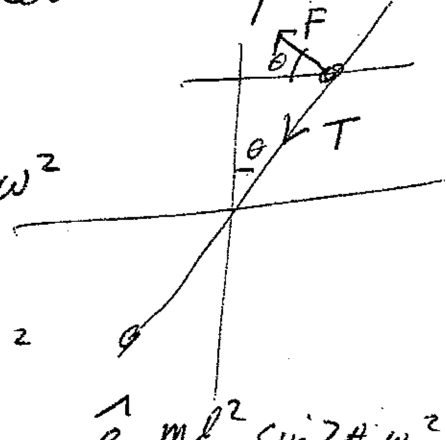
$$\Rightarrow (I_3 - I_2)\,\omega_2\omega_3 = N_1$$

$$\Rightarrow N_1 = -\frac{ml^2}{2}\omega^2 \sin\theta\cos\theta$$

$$\Rightarrow \boxed{\vec{N} = -\hat{e}_1 \frac{ml^2\omega^2}{4}\sin 2\theta}$$

② Space-based

To move the masses in circular orbits need a net centripetal force. Consider the fig:



$$F\cos\theta + T\sin\theta = m\frac{l}{2}\sin\theta\,\omega^2$$

$$F\sin\theta = T\cos\theta$$

$$\Rightarrow F = m\frac{l}{2}\sin\theta\cos\theta\,\omega^2$$

$$\rightarrow \vec{N} = 2\vec{r}\times\vec{F} = -\hat{e}_1\frac{ml^2}{4}\sin 2\theta\,\omega^2$$

Else, $\frac{d\vec{L}}{dt} = \vec{N}$, space based

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}, \quad \vec{\omega} = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

$$\Rightarrow \vec{L} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = (I_{xz}\omega + I_{yz}\omega + I_{zz}\omega)$$

each mass has $x(t), y(t), z(t)$; $\omega = \text{const}$

$$z = \frac{\ell}{2}\cos\theta, \quad x = \frac{\ell}{2}\sin\theta\cos(\omega t)$$
$$y = \frac{\ell}{2}\sin\theta\sin(\omega t)$$

$$\Rightarrow \vec{N} = \dot{\vec{L}} = \begin{pmatrix} \dot{I}_{xz} \\ \dot{I}_{yz} \\ 0 \end{pmatrix} \omega$$

$$I_{xz} = \sum(-m\,xz) = -2m\left(\frac{\ell}{2}\right)^2 \sin\theta\cos\theta\cos\omega t$$
$$I_{yz} = \sum(-m\,yz) = -2m\left(\frac{\ell}{2}\right)^2 \sin\theta\cos\theta\sin\omega t$$

$$\Rightarrow \vec{N} = -\frac{m\ell^2}{4}\omega\sin 2\theta\,(\hat{x}\cos\omega t + \hat{y}\sin\omega t)^{\cdot}$$
$$= -\frac{m\ell^2}{4}\sin 2\theta\,\omega^2(-\hat{x}\sin\omega t + \hat{y}\cos\omega t)$$

note: $\hat{e}_1 = -\hat{x}\sin\omega t + \hat{y}\cos\omega t$

$$\Rightarrow \boxed{\vec{N} = -\hat{e}_1\frac{m\ell^2\omega^2}{4}\sin 2\theta}$$

Body based is easier

Problem 2

This solution is reported in 2 ways:

First, take the equation at face value and find L and H using the E-L equ (as was suggested in the clarifying email).

Second, we do this problem by including $l(t)$ explicitly, as is more precise.

In both cases, we use $\dot{l}/l \ll \omega$.

In both cases, the bottom line is that the action is an **adiabatic invariant**, ie, A is constant = area enclosed.

Method (1) **First way**. $\ddot{\theta} = -\omega^2\theta$ is the E-L eqn.

Work backwards $\Rightarrow L = \frac{1}{2}\dot{\theta}^2 - \omega^2(t)\frac{1}{2}\theta^2$.

$\Rightarrow p = \dot{\theta} \Rightarrow H = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}\omega^2\theta^2$.

$\Rightarrow A = \oint d\theta\, p = \oint d\theta\, \dot{\theta}$.

2.1 Use the WKB eikonal method to find an approximate solution for θ(t), *correct to first order.* The lowest order solution is not acceptable. Your solution should satisfy the conditions θ(0)=0 and (dθ/dt)(0)=1. Assume that (dω/dt)(0)=0. in terms ω(t)

$$\ddot{\theta} + \omega^2(t)\theta = 0; \quad \theta(t); \quad \omega^2 \gg |\dot{\omega}|; \quad \dot{\omega}(0)=0, \ \omega(0)=1, \ \omega(\infty)=\sqrt{2}$$

WKB: Try $\theta(t) = e^{-iS}$, $|S| \gg 1$ ansatz

$$\Rightarrow -i\ddot{S} - \dot{S}^2 + \omega^2 = 0 \Rightarrow \boxed{\dot{S}^2 + i\ddot{S} = \omega^2}$$

lowest $\quad \dot{S}_0^2 = \omega^2, \quad \dot{S}_0 = \pm\omega \Rightarrow S_0 = \int^t dt'\,\omega(t')\,(\pm)$

first $\quad 2\dot{S}_0\dot{S}_1 + i\ddot{S}_0 = 0 \Rightarrow \dot{S}_1 = \frac{-i}{2}(\ln \dot{S}_0)^{\cdot}$

$$\Rightarrow -iS_1 = \ln\left(\frac{1}{\omega^{1/2}}\right) + \text{const}$$

$$\Rightarrow \boxed{\theta(t) \to \begin{Bmatrix} \cos[\int dt\,\omega] \\ \sin[\int dt\,\omega] \end{Bmatrix} * \omega^{-1/2}}$$

$\theta(0)=0, \ \dot{\theta}(0)=1 \Rightarrow$ Try

$$\boxed{\theta(t) = \sin\left[\int_0^t \omega(t')\,dt'\right] / \omega(t)^{1/2}}$$

$$\theta(0)=0; \quad \dot{\theta} = \cos[\ ]\frac{\omega}{\omega^{1/2}} + \frac{\sin[\ ]}{\omega^{3/2}}\frac{1}{2}\dot{\omega}$$

$$\dot{\theta}(0) = 1 \quad \text{since } \omega(0)=1$$

WKB solution to first order including initial conditions

$$\theta(t) = \frac{1}{(\omega(t))^{1/2}} \sin\left[\int_0^t ds\,\omega(s)\right]$$

2.2 Write down the Hamiltonian function for this system, $H(p,\theta,t)$. Using your solution for $\theta(t)$, calculate explicitly, *to lowest non-vanishing order only*, the energy of the mass, $h(t)$. The latter function is the "h" function as defined by GPS.

$$h(t) = p^2/2 + \omega^2\theta^2/2 = \dot{\theta}^2/2 + \omega^2\theta^2/2$$

$$\simeq \frac{1}{2}\omega\cos^2[\ ] + \frac{1}{2}\omega^2\sin^2[\ ]/\omega = \omega(t)/2$$

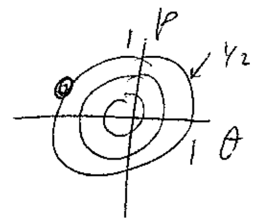
$H(p,\theta,t) = p^2/2 + \omega^2(t)\,\theta^2/2$

$h(t) = \left(\frac{1}{2}\right)\omega(t)$

Now, make a sketch of contours of constant H for the two cases $t = 0$ and $t >> \tau$. Calculate the value of $h(t)$ for these two cases; then mark the position of the particle on your H-contours, ie, identify which H surface it begins on and where it ends up. Your two H-contours should be approximately to scale, showing the distortion as well as H-values and where "h" is.

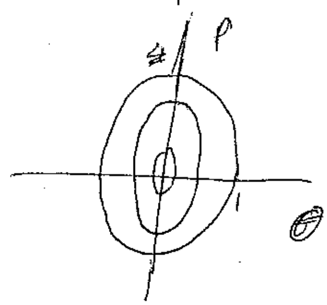
$\underline{t=0}$ $\quad H = p^2/2 + \theta^2/2$

$h = 1/2$

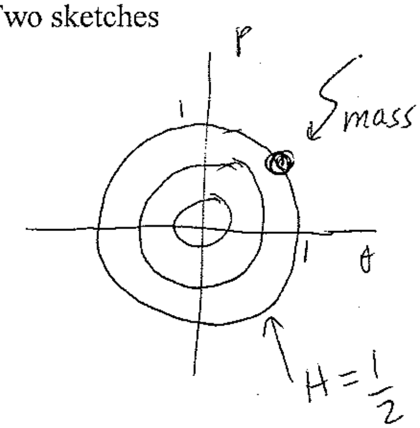


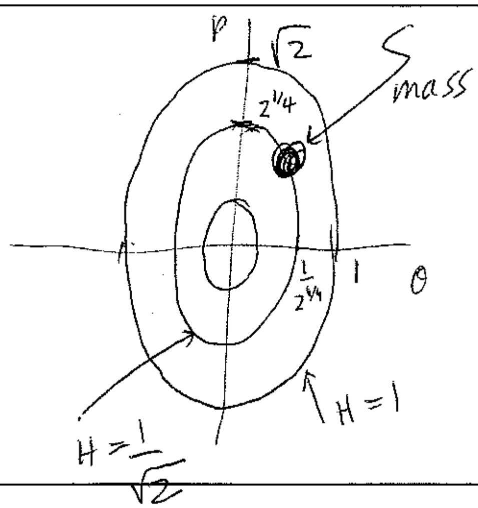
$\underline{t=\infty}$ $\quad H = p^2/2 + \theta^2$

$h = 1/\sqrt{2}$



Two sketches





2.3 From your WKB solution, calculate to lowest non-vanishing order the action, $A(t) = \int p\,d\theta$, for this system at any time, where the integral is over one full period. [The integral is doable even if omega is not known explicitly.] Show that, to the order of your calculation, the action is a constant of the motion. What is the interpretation of $\int p\,d\theta$ and its constancy in phase space? Complete the sentence below.

$$p = \dot{\theta} \simeq \omega^{1/2} \cos\left[\int dt\,\omega\right]$$

$$\theta \simeq \omega^{-1/2} \sin\left[\int dt\,\omega\right]$$

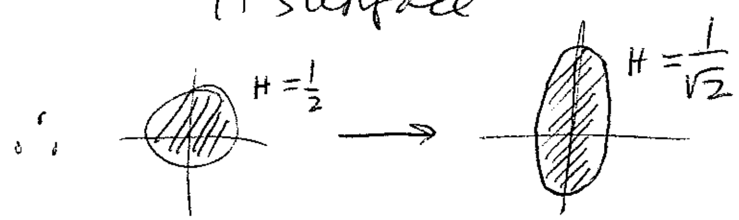
$$A = \oint p\,d\theta = \oint \dot{\theta}\,d\theta = \oint \dot{\theta}^2\,dt$$

$$A(t) = \oint dt\,\omega(t) \cos^2\left[\int_0^t dt'\,\omega(t')\right]$$

Let $dt\,\omega(t) \equiv ds \Rightarrow A = \oint ds \cos^2[s] = \int_0^{2\pi} ds \cos^2(s)$

$= 2\pi \frac{1}{2} = \pi \Rightarrow$ **$A = \pi$ = constant**

A = area ~~under a~~ inside a constant H surface



What is A(t)? $A = \pi$ = const

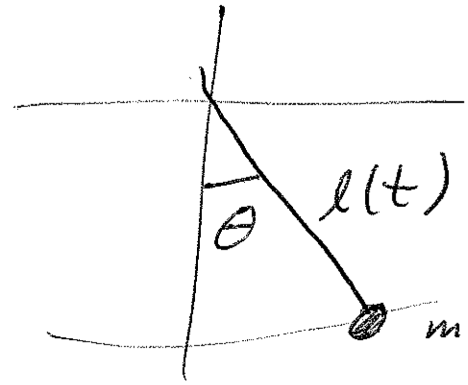
Complete the sentence: "As time evolves, the H surfaces slowly distort but the particle stays on those H surfaces such that area is preserved in phase space."

Method (2)

Solution of $\ell(t)$ H.O. Pbm including $\ell(t)$

$T = \frac{1}{2}\dot{\ell}^2 + \frac{1}{2}\ell^2\dot{\theta}^2$

$V = \frac{1}{2}\ell\theta^2$ $\quad g=1$, $m=1$



For this problem, we know $\dot{\ell}/\ell \ll \dot{\theta}/\theta$, so $T \simeq \frac{1}{2}\ell^2\dot{\theta}^2$. We are here making an error of $(\omega\tau)^{-2}$. But our WKB will be good to $(\omega\tau)^{-1}$, so this error is OK.

$\therefore L \simeq \frac{1}{2}\ell^2\dot{\theta}^2 - \frac{1}{2}\ell\theta^2$

$$\Rightarrow \boxed{(\ell^2\dot{\theta})^{\cdot} = -\ell\theta}$$

$\Rightarrow \ell^2\ddot{\theta} + 2\ell\dot{\ell}\dot{\theta} = -\ell\theta$

$\ell\ddot{\theta} + 2\dot{\ell}\dot{\theta} = -\theta$

This ($2\dot{\ell}\dot{\theta}$) is small, but we are keeping 1st order.

Try WKB $\quad \theta = e^{-iS} \qquad l^2$

$\Rightarrow \dot\theta = -i\dot S\theta \,,\quad \ddot\theta = -\dot S^2\theta - i\ddot S\theta$

$\Rightarrow +\dot S^2 + i\ddot S + \frac{2\dot l}{l}(+i\dot S) = +\frac{1}{l}$

Lowest $\quad \dot S^2 \simeq 1/l \Rightarrow \boxed{\dot S \simeq 1/l^{1/2}} \to \omega$

1st $\quad 2\dot S\dot S_1 + i\ddot S + 2i\frac{\dot l}{l}\dot S = 0$

$\Rightarrow \dot S_1 + \frac{i}{2}(\ln\dot S)^{\cdot} + i(\ln l)^{\cdot} = 0$

$\Rightarrow -S_1 = \frac{i}{2}\ln\dot S + i\ln l$

$= i\ln(\dot S^{1/2} l)$

$= i\ln(l^{3/4})$

$e^{-iS} \to e^{-iS}e^{-iS_1}$

~~$e^{-iS_1} = e^{-\ln\dot S^{1/2}} e^{-\ln l}$~~ $\qquad \omega^2 = 1/l$

$e^{-iS_1} = \frac{1}{l^{3/4}} = \omega^{3/2}$

$\therefore \theta \to \sin\left[\int_0^t dt\,\omega\right]\omega^{3/2}$

$\theta(0) = 0, \quad \dot\theta = \omega\,\omega^{3/2}\cos\left[\int_0^t\right]$

$\dot\theta(0) = 1 \qquad + \sin[\,]\,\frac{3}{2}\omega^{1/2}\dot\omega$

Try the Action.

$$p = \frac{\partial L}{\partial \dot\theta} = l^2 \dot\theta \qquad \omega^2 = \frac{1}{l}$$

$$A = \oint p \, d\theta = \oint l^2 \dot\theta^2 dt$$

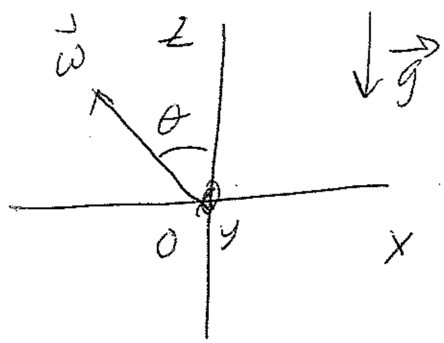
$$= \oint l^2 dt \; \omega^3 \omega^2 \cos^2\left[\int_0^t \omega \, dt\right]$$

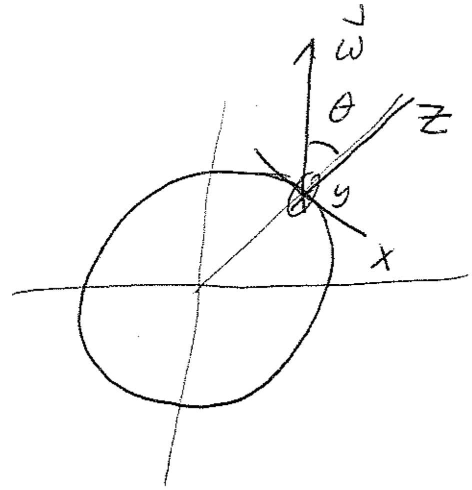
$$= \oint dt \; \omega \cos^2\left[\int_0^t \omega \, dt\right]$$

$$\text{let } dt\omega = ds \Rightarrow A = \oint ds \cos^2[s]$$

$$\Rightarrow A = \pi$$

Coriolis





$$\ddot{\vec{r}} = \vec{g} - 2\vec{\omega}\times\dot{\vec{r}} - \vec{\omega}\times(\vec{\omega}\times\vec{r}),\quad \vec{g} = -\hat{z}g$$

$$\vec{r} = (x, y, z+R) \text{ in local system}$$

To lowest order, $\vec{g}$ dominates ($|\vec{\omega}|^2 \to 0$)

$$\ddot{z}_0 = -g,\quad \ddot{x}_0 = 0,\quad \ddot{y}_0 = 0$$

$$= \dot{z}_0 = v_0 - gt,\quad z_0 = a_0 + v_0 t - \tfrac{1}{2}gt^2$$

First order

$$\ddot{x}_1 = +2\omega_z \dot{y} + \omega^2 R\cos\theta$$

$\dot{y}_0$ to lowest $= 0$

$$\ddot{y}_1 = 2\omega_x \dot{z}_0$$

↑ centrifugal deflection

↑ coriolis deflection

Note: Free fall time $\sim \sqrt{h/g}$

$\therefore$ centrifugal deflection $\sim x_1 \sim (\omega\tau)^2 R$

Coriolis deflection $\sim y_1 \sim (\omega\tau) h$

ratio $\sim \frac{x_1}{y_1} \sim (\omega\tau)\left|\frac{R}{h}\right| \Rightarrow$ depends on relative sizes

(arrows: $(\omega\tau)$ small; $\frac{R}{h}$ large)

ignore x_1

$$\therefore \dot{y}_1 = 2\omega_x (z_0 - a_0); \quad y_1(0) = 0$$

~~Note~~

Case ① $v_0 = 0$, $a_0 = h$

$$\Rightarrow (z_0 - a_0) = -\frac{1}{2} g t^2 \Rightarrow \tau = \sqrt{2h/g}$$

$$\Rightarrow \dot{y}_1 = -2\omega_x \frac{1}{2} g t^2, \quad y_1 = -\omega_x g \frac{t^3}{3}$$

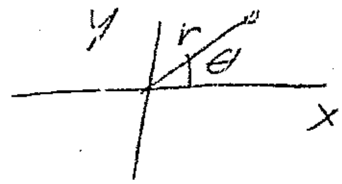
$$\boxed{y_1 = -\omega_x \frac{g}{3} \tau^3} \quad \tau = \sqrt{\frac{2h}{g}} \Rightarrow \boxed{y_1 = -\omega_x \tau \frac{2}{3} h}$$

Case ② $a_0 = 0$, $\frac{1}{2} v_0^2 = g h \Rightarrow \dot{y}_1 = 2\omega_x (v_0 t - g t^2/2)$

$$t = 2v_0/g \Rightarrow y_1 = 2\omega_x \frac{1}{3} \frac{v_0}{2} \frac{4 v_0^2}{g^2} = \frac{8}{3} \omega_x \tau h \Rightarrow \boxed{y_1 = \omega_x \tau \frac{8}{3} h}$$

$$\Rightarrow \boxed{\text{Ratio} = \frac{8/3}{-2/3} = -4} \quad \text{need } \boxed{\omega\tau \ll 1}$$

#3) Goldstein 10.6



$$\vec{A} = \frac{1}{2} B \hat{z} \times \hat{r}$$

$$A_r = 0, \quad A_\theta = \frac{1}{2} B r$$

$$\vec{B} = \hat{z} \frac{1}{r} \frac{\partial}{\partial r}(r A_\theta) = \hat{z} B$$

$$L = T - U = T + q \vec{v} \cdot \vec{A}$$

$$T = \frac{1}{2} m v^2$$

$$L = \frac{1}{2} m v^2 - q r^2 \dot{\theta} \frac{B}{2} - \frac{1}{2} k r^2$$

$$\boxed{L = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + q \frac{B}{2} r^2 \dot{\theta} - \frac{1}{2} k r^2}$$

$m = 1$, $qB = 1$ ($\Rightarrow$ gyrofrequency $= 1$) → $\frac{qB}{m}$

$$\boxed{L = \frac{1}{2} \dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 + \frac{r^2}{2} \dot{\theta} - \frac{1}{2} k r^2}$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = \dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = r^2 \dot{\theta} + \frac{r^2}{2}$$

$$\boxed{p_r = \dot{r}; \quad p_\theta = r^2 (\dot{\theta} + 1/2)}$$

$$H = \frac{1}{2} v^2 - \frac{1}{2}kr^2 = \frac{1}{2} \dot{r}^2 + \frac{1}{2} r^2 \dot{\theta}^2 - \frac{1}{2}kr^2$$

$$\Rightarrow H = \frac{p_r^2}{2} + \frac{1}{2r^2}\left(p_\theta - \frac{1}{2} r^2\right)^2 - \frac{1}{2}kr^2$$

(1) $$\boxed{H = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} - \frac{p_\theta}{2} + \frac{r^2}{8} + \frac{1}{2}kr^2}$$

$H(r, p_r, p_\theta)$, $\partial_\theta H = 0$

$$\Rightarrow \boxed{p_\theta = \text{const} = r^2(\dot{\theta} + 1/2)}$$

$$\boxed{\dot{r} = p_r} \quad \boxed{\dot{p}_r = + \frac{p_\theta^2}{r^3} - \frac{r}{4} - kr}$$

$$\boxed{\dot{\theta} = \frac{p_\theta}{r^2} - \frac{1}{2}}$$

(2) Suppose $p_\theta(0) = 0$, $\theta(0) = 0$, $r(0) = 1$, $\dot{r}(0) = 0$

$\Rightarrow \dot{\theta} = -1/2$, $\boxed{\theta = -t/2}$

$$\boxed{\ddot{r} = -\left(\frac{1}{2}\right)^2 r\,\Omega^2} \Rightarrow \boxed{r = \cos(\Omega t/2)}$$

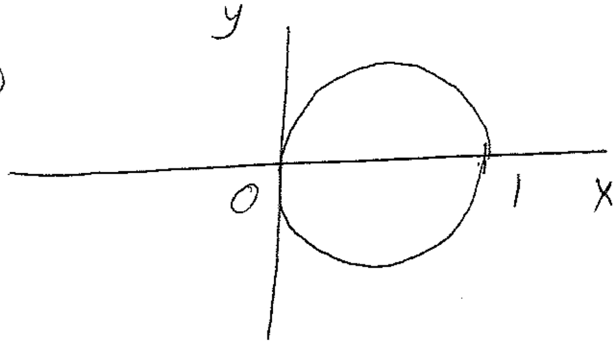
$\Omega^2 = 1 + 4k$

$\Rightarrow$

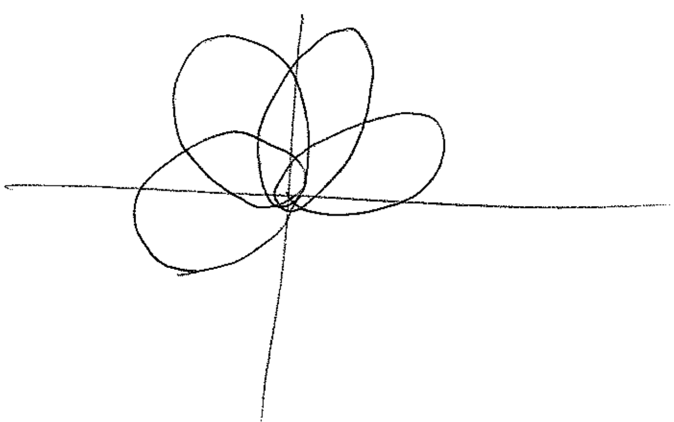
$$\Rightarrow \boxed{r = \cos(\Omega\theta)}$$

$$\Omega = \sqrt{1+4k}$$

$\underline{k=0}$



$\underline{k \neq 0}$



GPS 10.5

$m=1, \omega=1;$

$$S = \frac{1}{2}(q^2+\alpha^2)\cot(t) - q\alpha\csc(t)$$

$$\boxed{S = \frac{1}{2}(q^2+\alpha^2)\frac{1}{\tan t} - \frac{q\alpha}{\sin t}}$$

Solves H-J eqn for principal fn for $H = \frac{1}{2}(p^2+q^2)$.

ie $\partial_t S + H = 0$, where $p = \frac{\partial S}{\partial q}$

$$\Rightarrow \boxed{\partial_t S + \frac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + \frac{1}{2}q^2 = 0} \quad S(q,\alpha,t)$$

Check $\partial_t S = -\frac{1}{2}(q^2+\alpha^2)\frac{\sec^2 t}{\tan^2 t} + \frac{q\alpha\cos t}{\sin^2 t}$

$$\sin^2 t\, \partial_t S = -\frac{1}{2}(q^2+\alpha^2) + q\alpha\cos t$$

$$S_q = \frac{q}{\tan t} - \frac{\alpha}{\sin t}; \quad \sin t\, S_q = q\cos t - \alpha$$

Plug in $\sin^2 t\left(\partial_t S + \frac{1}{2}S_q^2\right) = -\frac{1}{2}(q^2+\alpha^2) + \alpha q\cos t$
$+\frac{1}{2}(q^2\cos^2 t + \alpha^2 - 2\alpha q\cos t)$

$= \frac{1}{2}q^2(\cos^2 t - 1) = -\frac{1}{2}q^2\sin^2 t.$ Works generates solution?

$$p = S_q \Rightarrow p\sin t = q\cos t - \alpha$$

$$\Rightarrow \boxed{\alpha = q\cos t - p\sin t} \quad ①$$

$$\beta = S_\alpha = \frac{\alpha}{\tan t} - \frac{q}{\sin t} \Rightarrow \beta\sin t = \alpha\cos t - q$$

$$\Rightarrow \boxed{q = \alpha\cos t - \beta\sin t} \quad ②$$

Insert ② → ① $\Rightarrow \alpha = (\alpha\cos t - \beta\sin t)\cos t - p\sin t$

$$= \alpha\cos^2 t - \beta\sin t\cos t - p\sin t$$

$$\Rightarrow \alpha\sin t = -\beta\cos t - p$$

$$\Rightarrow \boxed{p = -\alpha\sin t - \beta\cos t} \quad ③$$

Check $\quad \dot{q} = -\alpha\sin t - \beta\cos t = p$

$$\dot{p} = -\alpha\cos t + \beta\sin t = -q$$

Charles ✓